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# Some results on stable matchings and fixed points

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**Abstract**

In this survey paper, we explain some interconnections between fixed point theorems and the theory of stable matchings. Namely, we relate the bipartite matching problems to the Knaster-Tarski fixed point theorem and the nonbipartite ones to the Kakutani fixed point theorem. We study the natural lattice structure of stable matchings, and deduce some consequences of it, like linear characterizations of stable matching related polyhedra.

**Keywords:** stable marriages, lattices, matroids, polyhedra

## 1 Introduction

In the stable marriage problem of Gale and Shapley [18], there are $n$ men and $n$ women and each person ranks the members of the opposite gender by an arbitrary, strict preference order. A *marriage scheme* in this model is a set of marriages between different men and women. Such a scheme is *unstable* if there exist a man $m$ and a woman $w$ in such a way that $m$ is either unmarried or $m$ prefers $w$ to his wife, and at the same time, $w$ is either unmarried or prefers $m$ to her partner. A marriage scheme is *stable* if it is not unstable, and a natural problem is finding a stable marriage scheme if it exists at all.

Nowadays, it is already folklore that for any preference rankings of the $n$ men and $n$ women, there exists a stable marriage scheme. This theorem was proved first by Gale and Shapley in [18]. They constructed a special stable marriage scheme with the help of a finite procedure, the so called deferred acceptance algorithm. It also turned out that for the existence of a stable scheme it is not necessary that the number of men is the same as the number of women or that for each person, all members of the opposite group are acceptable. Moreover, a natural modification of the deferred acceptance algorithm solves the more general stable admissions problem that allows polygamy on one side of the marriage market.

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Although the paper of Gale and Shapley had a certain "recreational mathematics" flavour, later, the model turned out to be especially applicable both in theory and practice. Roth has observed in [29] that the two-sided market of medical students (intern candidates) and hospitals is fairly close to the stable admissions model of Gale and Shapley. Actually, in this particular segment of the economy, the "traditional" free competition approach failed to produce an equilibrium. The market has been stabilized only after a centralized program has been introduced that computes a stable admission scheme for volunteer participants. Roth lists several other practical applications of the framework on his homepage [28].

A most significant theoretical application of the stable marriage theorem is the solution of the Dinitz conjecture. Namely, Galvin proved that the list-chromatic index of any bipartite graph $G$ equals the maximum degree $\Delta$ of $G$ (see [20]). In other words, if for every edge of $G$ there is a list of $\Delta$ possible colours then we can choose a colour for each edge from its list in such a way that adjacent edges receive different colours. The conjecture was open for fifteen years and the key idea in its ingenious elementary solution is the application of the Gale-Shapley theorem. Another theoretically interesting result is the observation that Pym's linking theorem that proves that so called gammoids are matroids is an application of the stable marriage theorem (see [17] and Section 7 of this paper).

It seems that a vast amount of research on the theory of stable matchings was done by game theorists and economists. (For the state of art in the early 90's, see the book of Roth and Sotomayor [34].) Thanks to this, we have a wide knowledge about the structure of stable matchings, much work was devoted to different generalizations, algorithmic and optimizational aspects. In this present work, we discuss a fairly novel approach to the theory of stable matchings. Our framework views stable matchings as fixed points of a certain set function and by this, we can generalize known facts about stable matchings and give simple proofs for different, seemingly unrelated theorems.

Probably, Feder [13] and Subramanian [39] were the first ones that pointed out a connection between stable matchings and fixed points. Both of them were interested in the stable roommates problem (i.e. the nonbipartite generalization of the stable marriage problem). They reduced that to the so called network stability problem that is equivalent with the problem of deciding whether a certain set function on the edges has a fixed point. Subramanian has observed that in the bipartite (stable marriage) case there always exists a fixed point by the fixed point theorem of Tarski. Later, Adachi [5] reduced the stable marriage theorem and the lattice structure of stable matchings explicitly to Tarski's fixed point theorem. In this paper, we mainly follow the approach in [17] and describe a fixed point theorem based framework for stable matchings.

In Section 2, we define some basic notions and notations about stable matchings, recall Tarski's fixed point theorem and describe the basics of our framework. We discuss questions that are related to multiple partner stable matchings in Section 3. Section 4 is devoted to the lattice structure of stable admissions and to point out a connection between path independent choice functions and monotone functions. Theorem 4.5 of this section seems to be a new result. We generalize the lattice operations on stable matchings in Section 5 and discuss some consequences. In particular, we prove a generalization of a theorem of Teo and Sethuraman (see Theorem 5.5) based on the lattice structure. Certain nonbipartite versions of the stable matching problem are handled in Section 6 and we give a couple of new results about nonbipartite stable $b$-matchings (Theorem 6.3, 6.4 6.7 and 6.8). Section 7 contains results on graph paths that are related to stable marriages and Section 8 surveys linerar descriptions of stable matching related polyhedra.

# 2 Stable marriages

Let $G$ be a bipartite graph (parallel edges are allowed) with colour classes $M$ and $W$ and let $E$ denote the set of edges of $G$. (It might be convenient to think that vertices of $M$ and $W$ represent men and women, respectively, and edges are along possible marriages.) For vertex $v$ of $G$, let $D(v)$ denote the set of edges incident with $v$. (Note that $D(v)$ might not be finite, even if $V(G)$ is finite). We call $(G, \mathcal{O})$ a *bipartite preference system* if graph $G$ is as above and $\mathcal{O}$ is a set of orders $<_v$ for $v \in M \cup W$, such that $<_v$ is a well-order on $D(v)$. We say that edge $e$ of $G$ $M$-*dominates* ($W$-*dominates*) edge $f$ of $G$ if $e$ and $f$ are incident with the same vertex $m$ of $M$ ($w$ of $W$) and $e <_m f$ ($e <_w f$). Edge $e$ *dominates* edge $f$ if $e$ $M$-dominates or $W$-dominates $f$. Set $F$ of edges ($M/W$-)dominates edge set $H$ if each edge $h$ of $H$ is ($M/W$-)dominated by some edge $f$ of $F$.

Let $(G, \mathcal{O})$ be a bipartite preference system. Subset $S$ of $E$ is a *stable matching* if no edge of $S$ is dominated by $S$, and $S$ dominates $E \setminus S$. Note that any stable matching is necessarily a matching (i.e. a set of disjoint edges), as if two edges of a stable matching would share a vertex then one of them dominates the other.

