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Covering symmetric supermodular functions by uniform hypergraphs

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Abstract

We consider the problem of finding a uniform hypergraph that satisfies cut demands defined by a symmetric crossing supermodular set function. We give min-max formulas for both the degree specified and the minimum cardinality problem. These results include as a special case a formula on the minimum number of r-hyperedges whose addition to an initial hypergraph will make it k-edge-connected.

1 Introduction

The problem of making a graph k-edge-connected by the addition of a minimal number of new edges, which was originally solved by T. Watanabe and A. Nakamura [8], has many extensions that have been subject to considerable research. Some recent results showed that similar questions for hypergraphs are also worthy of interest. In [3], E. Cheng gave a formula on the minimum number of graph edges that can be added to an initial $(k-1)$-edge-connected hypergraph such that the resulting hypergraph is k-edge-connected; J. Bang-Jensen and B. Jackson [1] extended this result to the case when the initial hypergraph can be arbitrary. This min-max theorem was then further generalized by A. Benczúr and A. Frank in [2], where they considered the minimum number of graph edges that can cover a given symmetric, crossing supermodular set function. This more abstract setting provided a better insight into the combinatorial structure underlying the augmentation problem. Another generalization of Cheng's result due to T. Fleiner and T. Jordán [5] involved the addition of r-hyperedges to a $(k-1)$-edge-connected hypergraph to make it k-edge-connected.

The aim of the present paper is to provide a common generalization of the above mentioned results in [5] and [2], based on the approach of Benczúr and Frank. We give a min-max formula on the minimum number of r-hyperedges that can cover a given symmetric, crossing supermodular set function. As in [2], the substantial part of the proof is a solution of the degree-specified problem (i.e. when each node is contained

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in a prescribed number of new hyperedges, taking into account multiplicities), which then easily leads to a min-max formula on the minimum number of new hyperedges needed.

Let V be a finite ground set. For a function $m : V \to \mathbb{R}$ and a set $X \subseteq V$, we use the notation $m(X) := \sum_{v \in X} m(v)$. Throughout the paper we allow hyperedges to contain nodes with multiplicity. This means that a *hyperedge* can be defined as a function $e : V \to \mathbb{Z}_+$, and it is called an *r-hyperedge* if $e(V) = r$. An *r-uniform hypergraph* is a hypergraph $H = (V, \mathcal{E})$, where V is the ground set, and $\mathcal{E}$ is a collection of r-hyperedges, possibly with repetition. For an $(r-1)$-hyperedge e' and a node $w \in V$, $e' + w$ denotes the r-hyperedge e for which $e(v) = e'(v)$ if $v \neq w$, and $e(w) = e'(w) + 1$. A hyperedge e *enters* a set X if $e(X) > 0$ and $e(V - X) > 0$. We define $d_H(X) := |\{e \in \mathcal{E} \mid e \text{ enters } X\}|$, which has the following property:

$$d_H(X) + d_H(Y) \geq d_H(X \cap Y) + d_H(X \cup Y) \quad \text{for every } X, Y \subseteq V. \tag{1}$$

A set X *separates* a set Y if $Y \cap X \neq \emptyset$ and $Y - X \neq \emptyset$.

Let $p : 2^V \to \mathbb{Z}_+$ be a set function (we always assume that $p(\emptyset) = 0$). The hypergraph H is said to *cover* p if $d_H(X) \geq p(X)$ for every $X \subseteq V$. The set function p is *positively crossing supermodular* if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \tag{2}$$

holds whenever $p(X), p(Y) > 0$, and $X \cap Y, V - (X \cup Y) \neq \emptyset$. If p is also symmetric (i.e. p(X)=p(V-X) for every $X \subseteq V$), and (X, Y) is a pair such that $p(X), p(Y) > 0$, and $X - Y, Y - X \neq \emptyset$, then

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{3}$$

An example for this type of function is the set function that arises in the hypergraph edge-connectivity augmentation problem. Given an initial hypergraph $H_0 = (V, \mathcal{E}_0)$ and an integer k, we can define $p(X) := (k - d_{H_0}(X))^+$ $(\emptyset \neq X \subset V)$. It is easy to see using (1) that p is positively crossing supermodular and of course symmetric. A hypergraph H covers p if and only if $H_0 + H$ is k-edge-connected.

2 Degree-specified hypergraphs

Let V be a finite ground set, $p : 2^V \to \mathbb{Z}_+$ a symmetric, crossing supermodular set function, $r \geq 2$ an integer, and $m : V \to \mathbb{Z}_+$ a degree specification such that $r \mid m(V)$.

