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Constructive characterizations for packing and covering with trees

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Abstract

We give a constructive characterization of undirected graphs which contain k spanning trees after *adding* any new edge. This is a generalization of a theorem of Henneberg and Laman who gave the characterization for $k = 2$.

We also give a constructive characterization of graphs which have k edge-disjoint spanning trees after *deleting* any edge of them.

Keywords. graph, constructive characterization, packing and covering by trees

AMS subject classification. 05C40

1 Introduction

By a constructive characterization of a graph property, we mean a building procedure consisting of some simple steps so that the graphs obtained from a specified initial graph are precisely those having the property. For example, a graph is connected if and only if it can be obtained from a node by the operation: add a new edge connecting an existing node with either an existing node or a new one. A slightly less trivial known result is the so called ear-decomposition of 2-connected graphs, while W.T. Tutte's [17] constructive characterization of 3-connected graphs is much deeper. Note that no constructive characterization is known for general k-connectivity. As far as edge-connectivity is concerned, the situation is much better. A graph is said to be **k-edge-connected** if the deletion of at most $k-1$ edges leaves a connected graph. From now on, adding an edge means adding a new edge connecting two existing nodes. This new edge can be parallel to existing ones, but it cannot be a loop unless otherwise stated. In 1976 L. Lovász [9] proved the following result.

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Theorem 1.1 (Lovász). *An undirected graph $G = (V, E)$ is $2k$-edge-connected if and only if G can be obtained from a single node by the following two operations:*

(i) *add a new edge (possibly a loop),*

(ii) *add a new node z, subdivide k existing edges by new nodes, and identify the k subdividing nodes with z.*

A directed counterpart of this result is due to W. Mader [11]. A digraph is said to be k**-edge-connected** if the deletion of at most $k-1$ edges leaves a strongly connected digraph.

Theorem 1.2 (Mader). *A directed graph $G = (V, E)$ is k-edge-connected if and only if G can be obtained from a single node by the following two operations:*

(i) *add a new edge (possibly a loop),*

(ii) *add a new node z, subdivide k existing edges by new nodes and identify the k subdividing nodes with z.*

We call the operation (ii) in these theorems **pinching** k **edges** (**with** z).

This kind of characterizations can be very useful. For example, Lovász used his result to derive Nash-Williams' theorem [12] on k-edge-connected orientations of graphs, while Mader used his result to derive Edmonds' theorem [1] on disjoint arborescences.

k-edge-connectivity is the usual way to formulate one's intuitive feeling for high 'edge-connection' of an undirected graph but there may be other possibilities, as well. We call an undirected graph k**-tree-connected** if it contains k edge-disjoint spanning trees. In 1961 W.T. Tutte found the following characterization [16].

Theorem 1.3 (Tutte). *An undirected graph $G = (V, E)$ is k-tree-connected if and only if*

$$e(\mathcal{F}) \geq k(t-1) \tag{1}$$

for every partition $\mathcal{F} = \{X_1, X_2, \ldots X_t\}$ of V with non-empty subsets and $t \geq 2$, where $e(\mathcal{F})$ denotes the number of edges connecting distinct classes of $\mathcal{F}$.

By the definition itself, it is straightforward to construct all k-tree-connected graphs: take k edge-disjoint spanning trees and add some extra edges. In the spirit of Lovász' theorem, however, it would be desirable to find an operation which constructs a k-tree-connected graph from one which has one less node or edge. This is indeed possible as was pointed out in [4] by observing that a combination of Mader's Theorem 1.2 and Tutte's Theorem 1.3 gives rise to the following. For a direct proof, see Tay [15].

Theorem 1.4. *An undirected graph $G = (V, E)$ is k-tree-connected if and only if G can be built from a single node by the following two operations:*

(i) *add a new edge,*

(ii) *add a new node z and k new edges ending at z,*

(iii) *pinch i ($1 \le i \le k-1$) existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.*

C.St.J.A. Nash-Williams [14] proved the following counterpart of Tutte's theorem concerning coverings of trees rather than packing. For a graph $G = (V, E)$, $\gamma_G(X)$ denotes the number of the edges of G with both end-nodes in $X \subseteq V$.

Theorem 1.5 (Nash-Williams). *A graph $G = (V, E)$ is the union of k edge-disjoint forests if and only if $\gamma_G(X) \le k|X| - k$ for all nonempty $X \subseteq V$.*

One has the following constructive characterization for these graphs.

Theorem 1.6. *An undirected graph $G = (V, E)$ is the union of k edge-disjoint forests if and only if G can be built from a single node by the following two operations:*

(j) *add a new node z and at most k new edges ending at z,*

(jj) *pinch i ($1 \le i \le k-1$) existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.*

The proof of this theorem easily follows by a proof of Theorem 1.4. (Sketch: It is clear that a graph G which can be obtained by the operations is the union of k edge-disjoint forests. The other direction: if there is no node of degree at most k, then consider a node z of degree at most $2k-1$. We can add edges to G not incident to z so that G becomes k-tree-connected. Since the inverse of operation (iii) in Theorem 1.4 can be applied at any node of degree at most $2k-1$ so that it results in a k-tree-connected graph, we are done.)

In this paper we consider two variants of the notion of k-tree-connectivity. We call a graph G (with at least 2 nodes) **nearly k-tree-connected** if G is not k-tree-connected but adding any new edge to G results in a k-tree-connected graph.

A nearly k-tree-connected graph has a partition $\mathcal{F} = \{X_1, X_2, \dots X_t\}$ violating (1). In such a partition each set X_i must be a singleton for otherwise $\mathcal{F}$ would violate (1) even after adding an edge connecting two distinct elements of X_i. Hence a nearly k-tree-connected graph has exactly one partition $\mathcal{F}$ violating (1) and $\mathcal{F}$ consists of singletons. Therefore such a graph $G = (V, E)$ has exactly $k(|V| - 1) - 1$ edges.

Nearly 2-tree-connected graphs arose in connection with rigidity properties. L. Henneberg [6] described a way to generate all minimally rigid plane structures, while G. Laman [8] found a characterization of so-called minimally generically rigid graphs. By combining these results, one obtains the following.

Theorem 1.7 (Henneberg and Laman). *A graph G is nearly 2-tree-connected if and only if G can be constructed from one (non-loop) edge by the following two operations:*

(i) *add a new node z and connect z to two distinct existing nodes,*

(ii) *subdivide an existing edge uv by a node z and connect z to an existing node distinct from u and v.*

We are going to extend this result for arbitrary $k \geq 2$. Let K_2^{k-1} denote the graph on two nodes with $k-1$ parallel edges.

Theorem 1.8. *An undirected graph $G = (V, E)$ is nearly k-tree-connected if and only if G can be built from K_2^{k-1} by applying the following operations:*

(O1') *add a new node z and k new edges ending at z so that no k parallel edges can arise,*

(O2') *choose a subset F of i existing edges $(1 \leq i \leq k-1)$, pinch the elements of F with a new node z, and add $k-i$ new edges connecting z with other nodes so that there are no k parallel edges in the resulting graph.*

Actually, we will prove this result in a slightly more general form. A graph $G = (V, E)$ is said to be k**-sparse** if $\gamma_G(X) \leq k|X| - (k+1)$ for all $X \subseteq V, |X| \geq 2$. (By convention the graph with a single node is k-sparse.) By Theorem 1.5 of Nash-Williams, a graph $G = (V, E)$ with $|E| = k|V| - (k+1)$ is nearly k-tree-connected if and only if G is k-sparse. Therefore the following constructive characterization of k-sparse graphs is indeed a generalization of Theorem 1.8.

Theorem 1.9. *An undirected graph $G = (V, E)$ is k-sparse if and only if G can be built from a single node by applying the following operations:*

(O1) *add a new node z and at most k new edges ending at z so that no k parallel edges can arise,*

(O2) *choose a subset F of i existing edges $(1 \leq i \leq k-1)$, pinch the elements of F with a new node z, and add $k-i$ new edges connecting z with other nodes so that there are no k parallel edges in the resulting graph.*

We call a graph **highly k-tree-connected** if the deletion of any existing edge leaves a k-tree-connected graph. In [5] a constructive characterization was given (among others) for highly 2-tree-connected graphs. Here we extend this for arbitrary $k \geq 2$.