**Theorem 2.1 (Gale and Shapley 1962 [18]).** *For any finite bipartite preference system $(G, \mathcal{O})$, there exists a stable matching.*

The original proof of Theorem 2.1 is a construction of a special stable matching with the so called deferred acceptance algorithm. This is an iterative procedure that alternatively repeat two steps. It starts with a proposal step in which each vertex of $M$ selects its most preferred edge. In the next refusal step, certain edges that have been selected in the proposal step, get deleted. Namely, $m$'s edge $e$ gets deleted, if there is another edge $f$ selected by some other vertex with $f <_w e$ for some vertex $w$ of $W$. If no edge is deleted in a refusal step then output the edges selected in the last proposal step. Otherwise start the procedure all over for the reduced bipartite preference system that we get after the deletions of the refusal step. Gale and Shapley have proved that the deferred acceptance algorithm constructs a stable matching of $(G, \mathcal{O})$ which is "man-optimal", that is, no edge that has been deleted in a refusal step can be in any stable matching, or equivalently, each $M$-vertex gets the best possible partner and each $W$-vertex receives the worst possible partner that he/she can have in a stable matching.

In the literature, the usual setting and the definition of a stable matching is somewhat different from ours. We describe that terminology as well, so as to have a "dictionary" between the two languages. If $G$ is a graph, a *matching* $N$ of $G$ is a set of disjoint edges of $G$. Equivalently, we can regard a matching as an involution on the vertices of $G$, that is, a function $\mu : V(G) \to V(G)$ in such a way that $\mu(\mu(v)) = v$ for each vertex $v$ of $G$. Indeed, if $uv$ is an edge of matching $N$ then $\mu(u) = v$ and $\mu(v) = u$, and if $w$ is not covered by $N$ then $\mu(w) = w$ for the corresponding involution. From involution $\mu$, we can also reconstruct matching $N$ as $uv \in N$ if and only if $\mu(u) = v$.

The conventional counterpart of a bipartite preference system is the following. We have given disjoint sets $M$ and $W$. For vertex $m$ of $M$ ($w$ of $W$), let $\prec_m$ ($\prec_w$) be a linear order on $W \cup \{m\}$ ($M \cup \{w\}$). In this model, matching $\mu$ is a stable matching if

$$\mu(v) \leq_v v \text{ for each vertex } v \text{ of } M \cup W \text{ and} \tag{1}$$

$$\text{for all } m \in M \text{ and } w \in W,\ w \prec_m \mu(m) \text{ implies } \mu(w) \prec_w m\ . \tag{2}$$

That is, a matching is stable, if each person (vertex) is not worse off by his/her partner than by remaining single (or, in other words, matching $\mu$ is *individually rational*) and whenever man $m$ (i.e. a vertex in the $M$ side) prefers woman $w$ to his eventual partner, then $w$ must prefer $\mu(w)$ to the partnership with $m$. Note that traditionally, the preference orders are on the vertices, and not on the edges. (This makes the traditional model in some sense richer than ours, as it allows that a man $m$ can accept woman $w$ but $w$ prefers to remain single to marray $m$). For simple graphs however, preference order $\prec_v$ induces a linear order $<_v$ on $D(v)$, so a stable matching in our sense corresponds to a stable matching $\mu$ in the latter sense. Our terminology is more general in the sense that it readily allows graphs with parallel edges in a bipartite preference model. This feature is useful if we use a model where the vertices correspond to firms and workers, so parallel edges between a firm and a worker may mean different wages in case of an employment.

Let us now return to our terminology. If $S$ is a stable matching then we can partition $E$ into three disjoint parts as $E = S_M \cup S \cup S_W$, in such a way that $S_M$ (and $S_W$) is $M$-dominated ($W$-dominated, respectively) by $S$. (Note that such a partition is unique only if no edge of $E \setminus S$ is both $M$- and $W$-dominated by $S$.) From here, it is easy to see that sets $S^M := S \cup S_M$ and $S^W := S \cup S_W$ have the following properties.

$$S^M \cup S^W = E \tag{3}$$

$$\text{no edge of } S^M \cap S^W \text{ dominates another edge of } S^M \cap S^W \tag{4}$$

$$S^M \text{ is } M\text{-dominated by } S^M \cap S^W \text{ and } S^W \text{ is } W\text{-dominated by } S^M \cap S^W \tag{5}$$

For convenience, we call $(S^M, S^W)$ a *stable pair* if $S^M$ and $S^W$ have the above properties (3,4,5). Clearly, if $(S^M, S^W)$ is a stable pair then $S := S^M \cap S^W$ is a stable matching. Equivalently, we can say that pair $(S^M, S^W)$ is stable if besides (3) we have

$$\mathcal{C}_M(S^M) = S^M \cap S^W = \mathcal{C}_W(S^W)\ , \tag{6}$$

where $\mathcal{C}_M(S^M)$ ($\mathcal{C}_W(S^W)$) denotes those elements of $S^M$ ($S^W$) that are not $M$-dominated ($W$-dominated) by other elements of $S^M$ ($S^W$). This means that $(S^M, S^W)$ is a stable pair if and only if

$$S^W = E \setminus (S^M \setminus \mathcal{C}_M(S^M)) \text{ and} \tag{7}$$
$$S^M = E \setminus (S^W \setminus \mathcal{C}_W(S^W)) \tag{8}$$

holds. By substituting (7) into (8), we get that $(S^M, S^W)$ is a stable pair if and only if (7) holds with

$$f(S^M) := E \setminus \left[\left(E \setminus [S^M \setminus \mathcal{C}_M(S^M)]\right) \setminus \mathcal{C}_W\left(E \setminus [S^M \setminus \mathcal{C}_M(S^M)]\right)\right] = S^M \ . \tag{9}$$

A key observation in our treatment that function $f$ is *monotone*, that is, $f(A) \subseteq f(B)$ whenever $A \subseteq B \subseteq E$. (To see this, it is useful to observe that function $A \mapsto A \setminus \mathcal{C}(A)$ is monotone for $\mathcal{C} = \mathcal{C}_M$ and $\mathcal{C} = \mathcal{C}_W$.) Thus we can invoke the Knaster-Tarski fixed point theorem.

**Theorem 2.2 (Knaster and Tarski 1928 [24]).** *If $f : 2^E \to 2^E$ is a monotone function then $f$ has a fixed point.*

As a consequence of Theorem 2.2, the function $f$ that comes from the bipartite preference model according to (9) has a fixed point $S^M$. Using (7) as a definition for $S^W$, we can construct a stable pair $(S^M, S^W)$ from the above fixed point, and a stable matching $S$ from stable pair $(S^M, S^W)$. This proves the existence part of Theorem 2.1. What is more interesting, than this single reduction is that for finite ground sets, there is a most simple algorithmic proof of Theorem 2.2 that can be applied in our construction. (Note that in Theorem 2.2, $E$ can be an arbitrarily large set.) Namely, $f(\emptyset) \subseteq f(f(\emptyset)) \subseteq f(f(f(\emptyset))) \subseteq \ldots$ by the monotonicity of $f$, so if $E$ is finite then this increasing chain has to get stabilized at some fixed point $A$ of $f$. This fixed point $A$ is contained in any fixed point of $f$ by the procedure. Similarly, chain $f(E) \supseteq f(f(E)) \supseteq f(f(f(E))) \supseteq \ldots$ must get stabilized at the inclusionwise maximal fixed point of $f$. It turns out that the deferred algorithm itself is essentially the iteration of $f$ starting from $E$. If we exchange the role of $M$ and $W$ then the deferred acceptance algorithm concludes with the woman-optimal stable matching that we can construct by iterating $f$ starting from the $\emptyset$. This observation is enough to prove that the deferred acceptance algorithm outputs the man-optimal stable matching.