We call a partition $\{V_1, \dots, V_l\}$ *p-full* if $l > r$ and

$$p(\cup_{i \in I} V_i) > 0 \quad \text{for every } \emptyset \neq I \subset \{1, \dots, l\}. \tag{4}$$

We always assume that the partition members are indexed so that $m(V_1) \leq m(V_2) \leq \ldots \leq m(V_l)$. Suppose that an r-uniform hypergraph covers p. If we contract the sets $V_1, \dots, V_l$, then the contracted hypergraph (which is still r-uniform since multiplicities are taken into account) must be connected, therefore it needs to have at least $\frac{l-1}{r-1}$ hyperedges. A p-full partition is called a *deficient partition* if

$$\frac{l-1}{r-1} > \frac{m(V)}{r}.$$

The main theorem of this section is the following:

Theorem 2.1. *Let $p : 2^V \to \mathbb{Z}_+$ be a symmetric, positively crossing supermodular set function, $r \geq 2$ an integer, and $m : V \to \mathbb{Z}_+$ a degree specification such that $r \mid m(V)$. There is an r-uniform hypergraph H covering p such that $d_H(v) = m(v)$ for every $v \in V$ if and only if the following are true:*

$$m(X) \geq p(X)\ \forall X \subseteq V, \tag{5}$$

$$\frac{m(V)}{r} \geq p(X)\ \forall X \subseteq V, \tag{6}$$

$$\text{There are no deficient partitions.} \tag{7}$$

Proof. The necessity of the conditions is easily verifiable. We prove sufficiency using induction on $|V| + m(V)$. An r-uniform hypergraph is called *feasible* if it matches the degree specification and covers p. First we show that if there is a set $X \subseteq V$ such that $m(X) = p(X) = 1$ and $|X| > 1$, then there exists a feasible hypergraph. The contraction of X leads to a modified problem: $V' := V - X + v_X$, $m'(v) := m(v)$ if $v \in V' - v_X$, $m'(v_X) := 1$, $p'(Y) := p(Y)$ if $v_X \notin Y$, and $p'(Y) := p((Y - v_X) \cup X)$ if $v_X \in Y$. Conditions (5)–(7) are satisfied by m' and p', and p' is symmetric and positively crossing supermodular, so by induction there is an r-uniform hypergraph $H' = (V', \mathcal{E}')$ with degree vector m', that covers p'. This hypergraph naturally defines an r-uniform hypergraph $H = (V, \mathcal{E})$ with degree vector m; we claim that H covers p. Suppose that $d_H(Y) < p(Y)$ for some $Y \subseteq V$. Y separates X, otherwise there would be a corresponding deficient set in the contracted problem. If $m(X \cap Y) > 0$, then we may assume that X and Y are crossing (because of the symmetry of p), but then $X \cup Y$ is deficient by (1) and (2), while if $m(X \cap Y) = 0$, then $Y - X$ is deficient according to (1) and (3).

From now on it is assumed that if $m(X) = p(X) = 1$ for some $X \subseteq V$, then $|X| = 1$; these singletons are called *special singletons*. The set of special singletons is denoted by S; we consider S as a subset of V.

We define an operation called *splitting*, which is an analogue of the splitting operation for graphs. For an r-hyperedge e for which $e(v) \leq m(v)$ for every $v \in V$, let

$$m^e(v) := m(v) - e(v), \tag{8}$$

$$p^e(X) := \begin{cases} \max(0, p(X) - 1) & \text{if } e \text{ enters } X, \\ p(X) & \text{otherwise.} \end{cases} \tag{9}$$

We say that (m^e, p^e) is obtained from (m, p) by *splitting off* the hyperedge e. A splitting operation is *feasible* if (5), (6), and (7) are true for m^e and p^e. It is easy to see that p^e is symmetric and positively crossing supermodular; so after the execution of a feasible splitting, by induction there exists an r-uniform hypergraph H' with degree vector m^e that covers p^e . By adding the hyperedge e to H' we obtain a feasible hypergraph H.

The rest of the proof consists of showing that a feasible splitting always exists. We define the following families of sets:

$$\begin{aligned}
\mathcal{B}_1 &:= \{X \subseteq V \mid m(X) - p(X) \leq r-2;\ p(Y) < p(X)\ \forall Y \subset X\},\\
\mathcal{B}_2 &:= \{X \subseteq V \mid m(X) - p(X) = r-1;\ p(Y) \leq p(X)\ \forall Y \subset X\},\\
\mathcal{B}_3 &:= \{X \subseteq V \mid p(X) = \frac{m(V)}{r};\ p(Y) < \frac{m(V)}{r}\ \forall Y \subset X\}.
\end{aligned}$$

The inequalities (5) and (6) hold for m^e and p^e if and only if

$$e(X) \leq m(X) - p(X) + 1 \quad \text{for every } X \in \mathcal{B}_1, \tag{10}$$

$$e(X) \leq r-1 \quad \text{for every } X \in \mathcal{B}_2, \tag{11}$$

$$e(X) \geq 1 \quad \text{for every } X \in \mathcal{B}_3. \tag{12}$$

In order to formulate necessary and sufficient conditions for m^e and p^e to satisfy (7), we call a p-full partition $\{V_1, \ldots, V_l\}$ *critical* if

$$\frac{l-1}{r-1} > \frac{m(V)}{r} - 1.$$

For a critical partition $\mathcal{F}$, let $s(\mathcal{F})$ denote the number of special singleton members of $\mathcal{F}$. A critical partition $\mathcal{F}$ is called *proper* if $s(\mathcal{F}) \geq 3$. Critical partitions have the following properties:

Claim 2.2. *If $\mathcal{F} = \{V_1, \ldots, V_l\}$ is a critical partition, then $2l - 2 \geq m(V)$, thus $s(\mathcal{F}) \geq m(V_l)$. In particular, $s(\mathcal{F}) \geq 2$ for every critical partition, and the partition is proper if $m(V_l) \geq 3$.*

Proof. The partition is critical and $\frac{m(V)}{r}$ is an integer, so

$$m(V) \leq r\left(\frac{l-2}{r-1} + 1\right) = 2r + \frac{r}{r-1}(l-r-1) \leq 2r + 2(l-r-1) = 2l-2.$$

□

Claim 2.3. *A partition $\{V_1, \ldots, V_l\}$ is critical if and only if $l > r$, $\frac{l-1}{r-1} > \frac{m(V)}{r} - 1$, $p(V_1) = 1$, and $p(V_1 \cup V_i) \geq 1$ $(i = 2, \ldots, l)$. If the partition is critical and U is the union of some partition members such that $V_1 \subseteq U$ and $V_2 \cap U = \emptyset$, then $p(U) = 1$.*

Proof. Let $\{V_1, \ldots, V_l\}$ be a partition with the above properties. If U is the union of at most $l-2$ partition members such that $V_1 \subseteq U$ and $V_j \cap U = \emptyset$, then (2) implies that $p(V_1 \cup V_j) + p(U) \leq p(V_1) + p(U \cup V_j)$, from which $p(U) \leq p(U \cup V_j)$. As a consequence, $p(U) > 0$ if U is the union of at most $l-1$ members including V_1. By the symmetry of p, the same is true if $U \cap V_1 = \emptyset$. If $V_1 \subseteq U$ and $V_2 \cap U = \emptyset$, then the above argument gives $1 = p(V_1) \leq p(U) \leq p(V - V_2) = 1$. □

Claim 2.4. *Let e be an r-hyperedge which satisfies (10)–(12). Then m^e and p^e satisfy (7) if and only if*

$$e(V - X) > 0 \text{ for any member } X \text{ of any proper critical partition.} \tag{13}$$

Proof. Clearly, only critical partitions can become deficient partitions after the splitting. If $e(V - X) > 0$ for every member X of a given critical partition, then Claim 2.3 guarantees that $p^e(U) = 0$ for some union U of members, so the partition is not p^e-full. If the critical partition is not proper, then a partition member X for which $e(V - X) = 0$ has a subset that violates (10) or (11). □

It suffices to show the existence of an r-hyperedge $e \leq m$ that satisfies (10), (11), (12), and (13). First we consider only (10) and (12):

$$\begin{aligned} Q &:= \{e : V \to \mathbb{Z}_+ \mid e \leq m;\ e(V) = r; \\ &\qquad e(X) \leq m(X) - p(X) + 1\ \forall X \in \mathcal{B}_1;\ e(X) \geq 1\ \forall X \in \mathcal{B}_3\}. \end{aligned}$$

Claim 2.5. *The family $\mathcal{B}_1 \cup \mathcal{B}_3$ is laminar. The sets in $\mathcal{B}_3$ are pairwise disjoint, and if $X \in \mathcal{B}_1$ and $Y \in \mathcal{B}_3$ are not disjoint, then $X \subseteq Y$.*

Proof. If $X, Y \in \mathcal{B}_1 \cup \mathcal{B}_3$, and $X - Y, Y - X, X \cap Y \neq \emptyset$, then $p(X) \leq p(X - Y)$ or $p(Y) \leq p(Y - X)$ by (3), which contradicts the definition of $\mathcal{B}_1$ and $\mathcal{B}_3$. If $X \in \mathcal{B}_1$ and $Y \in \mathcal{B}_3$, then $p(Y) \geq p(X)$, so $Y \not\subset X$ according to the definition of $\mathcal{B}_1$. □

It follows from Claim 2.5 that Q consists of the integral vectors of a g-polymatroid (which moreover is determined by a weak pair defined on a laminar family).

Claim 2.6. *Q is non-empty.*

Proof. The non-emptiness of Q is equivalent to the following (see e.g. [6], though in this case it is easy to show directly as well):

1. $|\mathcal{B}_3| \leq r$,
2. $m(V - \cup_{i=1}^t X_i) + \sum_{i=1}^t (m(X_i) - p(X_i) + 1) \geq r$ for every sub-partition $\{X_1, \ldots, X_t\}$.