Theorem 1.10. *An undirected graph $G = (V, E)$ is highly k-tree-connected if and only if G can be built up from a node by the following two operations:*

(j) *add a new edge (possibly a loop),*

(jj) *pinch i $(1 \leq i \leq k-1)$ existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.*

Section 2 includes some further notation and notions along with an important remark on why the proof of Theorem 1.9 is significantly more difficult than that of apparently similar results. We provide then some basic lemmas for proving Theorem 1.9 which already give rise to Theorem 1.7 of Henneberg and Laman.

Section 3 provides a necessary and sufficient condition for a given node to admit the inverse of the operation in Theorem 1.9, while Section 4 serves to prove that there always exists a node satisfying this condition, completing this way the proof of Theorem 1.9. In Section 5 we prove Theorem 1.10. An interesting feature of this proof is that first a constructive characterization is proved for so-called $(k, 1)$-edge-connected digraphs which is used then, via an earlier orientation theorem, for proving Theorem 1.10.

We will use the following common notations. $d_G(X, Y)$ denotes the number of edges with one end-node in $X - Y$ and other end-node in $Y - X$. For a node $z \in V$, $d_G(z) := d_G(z, V - z)$. For a graph $G = (V, E)$, $\gamma_G(X)$ denotes the number of the edges of G with both end-nodes in $X \subseteq V$. Let $\Gamma_G(u)$ denote the neighbour set of a node u in G. $\cup\mathcal{P} := \bigcup_{X\in\mathcal{P}} X$ for a set-system $\mathcal{P}$.

2 Splittings

In an undirected graph $G = (V, E)$ **splitting off a pair of edges** $e = zu, f = zv$ $(u \neq v)$ at a node $z \in V$ means the operation of replacing e and f by a new edge connecting u and v. The edge uv, which may be parallel to existing ones, will be called a **split edge**. When the degree of z is even, by a **complete splitting** at z we mean the following operation: pair the edges incident to z and split off all these pairs. Complete splitting may be viewed as the inverse of operation (ii) in Theorem 1.1 and hence Lovász' Theorem 1.1 can be formulated in terms of splittings. To this end, we call a K-edge-connected graph $G = (V, E)$ minimal if $G - e$ is not K-edge-connected for each edge $e \in E$. It is a known and easily provable property of minimally K-edge-connected graphs with $|V| \geq 2$ that

they always contain a node of degree K. (2)

Now (the non-trivial part of) Lovász' theorem is equivalent to the following.

Theorem 2.1. *A minimally $2k$-edge-connected graph $G = (V, E)$ with $|V| \geq 2$ contains a node z of degree $2k$ which admits a complete splitting preserving $2k$-edge-connectivity.*

That is, in order to prove Theorem 1.1 one has to show that, among the nodes of degree $2k$ guaranteed by property (2), *there is at least one* admitting a complete splitting preserving $2k$-edge-connectivity. Lovász' original proof however showed that, quite 'luckily', *every* node of degree $2k$ admits such a complete splitting.

It turned out that the situation is similar in Theorem 1.4 and 1.6. *Every* node of suitable degree admits the inverse of the operations preserving the graph property in question.

It will also turn out that the situation is again the same in Theorem 1.7 of Henneberg and Laman (which is the special case $k = 2$ of Theorem 1.8): every node of degree 2 or 3 admits the corresponding inverse operation preserving near 2-tree-connectivity. But in Theorem 1.8 for $k \geq 3$ a node with suitable degree does not necessarily admit the corresponding inverse operation preserving near k-tree-connectivity, as shown by an example of Z. Király. This is why the proof of Theorem 1.9 for $k \geq 3$ is significantly more difficult than that for $k = 2$.

In the remaining part of this section we give the basic tools for proving Theorem 1.9.

Let $k \geq 2$. A splitting off in G is **admissible** if the resulting graph on node set $V - z$ is k-sparse.

Definition 2.2. Let b_G denote the following function on the subsets of V with cardinality at least 2:

$$b_G(X) := k|X| - (k+1) - \gamma_G(X).$$

By this definition a graph $G = (V, E)$ is k-sparse if and only if $b_G(X) \geq 0$ for all subsets $X \subseteq V, |X| \geq 2$. If $b_G(X) = 0$ and $X \neq V$, then X is said to be a G-**tight** set. Furthermore G is nearly k-tree-connected if and only if G is k-sparse and $b_G(V) = 0$. We will abbreviate b_G by b.

Observation 2.3. Splitting off zu and zv at node z is non-admissible (that is, adding the edge uv to the induced subgraph of G on $V - z$ does not result in an k-sparse graph) if and only if there exists a tight subset in $V - z$ containing u and v.

We say that splitting off j disjoint pairs of edges ($1 \leq j \leq k - 1$) at node z is **admissible** if it consists of admissible splittings. Obviously the order of the pairs in a **splitting sequence** is irrelevant. The **length** of a splitting sequence $\mathcal{S}$ is the number of its pairs and it is denoted by $|\mathcal{S}|$. $G_{\mathcal{S}}$ denotes the graph obtained after applying the splitting sequence $\mathcal{S}$.

In proving Theorem 1.9 our goal will be to find a node at which applying the inverse of operation (O1) or (O2) results in a k-sparse graph. That is why an admissible splitting sequence at z of length $d_G(z) - k =: i$ is called a **full splitting** for $d_G(z) \geq k + 1$. For the sake of convenience, at a node z with degree at most k the inverse of operation (O1) (that is, the deletion of z and all of its adjacent edges) is also called a **full splitting**.

Note, that $b_G(X)$ is an upper bound for the number of split edges induced by $X \subseteq V - z$ provided by an admissible sequence of splittings at some node z.

The next four claims are about k-sparse graphs and will be crucial in the proof of Theorem 1.9.

Claim 2.4. *If $X, Y \subseteq V$ and $|X \cap Y| \geq 2$, then*

$$b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d(X, Y).$$

Proof. $b(X) + b(Y) = k|X| - (k+1) - \gamma_G(X) + k|Y| - (k+1) - \gamma_G(Y) = k(|X| + |Y|) - 2(k+1) - (\gamma_G(X \cap Y) + \gamma_G(X \cup Y) - d_G(X, Y)) = k|X \cap Y| - (k+1) - \gamma_G(X \cap Y) + k|X \cup Y| - (k+1) - \gamma_G(X \cup Y) + d_G(X, Y) = b(X \cap Y) + b(X \cup Y) + d(X, Y)$. □

Claim 2.5. *If $X, Y \subseteq V$ and $|X \cap Y| = 1$, then*

$$b(X) + b(Y) = b(X \cup Y) - 1 + d(X, Y).$$

Proof. $b(X)+b(Y) = k|X|-(k+1)-\gamma_G(X)+k|Y|-(k+1)-\gamma_G(Y) = k(|X|+|Y|-1)-(k+1)-1-(\gamma_G(X)+\gamma_G(Y)) = k|X\cup Y|-(k+1)-1-(\gamma_G(X\cup Y)-d_G(X,Y)) = b(X\cup Y)-1+d(X,Y)$.

□

Claim 2.6. *If $X_1, X_2, X_3 \subseteq V$ and $|X_j \cap X_l| = 1$ for $1 \leq j < l \leq 3$ and $|X_1 \cap X_2 \cap X_3| = 0$, then*

$$b(\bigcup_{j=1}^{3} X_j) \leq \sum_{j=1}^{3} b(X_j) - k + 2.$$

Proof. $b(\bigcup_{j=1}^3 X_j) = k|\bigcup_{j=1}^3 X_j| - (k+1) - \gamma_G(\bigcup_{j=1}^3 X_j) \leq k(\sum_{j=1}^3 |X_j| - 3) - (k+1) - \sum_{j=1}^3 \gamma_G(X_j) = \sum_{j=1}^3 (k|X_j| - (k+1) - \gamma_G(X_j)) - k + 2 = \sum_{j=1}^3 b(X_j) - k + 2$. □

Remark. Especially, all of X_1, X_2, X_3 cannot be tight at the same time for $k \geq 3$.