# 3 Multiple partner matchings

The stable marriage theorem of Gale and Shapley (Theorem 2.1) has several extensions and generalizations that nicely fit into our framework. We start with the stable admissions problem described in [18]. We have a bipartite preference system $(G, \mathcal{O})$ and a function $b : M \cup W \to \mathbb{N}$ with $b(w) = 1$ for all $w \in W$. Set $F$ of edges $(M, b)$-*dominates* ($(W, b)$-*dominates*) edge $e$ of $G$ if there is a vertex $m$ of $M$ ($w$ of $W$) and different edges $f_1, f_2, \ldots, f_{b(m)}$ ($f_1, f_2, \ldots, f_{b(w)}$) of $F$ so that $f_i <_m e$ for $i = 1, 2, \ldots, b(m)$ ($f_i <_w e$ for $i = 1, 2, \ldots, b(w)$). Set $F$ of edges $b$-dominates edge set $H$ if each edge $h$ of $H$ is $(M, b)$-dominated or $(W, b)$-dominated by $F$. Subset $S$ of the edges of $G$ is a *stable admission scheme* if no edge of $S$ dominates another edge of $S$, and $S$ dominates $E \setminus S$. By definition, each vertex in $W$ is incident with at most one edge and vertex $m$ of $M$ is incident with at most $b(m)$ edges of a stable admission scheme. (The story is, that vertices of $M$ represent colleges that offer admission, $W$ stands for the set of students that look for admission in a college, and $b$ is the quota of the colleges. We look for a stable admission scheme in which no college-student pair mutually prefer each other to their assignments.)

The same argument that we gave for the stable marriage problem goes through for the above admissions case, even in the more general case in which we do not require that $b(w) = 1$ for $w \in W$. (The generalization of a stable admission scheme to this case we call a *stable b-matching.*) The iteration of the corresponding monotone function describes the modified deferred acceptance algorithm that finds the optimal stable admissions. This implies the following theorem.

**Theorem 3.1.** *For any bipartite preference system* $(G, \mathcal{O})$ *and* $b : V(G) \to \mathbb{N}$ *there exists a stable b-matching. If* $M$ *and* $W$ *are the colour classes of* $G$ *then there is an* $M$*-optimal stable b-matching* $S$*, in which each vertex* $m$ *of* $M$ *is incident with the most preferred* $b(m)$ *edges of* $D(m)$ *that can be in a stable b-matching. Simultaneously, each vertex* $w$ *of* $W$ *is incident with the least preferred* $b(w)$ *edges of* $D(w)$ *that can appear in a stable b-matching.*

We can generalize the notions of stable matchings and stable admissions. Let $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ be set functions. Pair $(S^M, S^W)$ is a $\mathcal{C}_M\mathcal{C}_W$*-stable pair* if (3,6) holds. Subset $S$ of $E$ is a $\mathcal{C}_M\mathcal{C}_W$*-stable set* if $S$ dominates exactly the elements of $E \setminus S$, that is, if

$$\mathcal{C}_M(S) = \mathcal{C}_W(S) = S \text{ and} \tag{10}$$

$$\mathcal{C}_M(S \cup \{e\}) = S \text{ or } \mathcal{C}_W(S \cup \{e\}) = S \text{ for any element } e \text{ of } E \tag{11}$$

What are the crucial properties of a dominance function $\mathcal{C}$ that make our argument work? These are that

$$\mathcal{C}(A) \subseteq A \text{ for any } A \tag{12}$$

and that function

$$\overline{\mathcal{C}}(A) := A \setminus \mathcal{C}(A) \text{ is monotone.} \tag{13}$$

We call function $\mathcal{C}$ *comonotone* if (12,13) hold. These properties imply that function $f$ defined in (9) is monotone and there is a stable pair. Still, it might happen that $\mathcal{C}_M$ and $\mathcal{C}_W$ are comonotone so there is a stable pair $(S^M, S^W)$, but no stable set exists. However, $S := S^M \cap S^W$ is a stable set if $\mathcal{C}_M$ and $\mathcal{C}_W$ have the additional property that

$$\mathcal{C}(A) = \mathcal{C}(B) \text{ whenever } \mathcal{C}(A) \subseteq B \subseteq A\ . \tag{14}$$

Formally, we have the following theorem.

**Theorem 3.2 (Fleiner 2000 [17]).** *If $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are comonotone set functions then there exists a $\mathcal{C}_M\mathcal{C}_W$-stable pair $(S^M, S^W)$. If, moreover, both $\mathcal{C}_M$ and $\mathcal{C}_W$ have property* (14) *then there exists a $\mathcal{C}_M\mathcal{C}_W$-stable set $S$.*

We give two more interesting examples of comonotone functions with property (14), hence extend the stable marriage theorem in two different directions. A *partial well-order* is a partial order so that each subset of the ground set has a minimal element in the induced order.

**Observation 3.3.** If $<$ is a partial order on $E$ and $\mathcal{C}(A)$ denotes the set of $<$-minima of $A$ for subset $A$ of $E$ then $\mathcal{C}$ is comonotone. If $<$ is a partial well-order then $\mathcal{C}$ has property (14).

Theorem 3.2 and Observation 3.3 implies the following property of partial orders.

**Corollary 3.4 (see Fleiner [17, 14]).** *If $<_1$ and $<_2$ are partial orders on $E$ then there are subsets $E_1, E_2$ of $E$ such that*

$$E_1 \cup E_2 = E \text{ and}$$
$$E_1 \cap E_2 \text{ is the set of } <_1\text{-minima of } E_1 \text{ and the set of } <_2\text{-minima of } E_2$$

*If both $<_1$ and $<_2$ are partial well-orders then there is a common antichain $S$ of $<_1$ and $<_2$ (i.e. no two elements of $S$ are comparable in any of the orders) so that for any $e \in E$ there is an $s \in S$ with $s <_1 e$ or $s <_2 e$.*

**Observation 3.5.** Let $<$ be a linear order on finite set $E$, and $\mathcal{M}$ be a matroid on $E$. Denote by $\mathcal{C}^<_{\mathcal{M}}(A)$ the output of the greedy algorithm on input $A$, where the algorithm goes through the elements of $A$ in the order given by $<$. Then $\mathcal{C}^<_{\mathcal{M}}$ is comonotone and has property (14).

Theorem 3.2 and Observation 3.5 extends the stable matching theorem to matroids.