The first condition holds since $m(X) \geq \frac{m(V)}{r}$ for every $X \in \mathcal{B}_3$. The second condition is clearly true if $t \geq r$. If $t < r$, then $m(V - \cup X_i) + \sum(m(X_i) - p(X_i) + 1) \geq m(V - \cup X_i) + \sum(m(X_i) - \frac{m(V)}{r} + 1) = (r - t)\frac{m(V)}{r} + t \geq r$, the last inequality being valid because $m(V) \geq r$. □

Obviously, if an r-hyperedge e defines a feasible splitting, then it is in Q. The converse is generally not true; however, it turns out to be true when $\mathcal{B}_3 \neq \emptyset$:

Lemma 2.7. *If $\mathcal{B}_3 \neq \emptyset$, then any r-hyperedge $e \in Q$ defines a feasible splitting.*

Proof. Let $X \in \mathcal{B}_3$ and $e \in Q$. First suppose that there is a set $Y \in \mathcal{B}_2$ such that $e(Y) = r$. Then $X \cap Y \neq \emptyset$, and one of $X - Y$ and $Y - X$ is empty, otherwise (3) would imply that either $p(X - Y) \geq p(X)$ or $p(Y - X) > p(Y)$, contrary to the definition of $\mathcal{B}_3$ and $\mathcal{B}_2$. If $Y \subseteq X$, then there is a set $X' \in \mathcal{B}_3$ such that $X' \subseteq V - X \subseteq V - Y$; if $X \subseteq Y$, then $Y \in \mathcal{B}_2$ implies that $p(V - Y) = p(Y) \geq p(X) = \frac{m(V)}{r}$, so $\mathcal{B}_3$ would again contain a set $X' \subseteq V - Y$. This is impossible since $e \in Q$, which requires $e(X') \geq 1$.

Now suppose that a proper critical partition $\mathcal{F} = \{V_1, \ldots, V_l\}$ becomes deficient after the splitting. Since $\mathcal{B}_3$ contains at least two disjoint sets, it contains a set X that is disjoint from at least two special singleton members of $\mathcal{F}$, say V_1 and V_2. First we show that a member of $\mathcal{F}$ can not separate X. If V_i separates X, then $p(V_1 \cup V_i - X) \geq p(V_1 \cup V_i) + p(X) - p(X - V_i) > p(V_1 \cup V_i) = 1$. Let $U := V - V_i - V_2$; then $p(U \cup X) \geq p(U) + p(X) - p(X \cap U) > p(U) = 1$. Thus $p(V_i \cup U) \geq p(V_1 \cup V_i - X) + p(U \cup X) - p(V_1) > 1$, which contradicts Claim 2.3.

We can conclude that there is a partition member V_i such that $X \subseteq V_i$. This implies $m^e(V_i) \geq m^e(X) \geq p^e(X) = \frac{m^e(V)}{r}$, so $m^e(V) \geq l - 1 + \frac{m^e(V)}{r}$. But then $\frac{l-1}{r-1} \leq \frac{m^e(V)}{r}$, so $\mathcal{F}$ could not become deficient after the splitting. □

By Claim 2.6 and Lemma 2.7 we may assume that $\mathcal{B}_3 = \emptyset$. To handle condition (13), we will use some information on the structure of proper critical partitions. This information is based on an auxiliary graph G defined on the special singletons: $G = (S, F)$, where $uv \in F$ if and only if $p(\{u, v\}) = 1$.

Claim 2.8. *If $X \in \mathcal{B}_1$ and $|X| \geq 2$, then $d_G(X) = 0$.*

Proof. Suppose that uv is an edge of G such that $u \in X$, $v \notin X$. Then by (3), $p(X - u) \geq p(X) + p(\{u, v\}) - p(v) = p(X)$, which contradicts $X \in \mathcal{B}_1$. □

Claim 2.9. *Every proper critical partition $\mathcal{F} = \{V_1, \ldots, V_l\}$ can be refined by separating some special singletons in such a way that the resulting critical partition $\mathcal{F}_S$ (the S-*refinement *of $\mathcal{F}$) has the following properties:*

- *If $d_G(X) > 0$ for a member $X \in \mathcal{F}_S$, then X is a special singleton,*
- *The set of special singleton members of $\mathcal{F}_S$ defines a component of G that is a clique (which we call the S-*clique *of $\mathcal{F}$).*

Proof. We prove that if there is an edge uv of G such that $u \in V_i$, $v \notin V_i$ for some non-singleton partition member V_i, then a critical partition is obtained if we replace V_i by $\{u\}$ and $V_i - u$. $\mathcal{F}$ is proper, so we can assume that $v \neq V_1, V_2$. According to Claim 2.3, we have to show that $p(V_1 \cup V_i - u) > 0$ and $p(V_1 + u) > 0$.

By Claim 2.3, $p(V_1 \cup V_i) = 1$, so $p(V_1 \cup V_i - u) \geq p(V_1 \cup V_i) + p(\{u, v\}) - p(\{v\}) = 1$. Similarly, $p(V_2 \cup V_i - u) \geq 1$; thus $p(V_1 + u) \geq p(V_1 \cup V_i) + p(V_2 \cup V_i - u) - p(V_2) \geq 1$. As a consequence, the replacement of V_i by $\{u\}$ and $V_i - u$ results in a critical partition.