Claim 2.7. *Let $X \subset V$ be a maximal tight set containing the distinct nodes c_1, c_2. Let d be a node in $V - X$. If there is a tight set containing c_1 and d, then there is no tight set containing c_2 and d.*

Proof. According to Claim 2.4, $P \cap X = \{c_1\}$ since X is maximal. By Claims 2.4 and 2.6 we obtain that there is no tight set containing c_2 and d. □

Let G be a k-sparse graph. Since $\sum_{v\in V} d_G(v) = 2|E| \leq 2k|V| - 2(k+1) < 2k|V|$, it follows that there is a node z of G with $d_G(z) \leq 2k-1$.

Claim 2.8. *Let $G = (V, E)$ be a k-sparse graph. $d_G(u, v) \leq k - 1$ for any two nodes u, v.*

Proof. By the definition of k-sparse graphs, $\gamma_G(\{u, v\}) \leq k|\{u, v\}| - (k+1) = k-1$ for set $\{u, v\}$. □

3 Full splitting

The main task in proving Theorem 1.9 will be to show the existence of a node admitting a full splitting. This will be done in two steps. In this section we derive a necessary and sufficient condition for an arbitrary specified node to admit a full splitting, while in the next section we show that a k-sparse graph always has a node satisfying this condition.

Let G be a k-sparse graph.

Proposition 3.1. *If a node z of G has degree at most $k + 1$, then z admits a full splitting.*

Proof. If $d_G(z)$ is at most k, then if we delete z with its adjacent edges, then we obviously get a k-sparse graph, that is, z admits a full splitting.

We claim that there always exists a full splitting at a node z with degree $k+1$. We will find a pair of edges zu and zv with $u \neq v$ such that $G - z + uv$ is a k-sparse graph.

There is no tight set $X \subseteq V - z$ which contains all the neighbours of z because, if there was one, then $b_G(X + z) = b_G(X) + k - d_G(z) = 0 + k - (k+1) < 0$ which contradicts that G is k-sparse. Since there are no edges with multiplicity greater than $k-1$ by Claim 2.8, the neighbour-set of z in G has at least two elements, hence by Claim 2.7 there is an admissible splitting off at z, which is full at the same time for a node with degree $k+1$. $\square$

If G is nearly 2-tree-connected, then $|E| = 2|V| - 3$ and hence there exists a node of degree 2 or 3. Therefore Proposition 3.1 immediately gives the proof of the Theorem 1.7 of Henneberg and Laman: every node with degree 2 or 3 admits a full splitting. Similarly, a 2-sparse graph has a node of degree at most 3, hence Theorem 1.9 follows for $k = 2$. It is not true that a k-sparse graph always contain a node of degree at most $k+1$, there is a graph (on 8 nodes) showing that such a node z does not necessarily exist. Hence Proposition 3.1 does not prove Theorem 1.9. From now on let $k \geq 3$. For $k = 3$ Z. Király observed [7] that a node z with degree $k + 2 = 5$ does not necessarily admit a full splitting. His example is shown in Figure 1.

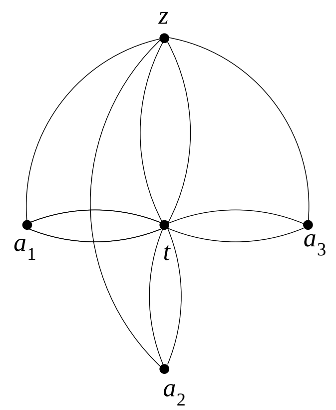

Figure 1: $k = 3$, z does not admit a full splitting

Call a node z **small** if $k + 2 \leq d_G(z) \leq 2k - 1$. For a node z, let i denote $d_G(z) - k$.

Theorem 3.2. *A small node z of G does not admit a full splitting if and only if z has a neighbour t and there is a family $\mathcal{P}_z$ of subsets of $V - z$ with at least two elements such that:*

$$X \cap Y = \{t\} \quad \textit{for} \quad X, Y \in \mathcal{P}_z, \tag{3a}$$

$$\sum_{X \in \mathcal{P}_z} b(X) < d_G(z,t) - (k-i) - d_G(z, V - z - \cup\mathcal{P}_z). \tag{3b}$$

Remark. In the graph of Figure 1, $X_j := \{t, a_j\}$ for $j = 1, 2, 3$.

Proof. Suppose first that t and $\mathcal{P}_z$ satisfy $(3a)$, $(3b)$ and let $\mathcal{S}$ be an admissible splitting sequence. The number of split edges incident to t with other end-nodes outside of $\cup\mathcal{P}_z$ is at most $d_G(z, V - z - \cup\mathcal{P}_z)$. The number of split edges incident to t with their other end-nodes in $\cup\mathcal{P}_z$ is at most $\sum_{X\in\mathcal{P}_z} b(X)$. In a full splitting we would have at least $d_G(z,t) - (k - i)$ split edges incident to t which implies by $(3b)$ that $\mathcal{S}$ is not full.

To see the other direction, let $\mathcal{S}$ be a longest admissible splitting sequence at z for which the following pair is lexicographically maximal: $(|\Gamma_{G_\mathcal{S}}(z)|, |P_{\max}|)$ where $P_{\max}$ denotes a maximal tight set in $G_\mathcal{S}$ which includes $\Gamma_{G_\mathcal{S}}(z)$ but does not contain z. If there is no such a tight set, then let $P_{\max} := \emptyset$. Since z does not admit a full splitting, $|\mathcal{S}| < i$. From now on $G_\mathcal{S}$-tight is abbreviated by tight.

CASE 1. $|\Gamma_{G_\mathcal{S}}(z)| \geq 2$.

Claim 3.3. *There exists a maximal tight subset $P_{\max}$ of $V - z$ containing all the neighbours of z.*

Proof. Let za and zb denote two non-parallel edges in $G_\mathcal{S}$. Since (za, zb) is not an admissible splitting off, there is a tight set $X \subseteq V - z$ containing a and b. According to Claim 2.4, there is a maximal tight set $P_{\max} \subseteq V - z$ containing a and b.

If there is another neighbour c of z which is not in $P_{\max}$, then there is a tight set $Y \subseteq V - z$ containing a and c, since (za, zc) is not an admissible splitting off. Since $P_{\max}$ is maximal, $Y \cap P = \{a\}$. By Claim 2.7 (zb, zc) is an admissible splitting off, a contradiction, that is, $P_{\max}$ contains all the neighbours of z. □

Claim 3.4. *There exists a split edge which is disjoint from the nodes of $P_{\max}$.*

Proof. Since there is no admissible splitting off at z in $G_\mathcal{S}$, according to Claim 3.3 there exists a maximal tight set $P_{\max} \subseteq V - z$. Let j, h, m denote the number of split edges with exactly, respectively, 2, 1, 0 end-node in $P_{\max}$. $j + h + m = |\mathcal{S}| < i$ since $\mathcal{S}$ is not full.

$$k|P_{\max} + z| - (k+1) \geq \gamma_G(P_{\max} + z) = \gamma_{G_\mathcal{S}}(P_{\max}) + j + h + d_{G_\mathcal{S}}(z, P_{\max})$$

$$= \gamma_{G_\mathcal{S}}(P_{\max}) + j + h + (k + i - 2(j + h + m))$$

$$= \gamma_{G_\mathcal{S}}(P_{\max}) + k + (i - (j + h + m)) - m > k|P_{\max}| - (k+1) + k - m$$

$$= k|P_{\max} + z| - (k+1) - m,$$

which implies $m > 0$. □

Claim 3.5. $|\Gamma_{G_\mathcal{S}}(z)| = 2$.

Proof. Suppose indirectly that $|\Gamma_{G_\mathcal{S}}(z)| \geq 3$ and let a_1, a_2, a_3 denote three of these neighbours. By Claim 3.4, there is a split edge uv disjoint from $P_{\max}$. Let $J = \{1, 2, 3\}$.