**Corollary 3.6 (see Fleiner [17, 14]).** *Let $<_1$ and $<_2$ be linear orders on finite set $E$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids on $E$. Then there is a common independent set $S$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ so that for any element $e$ of $E$*

$$e \in \operatorname{span}_i\{s \in S : s \leq_i e\} \text{ for some } i \in \{1, 2\} .$$

*There also is a common spanning set $T$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ such that for any element $t$ of $T$, subset $T$ is the lexicographically $<_i$-minimal spanning subset of $\mathcal{M}_i$ that contains $T \setminus \{t\}$ for some $i \in \{1, 2\}$.*

See [14, 17] for a reduction of the stable $b$-matching theorem to the first part of Corollary 3.6.

# 4 The lattice of stable sets and path independent choice functions

The marriage model of Gale and Shapley has attracted some interest in the theory of social choice. There are related results to the ones that we discussed so far. Again, the terminology is different from ours, so in this section we attempt to cover some basics and point out an interesting connection of this approach to the monotone function based framework.

Let $(G, \mathcal{O})$ be a bipartite preference system and let $F$ and $W$ be the two colour classes of $G$. We will identify vertices of $F$ with different firms and the vertices of $W$ with workers. Edge $fw$ of $G$ represents that firm $f$ considers $w$ as a potential employee and worker $w$ can accept $f$ as an employer. Firms would like to get certain specific jobs to be done, and this is why they have a more complex preference function on workers than plain ranking. Namely, each firm $f$ has a so called choice function $\mathcal{C}_f$ that selects from any set $W'$ of workers a subset $\mathcal{C}_f(W')$ of $W'$ that firm $f$ would hire if on the labour-market only firm $f$ and workers of $W'$ would be present. Set function $\mathcal{C} : 2^W \to 2^W$ is a *choice function* if there is a well-order $<$ on $2^W$ such that $\mathcal{C}(W')$ is the $<$-minimal subset of $W'$, for any subset $W'$ of $W$.

Continuing on a paper of Crawford and Knoer [12], Kelso and Crawford [23] extended the admissions model to a model where each firm has a choice function and each worker has an ordinary preference ranking on firms.

An assignment of workers to firms is called *stable* if it is not blocked by a worker-firm pair. Worker-firm pair $(w, f)$ *blocks* an assignment if $w$ prefers $f$ to his/her employer and in the meanwhile firm $f$ would take worker $w$ if $w$ would be available (that is $w \in \mathcal{C}_f(W_f \cup \{w\})$, where $W_f$ is the set of workers assigned to firm $f$).

Not surprisingly, in the above model there might be no stable assignment. However, if each choice function has the so-called substitutability property, then a stable assignment always exists. We say that set function $\mathcal{C}_f : 2^W \to 2^W$ of firm $f$ has the *property of substitutability*, if

$$w \in \mathcal{C}_f(W') \text{ implies } w \in \mathcal{C}_f(W' \setminus \{w'\}) \tag{15}$$

for any subset $W'$ of the set $W$ of workers and for any two different workers $w, w'$ of $W'$. This means that if a firm would like to employ a certain worker, then it still would like to hire him/her if some other worker leaves the labour-market.

**Theorem 4.1 (Kelso-Crawford 1982 [23]).** *If each firm's preference is a substitutable choice function in the worker-firm assignment model, then there is a stable assignment of workers to firms.*

The proof of Crawford and Kelso is via the accordingly modified deferred acceptance algorithm. They also observed that firm-proposing results in the firm-optimal assignment, and the worker-proposal based method leads to the worker-optimal situation. In [30, 31], Roth extended Theorem 4.1 to a many-to-many model. A *stable assignment* is a bipartite assignment graph $A$ with colour classes $F$ and $W$, such that for any $w \in W$ and $f \in F$ we have that $wf \in E(A)$ if and only if $f \in \mathcal{C}_w(\Gamma_A(w) \cup f)$ and $w \in \mathcal{C}_f(\Gamma_A(f) \cup w)$. (Notation $\Gamma_A(w)$ stands for the neighbours of $w$ in $A$.)

**Theorem 4.2 (Roth 1984 [30, 31]).** *Let $F$ and $W$ be disjoint finite sets, and for each $f \in F$ and $w \in W$ let $\mathcal{C}_w : 2^F \to 2^F$ and $\mathcal{C}_f : 2^W \to 2^W$ be choice functions with substitutability property (15). Then there is a stable assignment in the model.*

Clearly, the stable marriage theorem of Gale and Shapley is a special case of Theorem 4.2, where the choice functions simply select the highest ranked partner from the input. For the college model, the choice function of college $c$ selects the best $b(c)$ inputs.

A key observation in this section is that a choice function trivially has property (12) and (14), and a little effort shows that for finite ground sets substitutability property (15) implies property (13), so choice functions in Theorems 4.1 and 4.2 are comonotone. A fairly trivial construction shows that these theorems are special cases of Theorem 3.2. Thus, our monotone framework is relevant for these results and it proves that there exists a stable set, that is, a stable assignment.

In [31], Roth studied three models: the one-to-one, the many-to-one and the many-to-many with substitutable preferences. He showed that for all three models there is a firm-optimal, "worker-pessimal" and a worker-optimal, "firm-pessimal" stable assignment. The name "polarization of interests" refers to this property. Further on, Roth introduced the notion of the *consensus property*, that means the following. If each agent on one side of the market combines his/her most preferred assignment from a set of stable assignments, then this way another stable assignment is constructed. This is a generalization of the lattice property of stable schemes in the marriage model. (The observation that stable marriages in the marriage model have a natural lattice structure is attributed to John Conway.) Unfortunately, this property does not always hold in Theorem 4.2. In [31], Roth asked whether some lattice structure can still be defined on stable assignments. Blair answered this question positively in [11]. His idea was that instead of defining lattice operations, he introduced a more or less natural partial order on stable assignments. This partial order turned out to be a lattice order hence generalizing the lattice property of bipartite stable matchings.

**Theorem 4.3 (Blair 1988 [11]).** *Let $F$ be a set of firms and $W$ be a set of workers. Let, for each $f \in F$ ($w \in W$), set function $\mathcal{C}_f : 2^W \to 2^W$ ($\mathcal{C}_w : 2^F \to 2^F$) be given with properties (12,14,15). Let $\mathcal{A}$ be the set of stable assignments of the model, and define for $A_1, A_2 \in \mathcal{A}$*

$$A_1 < A_2 \text{ if } \Gamma_{A_1}(f) = \mathcal{C}_f(\Gamma_{A_1}(f) \cup \Gamma_{A_2}(f))$$

*holds for each firm $f$. (That is, each firm would choose $A_1$ if all choices provided by $A_1$ and $A_2$ would be offered.) Then $\mathcal{A}$ is nonempty and $<$ is a lattice order, that is, any two stable assignments have a $<$-maximal lower bound and a $<$-minimal upper bound.*

We have mentioned earlier that (15) implies (13) for functions on finite ground sets, so the functions that describe the choice of the agents of the market are comonotone with property (14). To prove the lattice property in our framework, we have to go back to the roots, that is to the fixed point theorem. Recall that we have cited the Knaster-Tarski fixed point theorem on monotone set functions. This theorem is often attributed to Tarski for the reason that 27 years after Knaster's paper, he published a lattice theoretical generalization of the result in English in a much easier reachable journal [41]. (Actually, Tarski has also formulated a corollary of the fixed point theorem there in terms of Boolean algebras that is more general than Theorem 3.2.) Note that the proof of the set function theorem is just as difficult as the proof of the lattice generalization, but in the latter paper Tarski has explicitly stated the lattice property of fixed points that we need now.