By repeating this step as many times as possible, we obtain a critical partition $\mathcal{F}_S$ with the required properties. □

Among the components of G that are cliques, let $K^* \subseteq S$ be one of maximal size. We choose an r-hyperedge $e^* \in Q$ such that $e^*(K^*)$ is maximal, and then choose an $(r-1)$-hyperedge $e' \leq e^*$ such that $e'(K^*)$ is maximal. Let $W := \{v \in V : m(v) > e'(v)\}$. We will show that there exists a node $w \in W$ such that the r-hyperedge $e = e' + w$ defines a feasible splitting.

Lemma 2.10. *If the splitting off of a hyperedge $e = e' + w$ results in a deficient partition, then (10) or (11) is violated for a set $X \in \mathcal{B}_1 \cup \mathcal{B}_2$.*

Proof. We can assume that $K^* \neq \emptyset$, otherwise there are no critical partitions. Since e^* can be chosen by the greedy algorithm, and Claim 2.8 implies that K^* is either disjoint from or subset of any non-singleton set in $\mathcal{B}_1$, $e'(K^*)$ is the minimum of the following three values:

$$|K^*|, \tag{14}$$

$$r - 1, \tag{15}$$

$$\min_{K^* \subseteq X \in \mathcal{B}_1} (m(X) - p(X) + 1). \tag{16}$$

Let V_i be the member of the would-be deficient partition $\mathcal{F}$ for which $e(V - V_i) = 0$; then $|K^* \cap V_i| \geq e'(K^*) > 0$. We can assume that $m(V_i) > 2$, since otherwise a subset of V_i would violate (10) or (11). If $K^* \subseteq V_i$ and $m(V_i - K^*) > 0$, then $s(\mathcal{F}) < m(V_i)$, given that the special singleton members form a clique in G whose size is at most $|K^*|$. This contradicts the criticality of $\mathcal{F}$ according to Claim 2.2. If $K^* \subseteq V_i$ and $m(V_i - K^*) = 0$, then we consider the set $X := \{v \in V_i : \ e(v) > 0\}$. The crossing supermodularity of p easily implies that $p(Y) > 0$ for every $Y \subseteq K^*$, hence $p(X) > 0$. But then a subset of X violates (10) or (11).

Now suppose that $K^* \not\subseteq V_i$, and let $\mathcal{F}_S$ denote the S-refinement of $\mathcal{F}$. By Claim 2.9, K^* is the set of special singletons that are members of $\mathcal{F}_S$. If $e'(K^*) = r - 1$, then $\frac{|\mathcal{F}_S|-1}{r-1} \geq \frac{|\mathcal{F}|+(r-1)-1}{r-1} \geq \frac{|\mathcal{F}|-1}{r-1} + 1 > \frac{m(V)}{r}$, which means that $\mathcal{F}_S$ would violate (7).

If $e'(K^*)$ is determined by (16), then there is a set $X \in \mathcal{B}_1$ such that $K^* \subseteq X$, $V_i \not\subseteq X$, and $|K^* \cap V_i| \geq m(X) - p(X) + 1$. Let U denote the union of the members of $\mathcal{F} - \{V_i\}$ that are not special singletons. Then $s(\mathcal{F}) = |K^* - U| - |K^* \cap V_i| \leq |K^* - U| - m(X) + p(X) - 1 < p(X) - m(X \cap U)$. By Claim 2.2, this would imply that $m(V_i) < p(X) - m(X \cap U)$. However, if $U \subseteq X$, then $m(V_i) \geq p(V_i - X) = p(X)$, while if $U \not\subseteq X$, then $m(V_i) \geq p(V_i - X) = p(U \cup X) \geq p(X) + p(V_1 \cup U) - p(X \cap (V_1 \cup U)) \geq p(X) + 1 - m(X \cap (V_1 \cup U)) = p(X) - m(X \cap U)$. □

Lemma 2.10 implies that condition (13) can be ignored when choosing an appropriate node $w \in W$. We have already seen that $e = e' + w \in Q$ is a necessary condition for the feasibility of the splitting. In addition to that, $e(X) \leq r - 1$ must hold for sets in $\mathcal{B}_2$. A set $X \in \mathcal{B}_2$ is called *critical* if $e'(X) = r - 1$, and there is no $Y \in \mathcal{B}_1$ such that $X \subseteq Y$ and $m(Y) - p(Y) = r - 2$. An r-hyperedge $e = e' + w$ defines a feasible splitting if and only if $e \in Q$ and there is no critical set containing w. Since $e^* \in Q$, we can assume that there is at least one critical set in $\mathcal{B}_2$.