By Claim 2.7, $\mathcal{S} - (zu, zv) \cup (zu, za_j)$ is an admissible splitting sequence for at least two elements j of J. We may suppose that $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, za_1)$ is an admissible splitting sequence. But then the maximal $G_{\mathcal{S}'}$-tight set containing all the remaining neighbours of z in $G_{\mathcal{S}'}$ contains v and includes $P_{\max}$, that is, $\mathcal{S}'$ contradicts the choice of $\mathcal{S}$. $\square$

Let s and t denote the two neighbours of z in $G_{\mathcal{S}}$. Since $\mathcal{S}$ is not a full splitting: $d_{G_{\mathcal{S}}}(z) \geq d_G(z) - 2(i-1) = k + i - 2(i-1) = k - i + 2 \geq 3$. Therefore one of the nodes t and s, say t, is connected to z by at least two paralell edges.

Claim 3.6. $d_{G_{\mathcal{S}}}(z, s) = 1$.

Proof. Let us suppose indirectly that $d_{G_{\mathcal{S}}}(z, s) \geq 2$. By Claim 3.4, there is a split edge uv disjoint from $P_{\max}$. According to Claim 2.7, $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, zt)$ or $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, zs)$ is an admissible splitting sequence. But then $|\Gamma_{G'_{\mathcal{S}}}(z)| = 3$, a contradiction. $\square$

An edge not incident to t is called **t-disjoint**. Let $u \in V - t - s$ be an arbitrary node for which there is a t-disjoint split edge uv. There is a tight set $X \subseteq V - z$ containing u and t, otherwise $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, zt)$ is an other longest admissible splitting sequence for which if $v \neq s$, then $|\Gamma_{G_{\mathcal{S}'}}(z)| = 3$, if $v = s$, then $d_{G_{\mathcal{S}'}}(z, t) \geq d_{G_{\mathcal{S}'}}(z, s) \geq 2$, which contradicts the choice of $\mathcal{S}$ by Claims 3.5 and respectively 3.6. Let P_u be such a tight set containing minimal number of t-disjoint split edges which is inclusionwise maximal. Similarly, there is a tight set $X \subseteq V - z$ containing s and t, otherwise $\mathcal{S} \cup (zs, zt)$ is a longer admissible splitting sequence than $\mathcal{S}$. Let P_s be such a tight set containing minimal number of t-disjoint split edges which is inclusionwise maximal.

Let $\mathcal{P}_z := \{X \subseteq V - z : \exists u \in V$ incident to a t-disjoint split edge such that $X = P_u$ or $X = P_s\}$. For nodes $u \neq v$, P_u can be equal to P_v, but there is only one copy of them in $\mathcal{P}_z$. Now we prove some essential properties of $\mathcal{P}_z$.

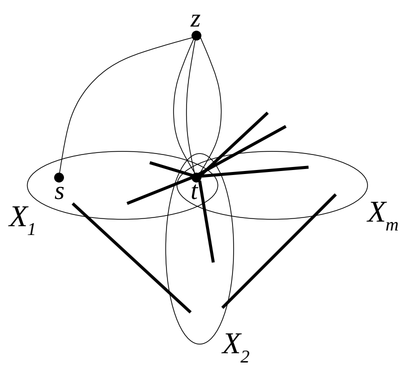

Figure 2: A set-system $\mathcal{P}_z$.

Proposition 3.7. *There is no t-disjoint split edge in a member X of $\mathcal{P}_z$.*

Proof. First let us assume that $X = P_s$. Let us suppose indirectly that there is a t-disjoint split edge ab in P_s. $\mathcal{S}' := \mathcal{S} - (za, zb) \cup (zt, zs)$ is an admissible splitting sequence with three remaining neighbours of z in $G_{\mathcal{S}'}$, which contradicts the choice of $\mathcal{S}$ by Claim 3.5.

Now let us assume $X = P_u$ and $u \neq s$. By the definition of P_u we have a t-disjoint split edge uv. Let us suppose indirectly that there is a t-disjoint split edge ab in P_u. We may suppose that $b \neq u$.

If $v \neq s$, then $v \notin P_u$ (if $v \in P_u$, then $\mathcal{S} - (zu, zv) \cup (zt, zu)$ is an admissible splitting sequence with the same length but with one more remaining neighbour of z). $P_v \cap P_u = \{t\}$ according to Claim 2.4. $\mathcal{S} - (za, zb) - (zu, zv) \cup (zt, zu) \cup (zv, za)$ is an other longest splitting sequence with one more remaining neighbour of z, so it cannot be admissible, that is, there is a set $Y \subseteq V - z$ containing a, u, v, t, which is tight in $G_{\mathcal{S}}$. Y does not contain b, hence the tight set $Y \cap P_u$ contains a smaller number of split edges than P_u, a contradiction. If $v = s$ and $v \notin P_u$, then the proof is the same.

Supppose that $v = s$ and $v \in P_u$. Let us consider a split edge cd which is disjoint from $P_{\max}$ and hence from P_u (such an edge exists according to Claim 3.4). By the previous paragraph tight sets P_c and P_d do not contain t-disjoint split edges. According to Claim 2.4, $P_c \cap P_{\max} = \{t\}$.

According to Claim 2.7, $\mathcal{S}' := \mathcal{S} - (zc, zd) \cup (zc, zs)$ is an admissible splitting sequence. For $\mathcal{S}'' := \mathcal{S}' - (zu, zv) \cup (zt, zu)$, the cardinality of $\Gamma_{G_{\mathcal{S}''}} = \{t, s, d\}$ is 3, hence $\mathcal{S}''$ cannot be admissible, that is, there is a tight set $Y \subseteq V - z$ containing c, s, u, t in $G_{\mathcal{S}}$. $Y \cup P_{\max}$ contradicts the maximality of $P_{\max}$.

□

Now it follows that $(3b)$ holds for $\mathcal{P}_z$.

Claim 3.8. *Let X, Y be two distinct members of $\mathcal{P}_z$. $X \cap Y = \{t\}$.*

Proof. Let us suppose $X = P_u$ and $Y = P_v$ for some $u, v \in V$. By Proposition 3.7, $P_u \not\subset P_v$. If $|P_u \cap P_v| \geq 2$, then by Claim 2.4 $d_{G_{\mathcal{S}}}(P_u, P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any t-disjoint split edge, it contradicts the maximal choice of P_u. □

Hence $(3a)$ holds for $\mathcal{P}_z$.

CASE 2. $|\Gamma_{G_{\mathcal{S}}}(z)| = 1$. Let t denote the only neighbour of z in $G_{\mathcal{S}}$.

Claim 3.9. *There exists a t-disjoint split edge.*

Proof. Let l and m be the number of split edges incident to, respectively, not incident to t. Since $\mathcal{S}$ is not full, $l + m = |\mathcal{S}| < i$. In the original graph G by Claim 2.8:

$$k - 1 \geq d_G(z, t) = d_G(z) - l - 2m = k + i - l - 2m = k + (i - l - m) - m > k - m,$$

which implies that $m > 1$. □

Since $\mathcal{S}$ is not a full splitting: $d_{G_{\mathcal{S}}}(z) \geq k + i - 2(i - 1) = k - i + 2 \geq 3$. Now we define $\mathcal{P}_z$. Let $u \in V - t$ be an arbitrary node for which there is a t-disjoint split edge uv. There is a tight set $X \subseteq V - z$ containing u and t, otherwise $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, zt)$ is an other longest admissible splitting sequence for which $|\Gamma_{G_{\mathcal{S}'}}(z)| = 2$, which contradicts the choice of $\mathcal{S}$. Let P_u be such a tight set containing minimal number of

t-disjoint split edges which is inclusionwise maximal. Let $\mathcal{P}_z := \{X \subseteq V - z : \exists u \in V$ incident to a t-disjoint split edge such that $X = P_u\}$. (The only difference to the case above is that there is no set P_s here.)

Proposition 3.10. *There is no t-disjoint split edge in an arbitrary element of $\mathcal{P}_z$.*

Proof. Assume $X = P_u$. By the definition of P_u we have a t-disjoint split edge uv. Let us suppose indirectly that there is a t-disjoint split edge ab in P_u. We may suppose that $b \neq u$. $v \notin P_u$, otherwise $\mathcal{S} - (zu, zv) \cup (zt, zu)$ is an admissible splitting sequence with the same length but with one more remaining neighbour of z. $P_v \cap P_u = \{t\}$ according to Claim 2.4. $\mathcal{S} - (za, zb) - (zu, zv) \cup (zt, zu) \cup (zv, za)$ is an other longest splitting sequence with one more remaining neighbour of z, so it cannot be admissible, that is, there is a set $Y \subseteq V - z$ containing a, u, v, t, which is tight in $G_{\mathcal{S}}$. Y does not contain b, hence the tight set $Y \cap P_u$ contains a smaller number of split edges than P_u, a contradiction. $\square$

Now it follows that $(3b)$ holds for $\mathcal{P}_z$.