If $L$ is a lattice on $E$ with partial order $<$ and lattice operations $\wedge, \vee$ then function $f : E \to E$ is *monotone* if $a < b$ implies $f(a) < f(b)$. Lattice $L'$ is a *sublattice* of $L$ if its ground set $E'$ is a subset of $E$ and the lattice operations on $L'$ are $\wedge$ and $\vee$ restricted to $E'$. Lattice $L'$ is a *lattice subset* of $L$ if its ground set $E'$ is a subset of $E$ and the lattice order on $L'$ is the restriction of $<$ to $E'$. Lattice $L$ is complete if any subset $E'$ of its ground set has a meet (greatest lower bound) $\bigwedge E'$ and a join (least upper bound) $\bigvee E'$. (In particular, these lattices have a minimal and maximal element.) Note that any finite lattice is complete.

**Theorem 4.4 (Tarski 1955 [41]).** *Let $L$ be a complete lattice on ground set $E$ and $f : E \to E$ be a monotone function. Then the fixed points of $f$ form a nonempty complete lattice subset of $L$.*

It turns out that Theorem 4.4 is relevant in the setting of Theorem 4.3 and it implies that stable assignments exist and form a lattice as described. For the details, see [14]. In the next section, we discuss a property that implies that the lattice subset in Theorem 4.4 is a sublattice. If that is the case then stable pairs have a very rich structure that allows us to prove further results.

We finish this section with pointing out a connection between properties of set functions we have used so far. Set function $\mathcal{C} : 2^E \to 2^E$ is *path independent* if

$$\mathcal{C}(A \cup B) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B)) \text{ for any subsets } A, B \text{ of } E \tag{16}$$

The central notion of path independence has been introduced by Plott in 1973 [25] into the theory of social choice. Our observation is the following.

**Theorem 4.5.** *Set function $\mathcal{C} : 2^E \to 2^E$ is path independent with property* (12) *if and only if it is comonotone and has property* (14)*.*

**Lemma 4.6 (see [17, 14]).** *Set function $\mathcal{C} : 2^E \to 2^E$ is comonotone if and only if $\mathcal{C}$ has property* (12) *and*

$$\mathcal{C}(B) \cap A \subseteq \mathcal{C}(A) \textit{ whenever } A \subseteq B \subseteq E \tag{17}$$

*Proof.* If (12) holds for $\mathcal{C}$ then (17) is equivalent with the monotonicity of $\overline{\mathcal{C}}$. □

*Proof of Theorem 4.5.* Assume that $\mathcal{C}$ is path independent. Then

$$\mathcal{C}(A) = \mathcal{C}(A \cup A) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(A)) = \mathcal{C}(\mathcal{C}(A)) ,$$

hence for $\mathcal{C}(A) \subseteq B \subseteq A$

$$\mathcal{C}(B) = \mathcal{C}(\mathcal{C}(A) \cup B) = \mathcal{C}(\mathcal{C}(\mathcal{C}(A)) \cup \mathcal{C}(B)) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B)) = \mathcal{C}(A \cup B) = \mathcal{C}(A) \ ,$$

showing (14). For $A \subseteq B$ from (12) we get

$$\mathcal{C}(B) = \mathcal{C}(A \cup (B \setminus A)) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B \setminus A)) \subseteq \mathcal{C}(A) \cup \mathcal{C}(B \setminus A) \subseteq \mathcal{C}(A) \cup (B \setminus A) \ ,$$

implying

$$\mathcal{C}(B) \cap A \subseteq (\mathcal{C}(A) \cup (B \setminus A)) \cap A = \mathcal{C}(A) \ ,$$

so $\mathcal{C}$ is comonotone by Lemma 4.6.

Let now $\mathcal{C}$ be comonotone with property (14) and $A$ and $B$ be subsets of $E$. From (17) and (12), we get that

$$\mathcal{C}(A \cup B) = (\mathcal{C}(A \cup B) \cap A) \cup (\mathcal{C}(A \cup B) \cap B) \subseteq \mathcal{C}(A) \cup \mathcal{C}(B) \subseteq A \cup B \ , \tag{18}$$

and (14) and (18) implies (16). □

We give an example showing that comonotonicity and (14) is necessary in Theorem 4.5. That is, there is a function with properties (12) and (14) that is not path independent.

**Example 4.7.** Let $|E| > k \geq 1$, fix element $x$ of $E$ and define $\mathcal{C} : 2^E \to 2^E$ by

$$\mathcal{C}(A) = \begin{cases} A & \text{if } |A| > k \\ A \setminus \{x\} & \text{if } |A| \leq k \ . \end{cases}$$

Then $\mathcal{C}$ has properties (12,14) but it is neither path independent nor comonotone.

Example 4.7 and Theorem 4.5 together imply that there exists a comonotone function that is not path independent.

# 5 The stable matching lattice

In this section, we discuss the situation that the lattice subset of stable pairs in Theorem 3.2 and the lattice subset of fixed points in Theorem 4.4 are both sublattices. For a finite ground set $E$, we call function $f : 2^E \to 2^E$ *strongly monotone* if $f$ is monotone with property

$$|f(A \cup \{e\})| \leq |f(A)| + 1 \text{ for any subset } A \text{ and element } e \text{ of } E \ . \tag{19}$$

Set function $\mathcal{C} : 2^E \to 2^E$ is *increasing* if

$$A \subseteq B \subseteq E \text{ implies } |\mathcal{C}(A)| \leq |\mathcal{C}(B)| \ . \tag{20}$$

Note that $|\mathcal{C}^<_{\mathcal{M}}(A)| = \text{rank}(A)$ for comonotone function $\mathcal{C}^<_{\mathcal{M}}$ described in Observation 3.5, hence $\mathcal{C}^<_{\mathcal{M}}$ is increasing.

First we give a sufficient condition for a monotone function on subset lattices so that the lattice subset of its fixed points is a sublattice. The interested reader may find the details of an even more general treatment in [14].

**Theorem 5.1 (see [17, 14]).** *If $f : 2^E \to 2^E$ is a strongly monotone function on finite set $E$, then fixed points of $f$ form a nonempty sublattice of $(2^E, \cap, \cup)$.*

The following Lemma is a link between strongly monotone and increasing comonotone functions.

**Lemma 5.2 (see [17, 14]).** *If function $\mathcal{C} : 2^E \to 2^E$ is increasing and comonotone then $\overline{\mathcal{C}}$ is strongly monotone.*

Based on Lemma 5.2, we can give a sufficient condition for the property that stable pairs in Theorem 3.2 form a sublattice. Note that independently from our work, Alkan [8] has also found Theorem 5.3 (see also [19]). He used the name *cardinal monotonicity* for our increasing notion.