Claim 2.11. *Let $X \in \mathcal{B}_2$ be a critical set. If $Y \in \mathcal{B}_1$, then one of $X \cap Y$, $X - Y$, and $Y - X$ is empty. If $w \in W - X$, then $e' + w \in Q$.*

Proof. If $X \in \mathcal{B}_2$, $Y \in \mathcal{B}_1$, and $X \cap Y, X - Y, Y - X$ are non-empty, then $p(X - Y) \leq p(X)$ and $p(Y - X) < p(Y)$ by the definition of $\mathcal{B}_1$ and $\mathcal{B}_2$, which contradicts (3). If $w \in W - X$, then $e' + w \in Q$ unless there is a set $Y \in \mathcal{B}_1$ such that $w \in Y$ and $e'(Y) = m(Y) - p(Y) + 1$. But then $X \cap Y \neq \emptyset$, so $X \subseteq Y$, which contradicts the criticality of X. □

Suppose indirectly that every $w \in W$ is in a critical set. Consider a family $\mathcal{Z} = \{Z_1, \dots, Z_t\}$ of maximal critical sets, such that every $w \in W$ is in at least one of them, and the family has minimal number of members. The following sequence of claims establishes that the existence of such a family leads to a contradiction. Let $Z := \cap_{i=1}^t Z_i$.

Claim 2.12. *If $i \neq j$, then $m(Z_i \cap Z_j) = m(Z) = r - 1$, and $m(Z_i - Z_j) = p(Z_i - Z_j)$.*

Proof. We know that $m(Z_i \cap Z_j) \geq m(Z) \geq e'(Z) \geq r - 1$. On the other hand, (3) gives that $0 \leq p(Z_i - Z_j) + p(Z_j - Z_i) - p(Z_i) - p(Z_j) \leq m(Z_i - Z_j) + m(Z_j - Z_i) - p(Z_i) - p(Z_j) = 2(r - 1) - 2m(Z_i \cap Z_j)$. This is possible only if equality holds throughout. □

Clearly $|\mathcal{Z}| \geq 2$ and $\frac{m(V)}{r} \geq 2$, since $m(V - Z_i) > 0$ for every i. Suppose that $|\mathcal{Z}| = 2$. Then $p(Z_1) = m(Z_1 - Z_2) = m(V - Z_2) \geq p(Z_2)$ and vice versa, so $p(Z_1) = p(Z_2) = m(Z_1 - Z_2) = m(Z_2 - Z_1) = \frac{m(V)-(r-1)}{2}$. This value can be integer only if $r \geq 3$, but then $\frac{m(V)}{r} \leq \frac{m(V)-(r-1)}{2} = p(Z_1)$, contradicting $\mathcal{B}_3 = \emptyset$. Therefore we may assume that $|\mathcal{Z}| \geq 3$.

Claim 2.13. *$\cup_{i=1}^t Z_i = V$, $Z_j - Z$ is a special singleton for every j, and $p(Z_i \cup Z_j - Z) > 0$ for every i, j.*

Proof. For a set of indices $I \subseteq \{1, \dots, t\}$, let $U_I := \cup_{i \in I} Z_i$. Since $p(Z_i) = m(Z_i - Z)$ for every i, the repeated application of (2) gives that $p(U_I) \geq m(U_I - Z) - (|I| - 1) > 0$ if $U_I \neq V$. For $I = \{1, \dots, t\}$, this implies that $\cup_{i=1}^t Z_i = V$, since $p(\cup_{i=1}^t Z_i) \leq m(V - \cup_{i=1}^t Z_i) = 0$. Now let $I := \{1, \dots, t\} - \{i_0\}$, where i_0 is chosen so that $m(Z_{i_0} - Z)$ is minimal. Then $m(Z_{i_0} - Z) \geq p(U_I) \geq m(U_I - Z) - (t - 2)$. Since $|\mathcal{Z}| \geq 3$, this is only possible if $m(Z_i - Z) = 1$ for every i. For a fixed j, $p(Z_j - Z_i) = m(Z_j - Z_i) = 1$ for every i, so by repeatedly applying (2) we get $p(Z_j - Z) = p(\cup_{i \neq j}(Z_j - Z_i)) \geq 1$, hence $Z_j - Z$ is a special singleton. Finally, if $i \neq j$, then by setting $I := \{1, \dots, t\} - \{i, j\}$, we obtain $p(Z_i \cup Z_j - Z) = p(U_I) > 0$. □

Claim 2.14. *$\mathcal{F} := \{Z, Z_1 - Z, \dots, Z_t - Z\}$ is a proper critical partition.*

Proof. The partition has size $l = m(V) - (r - 1) + 1 \geq r + 2$ since $m(V) \geq 2r$, so $s(\mathcal{F}) \geq r + 1$, and $\frac{l-1}{r-1} = \frac{m(V)-(r-1)}{r-1} > \frac{m(V)}{r} - 1$. If X is the union of two partition members, then Claim 2.13 implies that $p(X) > 0$; therefore $\mathcal{F}$ is a proper critical partition by Claim 2.3. □

The special singleton members of $\mathcal{F}$ form a component of G that is a clique, and the size of this clique is at least $r+1$, therefore $|K^*| \geq r+1$. This means that $K^* \not\subseteq Z$, so K^* must be the S-clique of $\mathcal{F}$. The value of $e'(K^*)$ is not determined by (16): if $K^* \subseteq X \in \mathcal{B}_1$, then $Z_i \subseteq X$ for every i by Claim 2.11, which is not possible. It follows that $e'(K^*) = r-1$.