Claim 3.11. *Let X, Y be two distinct members of $\mathcal{P}_z$. $X \cap Y = \{t\}$.*

Proof. Let us suppose $X = P_u$ and $Y = P_v$ for some $u, v \in V$. By Proposition 3.7, $P_u \not\subset P_v$. If $|P_u \cap P_v| \geq 2$, then by Claim 2.4 $d_{G_{\mathcal{S}}}(P_u, P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any t-disjoint split edge, it contradicts the maximal choice of P_u. $\square$

Hence $(3a)$ holds for $\mathcal{P}_z$.

We have showed that if a small node z does not admit a full splitting, then the neighbour t of z and set-system $\mathcal{P}_z$ satisfy both $(3a)$ and $(3b)$. $\square$ $\square$

4 Construction of k-sparse graphs

Proof of Theorem 1.9. It is easy to see that any application of operation (O1) or (O2) in a k-sparse graph results in a k-sparse graph. Now we want to prove that a k-sparse graph G always can be built up in the way described in Theorem 1.9. By induction it suffices to show that there is a k-sparse graph $G' = (V', E')$ so that G arises from G' by one application of operation (O1) or (O2).

The existence of such a G' is clearly equivalent to the following statement. There is a node z of G for which

(α) $d_G(z) \leq 2k - 1$, and

(β) there is a full splitting at z (that is, $d_G(z) \leq k$ or there are $i := d_G(z) - k$ disjoint pairs of edges incident to z so that after splitting off these i pairs and deleting the remaining $k - i$ edges at z we obtain a k-sparse graph $G' = (V - z, E')$).

As was shown in Section 3, if $k = 2$, then any node satisfying (α) will automatically satisfy (β). The main ingredient of proving the existence of a node z satisfying both (α) and (β) is Theorem 3.2.

Let G be a k-sparse graph with at least two nodes. If there is a node z with degree at most $k+1$, then we are done by Proposition 3.1. Therefore we may assume that every node of G has degree at least $k+2$. Furthermore let us suppose indirectly that there is no small node admitting a full splitting. Recall the definition of a small node in Section 3.

By Theorem 3.2, for any small node z there exists a set-sestem $\mathcal{P}_z$. Let $\varphi(\mathcal{P}_z) := d_G(z,t) - (k-i) - d_G(z, V - z - \cup\mathcal{P}_z) - \sum_{X\in\mathcal{P}_z} b(X)$. Note that $\varphi(\mathcal{P}_z) + (k+i)$ is a lower bound for the number of parallel edges between z and t remaining after some admissible splittings. Property (3b) in Theorem 3.2 is equivalent to $\varphi(\mathcal{P}_z) > 0$.

Let us choose $\mathcal{P}_z$ to be lexicographically maximal with respect to the following triple: $(\varphi(\mathcal{P}_z), -|\mathcal{P}_z|, |\cup\mathcal{P}_z|)$. Let $\tau(z)$ denote the node t in Theorem 3.2 for a small node z, which is called the **blocking node** of z.

We will show that the degree of a blocking node is big, and by $|E| \le k|V| - (k+1)$ there must be a small node without a blocking node. First we prove two lemmas which will be useful for proving that the degree of a blocking node is big. Recall, that for a small node z, $i := d_G(z) - k$. From now on d_G is abbreviated by d.

Lemma 4.1. $|\mathcal{P}_z| \ge 3$ *for any small node z not admitting a full splitting.*

Proof. Suppose first that $\mathcal{P}_z = \{X\}$.

$$
\begin{gathered}
b(X+z) = b(X) + k - d(z,X) \quad \text{(by the definition of } b)\\
= b(X) + k - (d(z) - d(z, V-z-X))\\
= b(X) + k - (k + i - d(z, V-z-X)) = (b(X) + d(z, V-z-X) + (k-i)) - k\\
< d(z,t) - k \quad \text{(by (3}b\text{) of Theorem 3.2)}\\
< 0 \quad \text{(by Claim 2.8)},
\end{gathered}
$$

a contradiction.

Second, let $\mathcal{P}_z = \{X_1, X_2\}$. Since $\varphi(\mathcal{P}_z) > \varphi(\{X_1 \cup X_2\})$,

$$
\begin{gathered}
d(z,t) - (k-i) - d(z, V-z-(X_1\cup X_2)) - (b(X_1) + b(X_2)) >\\
> d(z,t) - (k-i) - d(z, V-z-(X_1\cup X_2) - b(X_1\cup X_2)),
\end{gathered}
$$

hence $b(X_1\cup X_2) > b(X_1) + b(X_2)$. By Lemma 2.5, this implies $b(X_1\cup X_2) = b(X_1) + b(X_2) + 1$ and $d(X_1, X_2) = 0$. By the definition of b we have

$$
\begin{gathered}
0 \le b(X_1\cup X_2 + z) = b(X_1\cup X_2) + k - d(z, X_1\cup X_2)\\
= (b(X_1) + b(X_2) + 1) + k - d(z, X_1\cup X_2)\\
= b(X_1) + b(X_2) + 1 + k - (d(z) - d(z, V-z-(X_1\cup X_2)))\\
= b(X_1) + b(X_2) + d(z, V-z-(X_1\cup X_2)) + (k-i) + i - d(z) + 1\\
< d(z,t) - d(z) + i + 1 \le (k-1) - (k+i) + i + 1 = 0, \text{by (3}b\text{) and Claim 2.8},
\end{gathered}
$$

a contradiction. □

Lemma 4.2. *If $P \in \mathcal{P}_z$ contains a small node z' such that $\tau(z) = \tau(z') = t$ and $P' \in \mathcal{P}_{z'}$ and $P' - P \neq \emptyset$, then $z \in P'$.*

Proof. Let us suppose indirectly that $z \notin P'$. Let $i := d(z) - k$ and $i' := d(z') - k$.
Case 1. $|P \cap P'| \geq 2$.

By the lexicographically maximal choice of $\mathcal{P}_z$, $\varphi(\mathcal{P}'_z) < \varphi(\mathcal{P}_z)$ holds for $\mathcal{P}'_z := \mathcal{P}_z - P + (P \cup P')$. By the definition of φ, we have $d(z,t) - (k-i) - d(z, V - z - \bigcup_{X \in \mathcal{P}'_z} X) - \sum_{X \in \mathcal{P}'_z} b(X) <$
$< d(z,t) - (k-i) - d(z, V - z - \bigcup_{X \in \mathcal{P}_z} X) - \sum_{X \in \mathcal{P}_z} b(X)$, and hence

$$-d(z, V - z - \bigcup_{X \in \mathcal{P}'_z} X) - \sum_{X \in \mathcal{P}'_z} b(X) < -d(z, V - z - \bigcup_{X \in \mathcal{P}_z} X) - \sum_{X \in \mathcal{P}_z} b(X).$$

By the definition of $\mathcal{P}'_z$, we obtain $-d(z, V - z - \bigcup_{X \in \mathcal{P}_z} X) + d(z, P' - P) - \sum_{X \in \mathcal{P}_z} b(X) + b(P) - b(P \cup P') < -d(z, V - z - \bigcup_{X \in \mathcal{P}_z} X) - \sum_{X \in \mathcal{P}_z} b(X)$, from which $d(z, P' - P) + b(P) - b(P \cup P') < 0$, and

$$b(P) < b(P \cup P') - d(z, P' - P) \leq b(P \cup P') \tag{4}$$

follows.