**Theorem 5.3 (Alkan 2000 [8], Fleiner 2000[17]).** *If $E$ is finite and $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are increasing, comonotone functions then $\mathcal{C}_M\mathcal{C}_W$-stable sets have the same cardinality and form a nonempty, complete lattice with lattice operations $S_1 \vee S_2 := \mathcal{C}_M(S_1 \cup S_2)$ and $S_1 \wedge S_2 := \mathcal{C}_W(S_1 \cup S_2)$. Moreover,*
*$S_1 \cap S_2 = (S_1 \wedge S_2) \cap (S_1 \vee S_2)$ and $S_1 \cup S_2 = (S_1 \wedge S_2) \cup (S_1 \vee S_2)$, or equivalently,*

$$\chi(S_1) + \chi(S_2) = \chi(S_1 \wedge S_2) + \chi(S_1 \vee S_2) \tag{21}$$

*holds for any two $\mathcal{C}_M\mathcal{C}_W$-stable sets $S_1, S_2$*

Theorem 5.3 can be regarded as a generalization of the "consensus property" observed by Roth in [31]. Namely, from Theorem 5.3, it follows that if $\mathcal{S}$ is a set of $\mathcal{C}_M\mathcal{C}_W$-stable sets then $\mathcal{C}_M(\bigcup \mathcal{S})$ (the *first choice of $\mathcal{C}_M$ from $\mathcal{S}$*) is a $\mathcal{C}_M\mathcal{C}_W$-stable set.

But more is true. Let us denote by $\mathcal{S}_i^M$ the *ith choice of $\mathcal{C}_M$ from $\mathcal{S}$* defined as the first choice of $\mathcal{C}_M$ from the support of

$$\sum_{S \in \mathcal{S}} \chi(S) - \sum_{j=1}^{i-1} \chi(\mathcal{S}_j^M) \ .$$

If $\mathcal{S}$ is a chain of $k$ $\mathcal{C}_M\mathcal{C}_W$-stable sets then $\mathcal{S}_i^M \in \mathcal{S}$ for $i = 1, 2, \ldots, k$. Otherwise, there are two uncomparable stable sets of $\mathcal{S}$, say $S_1$ and $S_2$ of $\mathcal{S}$ that we can exchange into $S_1 \wedge S_2$ and $S_1 \vee S_2$. By (21), this uncrossing operation does not change $\sum_{S \in \mathcal{S}} \chi(S)$. If we apply a sequence of uncrossing operations on $\mathcal{S}$ then we can transform collection $\mathcal{S}$ into a chain of $\mathcal{C}_M\mathcal{C}_W$-stable sets in finite number of steps (see [17] for the details). As the uncrossing steps do not change $\sum_{S \in \mathcal{S}} \chi(S)$, this chain must be the chain of the $i$th choices of $\mathcal{C}_M$. The above argument also holds for $\mathcal{C}_W$ and gives the following.

**Theorem 5.4.** *If $E$ is a finite ground set, $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are increasing comonotone functions and $\mathcal{S}$ is a set of $n$ (not necessarily different) $\mathcal{C}_M\mathcal{C}_W$-stable sets then the $i$th choice of $\mathcal{C}_M$ from $\mathcal{S}$ is a $\mathcal{C}_M\mathcal{C}_W$-stable set and coincides with the $(n + 1 - i)$th choice of $\mathcal{C}_W$ from $\mathcal{S}$.*

Theorem 5.4 generalizes the following nice structural result of Teo and Sethuraman on stable matchings. The original proof used linear programming tools.

**Theorem 5.5 (Teo and Sethuraman 1998 [42]).** *Let $(G, \mathcal{O})$ be a bipartite preference system with colour classes $M$ and $W$ of $G$ and let $S_i, S_2, \ldots, S_n$ be (not necessarily different) stable matchings. For each vertex $v$ of $M \cup W$ order edges of $\bigcup_{i=1}^n S_i \cap D(v)$ as $e_v^1 \leq_v e_v^2 \leq_v \ldots$ in such a way that each edge is listed as many times as it appears in an $S_i$. (So the length of this chain is the number of $S_i$'s that cover $v$.) Then*

$$S_M^k := \{e_m^k : m \in M\} \text{ and } S_W^k := \{e_w^k : w \in W\}$$

*are stable matchings for $k \in \{1, 2, \ldots, n\}$ and $S_M^k = S_W^{n+1-k}$ .*

We sketch a short direct proof of this result. We need the consequence of Theorem 5.3 that if each man chooses his best partner from a set of stable marriages then this induces a stable matching scheme in which each woman receives the worst husband from the given set.

*Proof.* For vertex $v$ of $G$ and stable matchings $S$ and $S'$ let $S \preceq_v S'$ denote that $v$ prefers $S$ to $S'$. List $S_1, S_2, \ldots, S_n$ as $S_v^1 \preceq_v S_v^2 \preceq_v \ldots \preceq_v S_v^n$ for each vertex $v$. Observe that $S_M^k = \bigwedge_{m \in M} \bigvee_{i=1}^k S_m^i$ and $S_W^k = \bigwedge_{w \in W} \bigvee_{i=1}^k S_w^i$ , Chains $S_M^1, S_M^2, \ldots, S_M^n$ and $S_W^1, S_W^2, \ldots, S_W^n$ are opposite, hence $S_M^k = S_W^{n+1-k}$ . □

Actually, it is straightforward to generalize the above direct proof of Theorem 5.5 to stable $b$-matchings. Instead of formulating that, we point out another interesting property of stable $b$-matchings. To this end, we use the generalization of the Comparability Theorem of Roth and Sotomayor [33] to the many-to-many model by Baïou and Balinski. The Comparability Theorem states that in a fixed bipartite preference system, if two stable $b$-matchings are different for some agent $a$, then $a$ strictly prefers one $b$-matching to the other (that is, $a$ would choose one of the $b$-mathcings if all options of the two $b$-matchings were offered). For a short and direct proof see [16].

**Theorem 5.6 (Baïou and Balinski 2000 [9]).** *Let $S$ and $S'$ be two stable $b$-matchings for bipartite preference system $(G, \mathcal{O})$, let $v$ be a vertex of graph $G$ and $S_v := S \cap D(v)$ and $S'_v := S' \cap D(v)$. If $S_v \neq S'_v$ then $|S_v| = |S'_v| = b(v)$ and the $b(v)$ $<_v$-best edges of $S_v \cup S'_v$ are either $S_v$ or $S'_v$.*

A consequence of Theorem 5.6 that is interesting in itself is that in the polygamous stable marriage problem each participating person $p$ can partition the members of the other gender into as many groups as $p$'s quota in such a way that in any polygamous stable marriage scheme $p$ receives at most one partner from each group. This result turns out to be useful for the linear characterization of the stable $b$-matching polytope. See Section 8 for the details.