To prove that the existence of $\mathcal{Z}$ contradicts (7), we consider the S-refinement $\mathcal{F}_S$ of $\mathcal{F}$. The properties of S-refinements stated in Claim 2.9 imply that every member of $\mathcal{F}_S$ is a special singleton. However, such a p-full partition would be a deficient partition, given that $\frac{m(V)}{r} \geq 2$.

We proved that there is a node w such that the r-hyperedge $e = e' + w$ can be feasibly split off. This concludes the proof of Theorem 2.1. □

3 Hypergraphs with minimum number of hyperedges

As in the case of many edge-connectivity augmentation results including [2], the characterization of the degree-specified problem in Theorem 2.1 can be used in a straightforward way to prove a min-max theorem on the corresponding minimum cardinality problem. Recall that a partition $\{V_1, \dots, V_l\}$ is called *p-full* if $l > r$ and $p(\cup_{i \in I} V_i) > 0$ for every $\emptyset \neq I \subset \{1, \dots, l\}$.

Theorem 3.1. *Let $p : 2^V \to \mathbb{Z}_+$ be a symmetric, positively crossing supermodular set function, and $r \geq 2$ an integer. There is an r-uniform hypergraph with γ hyperedges that covers p if and only if the following hold:*

$$r\gamma \geq \sum_{i=1}^{t} p(X_i) \quad \textit{for every partition } \{X_1, \dots, X_t\}, \tag{17}$$

$$\gamma \geq p(X) \quad \textit{for every } X \subseteq V, \tag{18}$$

$$\gamma \geq \frac{l-1}{r-1} \quad \textit{if there is a p-full partition with } l \textit{ members.} \tag{19}$$

Proof. The conditions are clearly necessary for the existence of an r-uniform hypergraph that covers p. We prove sufficiency for a fixed γ. Let $m' : V \to \mathbb{Z}_+$ be a vector that satisfies (5) such that $m'(V)$ is minimal, and let $M' := \{v \in V : m'(v) > 0\}$. Then for every node $v \in M'$, there exists a set X for which $v \in X$ and $m'(X) = p(X)$; sets with the latter property are called *tight*. There is a family of tight sets covering every node $v \in M'$; let $\mathcal{F}$ be such a family with $|\mathcal{F}|$ minimal. If $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ are not disjoint, then $X \cup Y = V$, otherwise $X \cup Y$ would be tight according to (2), which would contradict the minimality of $|\mathcal{F}|$, as X and Y could be replaced by $X \cup Y$. The symmetry of p implies that $X - Y = V - Y$ and $Y - X = V - X$ are both tight and $m'(X \cap Y) = 0$, so $X - Y$ and $Y - X$ cover every node of M'. We can conclude that there is always a partition $\{X_1, \dots, X_t\}$ for which $\sum_{i=1}^{t} p(X_i) = m'(V)$. It follows from (17) that $r\gamma \geq m'(V)$.

We can obtain a degree specification $m : V \to \mathbb{Z}_+$ from m' by increasing m' on one arbitrary node by $r\gamma - m'(V)$. Then m satisfies (5), (6), (7), and $r \mid m(V)$, thus by Theorem 2.1 there exists an r-uniform hypergraph H with degree-vector m that covers p. The choice of m implies that H has γ hyperedges. □

As it was mentioned in the introduction, the k-edge-connectivity augmentation of an initial hypergraph $H_0 = (V, \mathcal{E}_0)$ corresponds to the case when $p(X) = (k - d_{H_0}(X))^+$ ($\emptyset \neq X \subset V$). This set function is symmetric and positively crossing supermodular, so Theorem 3.1 is applicable; furthermore, condition (19) concerning p-full partitions can be considerably simplified. For a hypergraph $H = (V, \mathcal{E})$, let $i_H(X) := |\{e \in \mathcal{E} : e(V - X) = 0\}|$, and let $c(H)$ denote the number of components of H.

Corollary 3.2. *Let $H_0 = (V_0, \mathcal{E}_0)$ be a hypergraph, and $r \geq 2$ an integer. There is an r-uniform hypergraph H with γ hyperedges such that $H_0 + H$ is k-edge-connected if and only if the following hold:*

$$r\gamma \geq tk - \sum_{i=1}^{t} d_{H_0}(X_i) \quad \text{for every sub-partition } \{X_1, \ldots, X_t\}, \tag{20}$$

$$\gamma \geq k - d_{H_0}(X) \quad \text{for every } X \subseteq V, \tag{21}$$

$$(r-1)\gamma \geq c(H_0 - \mathcal{E}'_0) - 1 \quad \text{for every } \mathcal{E}'_0 \subseteq \mathcal{E}_0 \text{ for which } |\mathcal{E}'_0| = k - 1. \tag{22}$$

Proof. Compared to Theorem 3.1, the only difference is that (19) is replaced by condition (22). Its necessity follows from the fact that the components of $H_0 - \mathcal{E}'_0$ form a p-full partition if $c(H_0 - \mathcal{E}'_0) > r$. We prove sufficiency by showing that if a p-full partition $\mathcal{F} = \{V_1, \ldots, V_l\}$ violates (19) while conditions (20) and (21) are satisfied, then an appropriate $\mathcal{E}'_0$ violates (22). Let $\mathcal{E}'_0$ be the set of hyperedges in $\mathcal{E}_0$ that enter at least one member of $\mathcal{F}$, and let $H'_0 := (V, \mathcal{E}'_0)$. The partition $\mathcal{F}$ violates (19), so $(r-1)\gamma < l - 1 \leq c(H_0 - \mathcal{E}'_0) - 1$.