On the other hand by the lexicographically maximal choice of $\mathcal{P}_{z'}$, $\varphi(\mathcal{P}'_{z'}) \leq \varphi(\mathcal{P}_{z'})$ holds for $\mathcal{P}'_{z'} := \mathcal{P}_{z'} - P' + (P \cap P')$. By the definition of φ we have $d(z',t) - (k - i') - d(z', V - z' - \bigcup_{X \in \mathcal{P}'_{z'}} X) - \sum_{X \in \mathcal{P}'_{z'}} b(X) \leq d(z',t) - (k - i') - d(z', V - z' - \bigcup_{X \in \mathcal{P}_{z'}} X) - \sum_{X \in \mathcal{P}_{z'}} b(X)$, and hence

$$-d(z', V - z' - \bigcup_{X \in \mathcal{P}'_{z'}} X) - \sum_{X \in \mathcal{P}'_{z'}} b(X) \leq -d(z', V - z' - \bigcup_{X \in \mathcal{P}_{z'}} X) - \sum_{X \in \mathcal{P}_{z'}} b(X).$$

By the definition of $\mathcal{P}'_{z'}$, we obtain $-d(z', V - z' - \bigcup_{X \in \mathcal{P}_{z'}} X) - d(z', P' - P) - \sum_{X \in \mathcal{P}_{z'}} b(X) + b(P') - b(P \cap P') \leq -d(z', V - z' - \bigcup_{X \in \mathcal{P}_{z'}} X) - \sum_{X \in \mathcal{P}_{z'}} b(X)$, from which $-d(z', P' - P) + b(P') - b(P \cap P') \leq 0$, and

$$b(P') \leq b(P \cap P') + d(z', P' - P) \tag{5}$$

follows.

By adding up (4) and (5) we have

$$b(P) + b(P') < b(P \cap P') + b(P \cup P') + d(z', P' - P) \leq b(P \cap P') + b(P \cup P') + d(P - P', P' - P),$$

which contradicts Claim 2.4.
Case 2. $|P \cap P'| = 1$.

By the lexicographically maximal choice of $\mathcal{P}_z$, for $\mathcal{P}'_z := \mathcal{P}_z - P + (P \cup P')$ we also have (4). On the other hand by the lexicographically maximal choice of $\mathcal{P}_{z'}$, $\varphi(\mathcal{P}'_{z'}) < \varphi(\mathcal{P}_{z'})$ holds for $\mathcal{P}'_{z'} := \mathcal{P}_{z'} - P'$. By the definition of φ we have $d(z',t) - (k - i') - d(z', V - z' - \bigcup_{X \in \mathcal{P}'_{z'}} X) - \sum_{X \in \mathcal{P}'_{z'}} b(X) < d(z',t) - (k - i') - d(z', V - z' - \bigcup_{X \in \mathcal{P}_{z'}} X) - \sum_{X \in \mathcal{P}_{z'}} b(X)$, and hence

$$-d(z', V - \bigcup_{X \in \mathcal{P}'_{z'}} X) - \sum_{X \in \mathcal{P}'_{z'}} b(X) < -d(z', V - \bigcup_{X \in \mathcal{P}_{z'}} X) - \sum_{X \in \mathcal{P}_{z'}} b(X).$$

By the definition of $\mathcal{P}'_{z'}$, we obtain $-d(z', V - \bigcup_{X \in \mathcal{P}_{z'}} X) - d(z', P' - P) - \sum_{X \in \mathcal{P}_{z'}} b(X) + b(P') < -d(z', V - \bigcup_{X \in \mathcal{P}_{z'}} X) - \sum_{X \in \mathcal{P}_{z'}} b(X)$, from which $-d(z', P' - P) + b(P') < 0$, and

$$b(P') \leq d(z', P' - P) - 1 \tag{6}$$

follows.

By adding up (4) and (6) we have:

$$b(P) + b(P') < b(P \cup P') + d(z', P' - P) - 1 \leq b(P \cup P') + d(P - P', P' - P) - 1,$$

which contradicts Claim 2.5. □

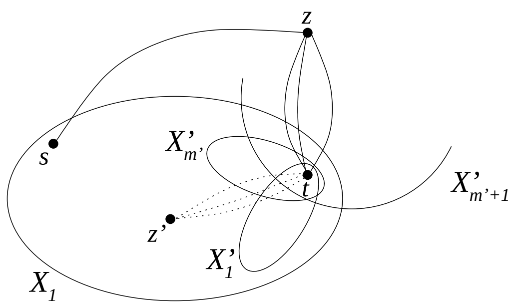

Figure 3: Set-systems $\mathcal{P}_z$ and $\mathcal{P}_{z'}$.

Claim 4.3. *For a node v in a set $X \subseteq V$,*

$$d(v, X - v) \geq (k - 1) - b(X).$$

Proof. If $|X| \geq 3$, then $0 \leq b(X - v) = b(X) - k + d(v, X - v)$. If $|X| = 2$, then by Claim 2.8 we are done. □

Proposition 4.4. *Let t be a blocking node. Let l_i $(2 \leq i \leq k - 1)$ denote the number of small nodes with degree $k + i$ whose blocking node is t. Then*

$$d_G(t) \geq \sum_{i=2}^{k-1} (k - i + 1) l_i + 3(k - 1).$$

Proof. For every node z^* with blocking node t ($i^* := d(z^*) - k$), according to (3b):

$$d_G(z^*, t) \geq k - i^* + 1. \tag{7}$$

Let z be a small node with degree $k + i$ whose blocking node is t. Let $m := |\mathcal{P}_z|$. Let l denote the number of the members $X_1, X_2, \ldots X_l$ of $\mathcal{P}_z$ containing no small node with blocking node t, then by ($3b$): $0 < d_G(z,t) - (k-i) - d_G(z, V - z - \cup_{j=1}^m X_j) - \sum_{j=1}^m b(X_j) \leq d_G(z,t) - (k-i) - \sum_{j=1}^m b(X_j)$, that is,

$$(k-i+1) + \sum_{j=1}^{l} b(X_j) \leq (k-i+1) + \sum_{j=1}^{m} b(X_j) \leq d_G(z,t). \tag{8}$$

By Claim 4.3,

$$\sum_{j=1}^{l} ((k-1) - b(X_j)) \leq d_G(t, \bigcup_{j=1}^{l} X_j - t). \tag{9}$$

By adding up (8) and (9):

$$l(k-1) + (k-i+1) \leq d_G(z,t) + d_G(t, \bigcup_{j=1}^{l} X_j - t). \tag{10}$$

Let L denote the set of small nodes with blocking node t.

Claim 4.5. *If $X \in \mathcal{P}_z$ contains at least one small node with blocking node t, then there exists a small node $z_0 \in X$ with blocking node t such that $d_G(z_0,t) + d_G(t, X - L - t) \geq (k - i_{z_0} + 1) + (k-1)$.*

To prove the claim, let z_0 be a small node in X with blocking node t such that the minimum member $Y_0 \subseteq X$ of $\mathcal{P}_{z_0}$ is minimal. Such a set and node exist by Lemmas 4.1 and 4.2, furthermore Y_0 does not contain any node of L. By ($3b$), $0 < d_G(z_0,t) - (k - i_0) - d_G(z_0, V - z_0 - \cup_{Y_j \in \mathcal{P}_{z_0}} Y_j) - \sum_{Y_j \in \mathcal{P}_{z_0}} b(Y_j) \leq d_G(z_0,t) - (k-i_0) - \sum_{Y_j \in \mathcal{P}_{z_0}} b(Y_j)$, that is,

$$(k - i_0 + 1) + b(Y_0) \leq (k - i_0 + 1) + \sum_{Y_j \in \mathcal{P}_{z_0}} b(Y_j) \leq d_G(z_0, t). \tag{11}$$

By Claim 4.3,

$$(k-1) - b(Y_0) \leq d_G(t, Y_0 - t) \leq d_G(t, X - L - t). \tag{12}$$

By adding up (11) and (12) we get the equality of the claim.

Now we have $d_G(t) \geq \sum_{v \in L} d_G(t,v) + d_G(t, \bigcup_{j=1}^m X_j - t - L)$

$$= \sum_{v \in L} d_G(t,v) + d_G(t, \bigcup_{j=1}^{l} X_j - t - L) + d_G(t, \bigcup_{j=l+1}^{m} X_j - t - L)$$

$$\geq \sum_{i=2}^{k-1} (k-i+1) l_i + l(k-1) + (m-l)(k-1) \quad \text{(by (7), (10) and Claim 4.5)}$$

$$= \sum_{i=2}^{k-1} (k-i+1) l_i + m(k-1),$$

by Lemma 4.1, $m \geq 3$. $\square$

It follows that a blocking node is not small because its degree is at least $3(k-1) = 3k-3$. Let L_{k+i} denote the subset of small nodes of degree $k+i$ whose blocking node is t.