**Corollary 5.7 (Fleiner 2002 [16]).** *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : V(G) \to \mathbb{N}$ be a quota function. Then for each vertex $v$ of $G$, there is a partition of $D(v)$ into $b(v)$ parts $D_1(v), D_2(v), \ldots, D_{b(v)}(v)$ so that $|S \cap D_i(v)| \leq 1$ for any stable $b$-matching $S$ and any integer $i$ with $1 \leq i \leq b(v)$.*

The last result in this section generalizes a well-known fact in the stable admissions model (that is also valid in the many-to-many case). In that model, those colleges that cannot fill up their quota in some stable admission scheme receive the very same set of students in any stable assignment. A special case of this property is that in any bipartite preference system always the same persons get married in each stable marriage scheme. Theorem 5.8 is a direct corollary of Theorem 5.3.

**Theorem 5.8 (Fleiner 2000 [17]).** *If $\mathcal{M}_1, \mathcal{M}_2$ are matroids on a common ground set and $S_1, S_2$ have the property of $S$ in Corollary 3.6 for linear orders $<_1, <_2$, then* $\mathrm{span}_{\mathcal{M}_i}(S_1) = \mathrm{span}_{\mathcal{M}_i}(S_2)$ *for $i \in \{1, 2\}$.*

# 6 The stable roommates problem

We have seen that the bipartite nature of the stable marriage theorem was crucial for the application of the Knaster-Tarski fixed point theorem. There is however a natural nonbipartite model in which similar questions can be asked. We discuss certain nonbipartite versions in this section. The interested reader is referred to [6] for further details.

A *graphic preference system* is a pair $(G, \mathcal{O})$ where $G$ is a graph and $\mathcal{O} = \{<_v : v \in V(G)\}$ so that $<_v$ is a a linear order on $D(v)$. Let $b : V(G) \to \mathbb{N}$. Set $F$ of edges of $G$ *$b$-dominates* edge $e$ of $G$ if there is a vertex $v$ of $e$ and different edges $f_1, f_2, \ldots, f_{b(v)}$ of $F$ so that $f_i <_v e$ for $i = 1, 2, \ldots, b(v)$. Set $F$ of edges $b$-dominates edge set $H$ if each edge $h$ of $H$ is $b$-dominated by $F$. A *stable $b$-matching* is a subset $S$ of the edges of $G$ such that no edge of $S$ is dominated by $S$, and $S$ dominates $E \setminus S$. By definition, each vertex $v$ of $G$ is incident with at most $b(v)$ edges of a stable $b$-matching. A *stable matching* is the short name of a stable $\mathbf{1}$-matching.

An important difference from the bipartite model is that in nonbipartite graphic preference systems there might exist no stable matching at all. The first efficient algorithm to decide the existence of a stable matching for this case is due to Irving [21]. Later on, based on Irving's proof, Tan gave a compact characterization of those models that contain a stable matching in [40]. In this section, we study Tan's result.

Recall that a function $w$ assigning non-negative weights to edges in $G$ is called a *fractional matching* if $\sum_{v \in e} w(e) \leq 1$ for every vertex $v$. A fractional matching $w$ is called *stable* if every edge $e$ contains a vertex $v$ such that $\sum_{v \in f, f \leq_v e} w(f) = 1$.

**Theorem 6.1 (Tan, 1991 [40]).** *In any graphic preference system, there exists a half-integral fractional stable matching. In other words, there exists a set $S$ of edges whose connected components are single edges and cycles, such that every edge $e$ of the graph contains a vertex $v$ of $V(S)$ such that $e \leq_v s$ for each $s \in S$ containing $v$.*

Tan's original proof is based on the algorithm of Irving [21] for testing the existence of an (integral) stable matching in a graphic preference system. As Irving's algorithm runs in polynomial time in the size of the input, Tan's paper describes a polynomial time algorithm for finding a stable fractional matching in graphic preference systems.

Observe that if the support of a half-integral fractional stable matching contains only even cycles then there obviously exists a stable matching: we only have to throw away each second edge of the cycle components of the support. Tan also proved the following curious fact. (For a short and direct proof that is independent from Irving's algorithm, see [6].)

**Theorem 6.2 (Tan 1991 [40]).** *Let $(G, \mathcal{O})$ be a graphic preference system. If an odd cycle appears in the support of some fractional stable matching of $(G, \mathcal{O})$, then this very cycle appears in the support of any fractional stable matching of $(G, \mathcal{O})$.*

As the characteristic vector of a stable matching is a half-integral fractional stable matching, the presence of an odd cycle in the support of a half-integral fractional stable matching is equivalent to the non-existence of a stable matching.

Theorem 6.1 follows directly from a well-known game theoretical lemma of Scarf. An advantage of this reduction is that Scarf's lemma is very flexible and it allows us to deduce a generalization of Theorem 6.1 to stable $b$-matchings in nonbipartite graphs. We call function $w : E(G) \to \mathbb{N}$ a *fractional $b$-matching* if $\sum_{v\in e} w(e) \leq b(v)$ for every vertex $v$ of $G$. A fractional $b$-matching $w$ is called *stable* if for every edge $e$ either $w(e) = 1$ or $e$ contains a vertex $v$ such that $\sum_{v\in f, f\leq_v e} w(f) = b(v)$.

**Theorem 6.3.** *If $(G, \mathcal{O})$ is a graphic preference system and $b : V(G) \to \mathbb{N}$ then there exists a half-integral fractional stable $b$-matching. In other words, there exist disjoint subsets $S$ and $S^{half}$ of edges such that*

- *the components of $S^{half}$ are cycles,*
- *for any vertex $v$ of $G$,*

$$|D(v) \cap S| + \frac{1}{2}|D(v) \cap S^{half}| \leq b(v) \tag{22}$$

- *if $S^{half}$ covers some vertex $v$ then (22) holds with equality and the $<_v$-maximal edge of $D(v) \cap (S \cup S^{half})$ belongs to $S^{half}$, and at last*
- *each edge $e$ of $E \setminus S$ has a vertex $v$ such that (22) holds with equality and $s <_v e$ for any $s \in D(v) \cap (S \cup S^{half})$ .*

Obviously, if all components of $S^{half}$ are even in Theorem 6.3 then $S$ together with each second edge of $S^{half}$ is a stable $b$-matching. Just like in case of nonbipartite stable matchings, a half-integral fractional stable $b$-matching characterizes the existence of a stable $b$-matching.