We claim that $|\mathcal{E}'_0| = k - 1$. By (20), $lk - \sum d_{H'_0}(V_i) = lk - \sum d_{H_0}(V_i) \leq r\gamma \leq \frac{r}{r-1}(l-2) \leq 2l - 4$, from which $\sum d_{H'_0}(V_i) \geq (k-2)l + 4$. This implies that $|\mathcal{E}'_0| \geq k - 1$, and there are at least 4 members of $\mathcal{F}$ (say V_1, V_2, V_3, V_4), for which $d_{H'_0}(V_i) = k - 1$. We can assume that $i_{H'_0}(V_1 \cup V_2) \leq i_{H'_0}(V_i \cup V_j)$ for every $i, j \in \{1, 2, 3, 4\}$.

If $i_{H'_0}(V_1 \cup V_2) > 0$, then $d_{H'_0}(V_1 \cup V_2) \geq d_{H'_0}(V_1) - i_{H'_0}(V_1 \cup V_2) + i_{H'_0}(V_2 \cup V_3) + i_{H'_0}(V_2 \cup V_4) \geq k$, contradicting the p-fullness of $\mathcal{F}$. So $i_{H'_0}(V_1 \cup V_2) = 0$, in which case there are $k-1$ hyperedges in $\mathcal{E}'_0$ that enter each of V_1, V_2 and $V_1 \cup V_2$. Suppose that $\mathcal{E}'_0$ contains a hyperedge besides these $k - 1$, which enters a partition member V_i. Then $d_{H'_0}(V_1 \cup V_i) \geq k$, contrary to the p-fullness of $\mathcal{F}$; hence $|\mathcal{E}'_0| = k - 1$. □

4 Remarks

It might be argued that Theorems 2.1 and 3.1 are not good characterizations, since it is not possible to check in polynomial time whether a given partition is p-full, hence it cannot be decided whether it is a deficient partition or not. If a partition has at least one member V_i with $p(V_i) = 1$, then its deficiency can be checked using the characterization in Claim 2.3. But in general, deciding whether a partition is p-full or not is NP-complete (see [2], where p-fullness was defined differently for this reason).

However, it is easy to see that if $p(V_i) \geq 2$ for every member of a deficient partition, then at least one of the partition members violates (5) (and the partition violates (17) in case of Theorem 3.1). This means that Theorems 2.1 and 3.1 give good co-NP characterizations.

If an oracle is available that can maximize the set function $p + x - d_H$ for any modular function x and any hypergraph H, then the proof of Theorem 2.1 provides a polynomial algorithm for finding consecutive feasible splittings, and the construction of a degree specification that has the properties described in the proof of Theorem 3.1 is also polynomial. Hence, given the appropriate oracles, polynomial algorithms can be constructed for the problems presented in this paper. In the case of k-edge-connectivity augmentation, the required oracles can be realized using network flow algorithms.

As it was mentioned in the introduction, the framework studied in this paper is a generalization of the problem of k-edge-connectivity augmentation of hypergraphs by r-hyperedges. It is a natural question to ask whether these results could be extended to local edge-connectivity augmentation. In a closely related problem this extension was possible: a result of Z. Szigeti [7] implies that if there is no restriction on the size of the hyperedges, and the objective is to minimize the sum of the sizes of the new hyperedges, then local edge-connectivity augmentation can be solved in polynomial time.

Though this raised hopes that the minimum cardinality problem, i.e the minimization of the number of new r-hyperedges added to the initial hypergraph, might also be tractable for local edge-connectivity augmentation, it turned out that this problem is NP-complete. Let $\lambda_H(u, v)$ denote the local edge-connectivity between u and v in a hypergraph H, i.e.

$$\lambda_H(u, v) := \min_{u \in X, v \notin X} d_H(X).$$

NP-completeness was proved by B. Cosh, B. Jackson, and Z. Király for the following special case:

Theorem 4.1 ([4]). *Let $H_0 = (V, \mathcal{E}_0)$ be a connected hypergraph, $\mathcal{F}$ a partition of V, and $\gamma \in \mathbb{Z}_+$ an integer. The following problem is NP-complete: decide whether there exists a graph $G = (V, E)$ with γ edges such that $\lambda_{H_0+G}(u, v) \geq 2$ whenever u and v are in the same member of $\mathcal{F}$.*

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