Claim 4.6. *The average degree in $W := \bigcup_{i=1}^{k-1} L_{k+i} + t$ is greater than $2k$.*

Proof. By Proposition 4.4, the sum of the degrees in W is the following.

$$f := d_G(t) + \sum_{i=2}^{k-1}(k+i)|L_{k+i}| \geq \sum_{i=2}^{k-1}(k-i+1)|L_{k+i}| + 3(k-1) + \sum_{i=2}^{k-1}(k+i)|L_{k+i}|$$

$$= 3k - 3 + (2k+1)\sum_{i=2}^{k-1}|L_{k+i}|.$$

Hence the average degree in W is:

$$\frac{f}{1+\sum_{i=2}^{k-1}|L_{k+i}|} \geq \frac{3k-3+(2k+1)\sum_{i=2}^{k-1}|L_{k+i}|}{1+\sum_{i=2}^{k-1}|L_{k+i}|}$$

$$= \frac{k-4+(2k+1)(1+\sum_{i=2}^{k-1}|L_{k+i}|)}{1+\sum_{i=2}^{k-1}|L_{k+i}|} = \frac{k-4}{1+\sum_{i=2}^{k-1}|L_{k+i}|} + (2k+1) > 2k, \quad \text{since } k \geq 3.$$

$\square$

In a k-sparse graph G the average degree is

$$\frac{2|E|}{|V|} \leq \frac{2(k|V|-(k+1))}{|V|} < 2k.$$

So there must be a small node z with no blocking node, that is, z admits a full splitting. **End of proof of Theorem 1.9.** $\square$ $\square$

With the same technique a bit stronger result can also be proved.

Theorem 4.7. *If G is k-sparse with at least two nodes, then there are at least two nodes admitting a full splitting.*

Proof. According to Theorem 1.9 there is a node s admitting a full splitting. 0 is a lower bound on the degree of s. Let us suppose indirectly that there is no other node admitting a full splitting, hence the degree of any other node is at least $k+2$ by Proposition 3.1. Let n_{k+i} denote the number of nodes distinct from s of degree $k+i$ ($2 \leq i \leq k-1$). Let $T \subseteq V$ be the set of the blocking nodes.

Now we have:

$$2(k|V|-(k+1)) = 2k|V| - 2k - 2 \geq 2|E| \geq$$

$$\geq 0 + \sum_{t\in T} d_G(t) + \sum_{i=2}^{k-1}(k+i)n_{k+i} + 2k(|V| - 1 - |T| - \sum_{i=2}^{k-1} n_{k+i}) \quad \text{(by Proposition 4.4)}$$

$$\geq (3k-3)|T| + \sum_{i=2}^{k-1}(k-i+1)n_{k+i} + \sum_{i=2}^{k-1}(k+i)n_{k+i} + 2k(|V|-1-|T|-\sum_{i=2}^{k-1}n_{k+i}) =$$

$$= 2k|V| + (k-3)|T| + \sum_{i=2}^{k-1}n_{k+i} - 2k \geq 2k|V| + (k-2)|T| - 2k \geq 2k|V| - 2k.$$

$\sum_{i=2}^{k-1} n_{k+i} \geq |T|$ holds obviously. We arrived at a contradiction and hence there exists another node admitting a full splitting. □

The following theorem can be proved by a slight modification of the above computation.

Theorem 4.8. *If G is nearly k-tree-connected and differs from K_2^{k-1}, then there are at least three nodes admitting a full splitting.* □

The following theorem characterizes the connected k-sparse graphs, which are the union of k forests after adding an arbitrary edge, according to Nash-Williams' Theorem 1.5.

Theorem 4.9. *A graph G is the union of k spanning trees after adding an arbitrary edge if and only if it is a connected subgraph of a nearly k-tree-connected graph.*

Proof. It is straightforward that any connected subgraph of a nearly k-tree-connected graph has this property.

By the theorem of Nash-Williams, $G = (V, E)$ is the union of k (not necessarily edge-disjoint) spanning trees after adding an arbitrary edge if and only if it is connected and $\gamma_G(X) \leq k|X| - (k+1)$ for all $X \subseteq V$, $|X| \geq 2$. We claim that if $|E| < k|V| - (k+1)$, then we can add an edge e such that $G + e$ is also the union of k forests after adding an arbitrary edge. This will prove the theorem.

Let us consider a maximal tight set X and node $u \in X$ and other node $v \notin X$. If we cannot add edge uv, then there exists a tight set Y containing u and v. According to Claim 2.6, for any node a in $X - Y$ and any node b in $Y - X$, $G + ab$ is k-sparse. □

5 Construction of $(k,1)$-edge-connected digraphs and $(k,1)$-partition-connected graphs

In a directed graph by **splitting off** a pair of edges $e = uz, f = zv$ we mean the operation of replacing e and f by a new directed edge from u to v. Suppose that the in-degree and the out-degree of z is the same, that is, $\varrho(z) = \delta(z)$. By a **complete splitting** at z we mean the following operation: pair the edges entering and leaving z and split off all these pairs.

For non-negative integers $l \leq k$, we call a digraph D (k,l)**-edge-connected** (in short, (k,l)-ec) if D has a node s so that there are k (resp., l) edge-disjoint paths from s to every other node (there are l edge-disjoint paths from every node to s). If there is an exceptional node z for which the existence of these edge-disjoint paths is not required, we say that D is **(k,l)-edge-connected apart from** z. When the role of s is emphasized, we say that D is (k,l)-ec with respect to root-node s. (k,k)-edge-connectivity is abbreviated by k-edge-connectivity and $(k,0)$-edge-connectivity is sometimes called rooted k-edge-connectivity. Note that by Menger's theorem a digraph is (k,l)-ec if and only if

$$\varrho(X) \geq k \text{ for every subset } \emptyset \subset X \subseteq V - s \tag{13}$$

and

$$\delta(X) \geq l \text{ for every subset } \emptyset \subset X \subseteq V - s \tag{14}$$

where $\varrho(X) := \varrho_D(X)$ and $\delta(X) := \delta_D(X)$ denote the number of edges entering and leaving the subset X, respectively.

We say an undirected graph $G = (V, E)$ is **(k,l)-partition-connected** if there are at least $k(t-1)+l$ edges connecting distinct classes of every partition of V into t $(t \geq 2)$ non-empty subsets. Note that $(k,1)$-partition-connectivity and highly k-tree-connectivity are equivalent notions.

The following result exhibits a link between the two concepts. It is a special case of a general orientation theorem appeared in [2].

Theorem 5.1. *Let $0 \leq l \leq k$ be integers. An undirected graph $G = (V, E)$ has a (k,l)-edge-connected orientation if and only if G is (k,l)-partition-connected.*

Mader's directed splitting off theorem [11] is as follows.

Theorem 5.2. *Let $D = (U + z, E)$ be a digraph which is k-edge-connected apart from z. If $\varrho(z) = \delta(z)$, then there is a complete splitting at z resulting in a k-ec digraph on node-set U.*

This result has been extended in [3] as follows.

Theorem 5.3. *Let $D = (U + z, E)$ be a digraph which is (k,l)-edge-connected apart from z. If $\varrho(z) = \delta(z)$, then there is a complete splitting at z resulting in a (k,l)-ec digraph on node-set U.*

We need the following corollary of Theorem 5.2.

Theorem 5.4. *Let $D = (U + z, E)$ be a digraph which is*

$$(k,0)\ \textit{-ec apart from } z\ (k \geq 1) \textit{ with respect to a root node } s \in U. \tag{15}$$

If $\varrho(z) > \delta(z)$, then there are $\varrho(z) - \delta(z)$ edges entering z so that (15) continues to hold after discarding these edges. If $\varrho(z) = \delta(z)$, then there is a complete splitting at z preserving (15).

Proof. For every node $v \in U+z$ for which $\varrho(v) > \delta(v)$, add $\varrho(v)-\delta(v)$ parallel edges from v to s. In the resulting digraph D' clearly $\varrho'(v) \leq \delta'(v)$ holds for every node $v \in U-s$. Hence $\delta'(X) \geq \varrho'(X) = \varrho(X) \geq k$ holds for every subset $X \subseteq U-s, X \neq \{z\}$, that is, D' is k-ec apart from z.

By Theorem 5.2 there is a complete splitting at z resulting in a k-ec digraph. It follows that in case $\varrho(z) = \delta(z)$ this complete splitting, when applied to D, preserves (15). If $\varrho(z) > \delta(z)$, then there are $\varrho(z) - \delta(z)$ edges entering z such that their pairs at the complete splitting are necessarily newly added edges from z to s. Therefore these edges can be deleted from D without destroying (15). $\square$

W. Mader used Theorem 5.2 to derive Theorem 1.2 on the constructive characterization of k-ec digraphs. Analogously, Theorem 5.4 may be used to derive the following.

Theorem 5.5. *A directed graph $D=(V,E)$ is $(k,0)$-edge-connected if and only if D can be obtained from a single node by the following two operations:*

(i) add a new edge,

(ii) add a new node z and add k edges entering z,

(iii) pinch j $(1 \leq j \leq k-1)$ existing edges with a new node z, and add $k-j$ new edges entering z.

Given these constructive characterizations of (k,k)-ec and $(k,0)$-ec digraphs, one may formulate the following general conjecture.

Conjecture 5.6. *A directed graph D is (k,l)-edge-connected $(0 \leq l \leq k-1)$ if and only if it can be built up from a node by the following two operations:*

(j) add a new edge,

(jj) pinch i $(l \leq i \leq k-1)$ existing edges with a new node z, and add $k-i$ new edges entering z and leaving existing nodes.

Pinching 0 edge with new node z simply means adding a new node z.

Conjecture 5.7. *An undirected graph G is (k,l)-partition-connected if and only if it can be built up from a node by the following two operations:*

(j) add a new edge,

(jj) pinch i $(l \leq i \leq k-1)$ existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.

By Theorem 5.1 the second conjecture follows from the first one. Theorem 5.5 asserts the truth of this conjecture for $l=0$. The conjecture was proved for $l=k-1$ in [5]. Here we verify the conjecture for $l=1$. Note that the special case $l=1$ of Conjecture 5.7 is Theorem 1.10. The proof relies on the following lemma.

Lemma 5.8. *Let $D = (V, E)$ be a $(k, 1)$-edge-connected digraph which is minimal in the sense that the deletion of any edge destroys $(k, 1)$-edge-connectivity ($k \geq 2, |V| \geq 2$). Then D has a node z with $k = \varrho(z) > \delta(z)$ for which there is a set F of $\varrho(z) - \delta(z)$ edges entering z so that $D - F$ is $(k, 1)$-edge-connected apart from z.*

Proof. We claim that there is a node z for which $k = \varrho(z) > \delta(z)$. Indeed, by (14), there is an edge e entering s. Since (13) cannot break down by deleting e, it follows from the minimality of D that e leaves a subset $X \subset V - s$ for which $\delta(X) = 1$. Since $\varrho(X) \geq k \geq 2$, there must be a node z in X for which $\varrho(z) > \delta(z)$.

Let us choose such a node z so that the distance of s from z is as large as possible.

Claim 5.9. *Let F be a subset of at most $k - 1$ edges entering z. Then $D' := D - F$ satisfies (14).*

Proof. Assume indirectly that there is a subset $X \subseteq V - s$ for which $\delta_{D'}(X) = 0$. As $\delta(X) \geq 1$, the elements of set of edges of D leaving X are all in F. Therefore $\delta(X) \leq |F| < k$ and, by $\varrho(X) \geq k$, X must contain a node z' for which $\varrho(z') > \delta(z')$. Since the head of each edge leaving X is z, we obtain that each path from z' to s must go through z contradicting the maximal-distance choice of z. $\Box$

Claim 5.10. $\varrho(z) = k$.

Proof. By Claim 5.9 property (14) cannot break down when an edge entering z is left out. Hence the minimality of D implies that every edge entering z enters a subset $X \subseteq V - s$ for which $\varrho(X) = k$. If X and Y are two subsets of $V - s$ containing z for which $k = \varrho(X) = \varrho(Y)$, then $\varrho(X) + \varrho(Y) \geq \varrho(X \cap Y) + \varrho(X \cup Y) \geq k + k$ from which $\varrho(X \cap Y) = k$ follows. This implies that there is a unique smallest subset Z containing z for which $\varrho(Z) = k$ such that every edge entering z enters Z as well. But then the in-degree of z cannot exceed k and hence $\varrho(z) = k$ as D is $(k, 1)$-ec. $\Box$ $\Box$

By Theorem 5.4 there is a subset F of edges of D entering z for which $|F| = \varrho(z) - \delta(z) < k$ and the digraph $D - F$ is $(k, 0)$-ec. Now Claim 5.9 implies that $D - F$ is actually $(k, 1)$-ec, completing the proof of the lemma.

Theorem 5.11. *A digraph $D_0 = (V, E)$ is $(k, 1)$-edge-connected if and only if D_0 can be built up from a node by the following two operations:*

(j) *add a new edge,*

(jj) *pinch i $(1 \leq i \leq k - 1)$ existing edges with a new node z, and add $k - i$ new edges entering z and leaving existing nodes.*

Proof. It is straightforward to see that the two operations preserve $(k, 1)$-edge-connectivity. To prove the reverse direction we use induction on the number of edges. If there is an edge e whose deletion preserves $(k, 1)$-edge-connectivity, then $D_0 - e$ has a required construction by the inductive hypothesis from which the construction of D_0 can be obtained by giving back e (operation (j)).

Therefore we may assume that D_0 is minimally $(k, 1)$-edge-connected with respect to edge deletion. We are done if $|V| = 1$ so assume that $|V| \geq 2$.

By Lemma 5.8 there is a node z with $k = \varrho(z) > \delta(z)$ for which there is a subset F of $\varrho(z) - \delta(z)$ edges entering z so that the digraph $D_0 - F$ is $(k, 1)$-ec apart from z. By Theorem 5.3 there is a complete splitting at z so that the resulting digraph $D_1 = (V - z, E_1)$ is $(k, 1)$-ec. By the inductive hypothesis D_1 can be constructed from a node by the two given operations. But then D_0 is also constructible this way as D_0 arises from D_1 by operation (ii). □

By combining this result with Theorem 5.1 we obtain Theorem 1.10, which is a special case of Conjecture 5.7.

6 Conclusion Algorithmic aspects

Inspired by earlier results of Lovász and Mader, which indicated that constructive characterizations of graph properties may serve as a powerful proof technique, we have described constructive characterizations for several variants of the notion of higher graph connections. One of these results extends a theorem of Henneberg and Laman while another one generalizes a theorem of the first named author and Z. Király. We also formulated some natural conjecture concerning further connectivity properties. Beyond these it remains an interesting research area to find applications of the present characterizations.

As far as algorithmic aspects are concerned, the proofs of the two main theorems (Theorem 1.9 and 1.10) give rise to polynomial algorithms. We should, however, emphasize a significant difference between these algorithms and the ones suggested by Lovász (or Mader's) splitting theorems.

Lovász' theorem asserts that if $G = (V + z, E)$ is a graph which is k-edge-connected (apart from z) and $d_G(z)$ is even, then there exists a pair of edges incident to z whose splitting preserves these properties. To check algorithmically whether the splitting of an arbitrarily chosen pair of edges at z preserves k-edge-connectivity needs some (at most n^2) max-flow-min-cut computations which is doable in polynomial time. That is, Lovász' theorem itself, without relying on any proof of it, gives rise to an algorithm to find a full splitting at z. In order to find algorithmically the constructive characterizaton of a k-sparse graph, as described in Theorem 1.9, one must find a small node admitting a full splitting. This can be done by trying each small node separately. To decide whether a particular small node admits a full splitting one may apply the procedure described in Theorem 3.2. Note that even if a node z is known to have a full splitting, this fact itself, unlike the situation in Lovász' theorem, does not give any clue of how one can find algorithmically such a full splitting.

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