**Theorem 6.4.** *Let $(G, \mathcal{O})$ be a graphic preference system, $b : V(G) \to \mathbb{N}$ and $C$ be an odd cycle of $G$. If $w$ is a half-integral stable $b$-matching for $(G, \mathcal{O})$ and $w(e) = \frac{1}{2}$ for each edge $e$ of $C$ then the edges of $C$ receive weight $\frac{1}{2}$ in any half-integral fractional stable $b$-matching.*

That is, if $S^{half}$ contains an odd cycle then this very odd cycle is contained in the half-support of any half-integral stable $b$-matching, hence no (integral) stable $b$-matching can exist. We prove Theorem 6.4 with the help of two constructions that reduce the stable $b$-matching problem to the stable matching problem. Let graphic prefenerce system $(G, \mathcal{O})$ and quota function $b : V(G) \to \mathbb{N}$ be given. Applying a *b-splitting* to preference system $(G, \mathcal{O})$ results in another preference system $(G^b, \mathcal{O}^b)$ such that

$$\begin{aligned} V(G^b) &:= \{v(i) : v \in V(G) \text{ and } i = 1, 2, \dots, b(v)\} \\ E(G^b) &:= \{u(i)v(j) : uv \in E(G) \text{ and } u(i), v(j) \in V(G^b)\} \\ \mathcal{O}^b &:= \{<_{v(i)} : v(i) \in V(G^b)\} \\ u(i)v(j) <_{u(i)} u(i)w(k) &\iff \begin{cases} uv <_u uw \text{ or} \\ v = w \text{ and } j < k \end{cases} \end{aligned}$$

That is, we substitute each vertex $v$ of $G$ by $b(v)$ different copies, and two vertices of $G^b$ are joined by an edge if the corresponding vertices of $G$ are different and adjacent. Preferences are inherited from $(G, \mathcal{O})$ we only have to take extra care of that if the two edges come from the same edge.

**Observation 6.5.** Let $(G, \mathcal{O})$ be a graphic preference system and $b : V(G) \to \mathbb{N}$ in such a way that $b(u) = 1$ or $b(v) = 1$ for any edge $uv$ of $G$. Then there is a one-to-one correspondance of half-integral fractional stable $b$-matchings of $(G, \mathcal{O})$ to half-integral fractional stable matchings of $(G^b, \mathcal{O}^b)$ in such a way that an odd cycle of weight half of the fractional stable $b$-matching corresponds to an odd cycle of weight half in the fractional stable matching and vice versa.

That is, there exists a stable $b$-matching of $(G, \mathcal{O})$ if and only if there is a stable matching of $(G^b, \mathcal{O}^b)$. The problem is that the condition on $b$ in Observation 6.5 does not hold in general. The next construction solves this difficulty. The *subdivided preference system of* $(G, \mathcal{O})$ is preference system $(G^s, \mathcal{O}^s)$ such that

$$\begin{aligned} V(G^s) &:= V(G) \cup \{v(e) : v \in V(G) \text{ and } e \in D(v)\} \\ E(G^s) &:= \{vv(e) : v \in V(G) \text{ and } e \in D(v)\} \cup \{e_{uv} : uv = e \in E(G)\}, \\ &\quad\; \text{where } e_{uv} \text{ joins } u(e) \text{ to } v(e) \\ \mathcal{O}^s &:= \{\prec_v : v \in V(G)\} \cup \{<_{v(e)} : v(e) \in V(G^s)\}, \text{ such that} \end{aligned}$$

$$uu(e) \prec_u uu(f) \text{ if } e <_u f, \text{ and } e_{uv} <_{v(e)} vv(e) <_{v(e)} e_{vu} \text{ for } uv = e \in E(G)\ .$$

That is, we subdivide each edge of $G$ by two vertices and introduce a new edge between the subdividing vertices. (In other words, we substitute each edge with a path on four vertices so that the middle edge has a parallel copy.) The preference order of the old vertices come from the original preference order and the preference order of a subdividing vertex is such that each of the two parallel edges is the best at one of its ends and the worst at the other end. Define $b^s : V(G^s) \to \mathbb{N}$ by $b^s(v) := b(v)$ if $v \in V(G)$ and $b^s(v(e)) := 1$ for $v \in V, e \in D_G(v)$.

**Observation 6.6.** Let $(G, \mathcal{O})$ be a graphic preference system and $b : V(G) \to \mathbb{N}$. Preference system $(G^s, \mathcal{O}^s)$ and quota function $b^s$ has the property needed in Observation 6.5, that is, $b^s(x) = 1$ or $b^s(y) = 1$ for any edge $xy$ of $G^s$.

Moreover, any half-integral fractional stable $b$-matching of $(G, \mathcal{O})$ induces a half-integral fractional stable $b^s$-matching of $(G^s, \mathcal{O}^s)$ and vice versa. In both constructions, an odd cycle of weithg half induces another odd cycle of weight half.

In particular, there is a stable $b$-matching of $(G, \mathcal{O})$ if and only if there is a stable matching of $(G^s, \mathcal{O}^s)$. Moreover, $G^s$ is 3-chromatic, hence any stable ($b$-)matching problem can be reduced to one on a 3-chromatic graph. If $G$ is bipartite then $G^s$ is also bipartite, and we will need this fact in Section 8.

*Sketch of the proof of Theorem 6.4.* Let $w$ and $C$ be as in Theorem 6.4. By Observations 6.5 and 6.6, there is a fractional stable matching $w'$ of $((G^s)^b, (\mathcal{O}^s)^b)$ that corresponds to $w$, hence $C$ induces an odd cycle $C'$ of $(G^s)^b$ with $w'$-weight $\frac{1}{2}$. By Theorem 6.2, any half-integral fractional stable matching of $((G^s)^b, (\mathcal{O}^s)^b)$ assigns weight $\frac{1}{2}$ to each edge of $C'$. This means that any half-integral fractional stable $b$-matching of $(G, \mathcal{O})$ must induce a half-integral stable matching of $((G^s)^b, (\mathcal{O}^s)^b)$ that assigns weight $\frac{1}{2}$ to $C'$, hence any half-integral stable $b$-matching of $(G, \mathcal{O})$ must assign weight $\frac{1}{2}$ to each edge of $C$. □

There is another interesting consequence of Theorem 6.3 on approximate stable $b$-matchings.

**Theorem 6.7.** *If $(G, \mathcal{O})$ is a graphic preference system and $b : V(G) \to \mathbb{N}$ then there is a subset $U$ of $V(G)$ with $|U| \leq \frac{1}{3}|V(G)|$ such that for any $b' : V(G) \to \mathbb{N}$ of the form*

$$b'(v) := \begin{cases} b(v) & \text{if } v \notin U \\ b(v) \pm 1 & \text{if } v \in U \end{cases}$$

*there is a stable $b'$-matching of $(G, \mathcal{O})$ .*

*Proof.* Let $S$, $S^{half}$ be as in Theorem 6.3 and construct $U$ by choosing one vertex from each odd cycle of $S^{half}$. As each odd cycle is of length at least 3, the size of $U$ is at most the third of $|V(G)|$. Construct subset $T$ of $S^{half}$ by throwing away each second edge of each even cycle of $S^{half}$ and by selecting each second edge of each odd component with the exception of points of $U$, where we select both or none of the edges depending on whether $b'(u) = b(u) + 1$ or $b'(u) = b(u) - 1$. Clearly, $S \cup T$ is a stable $b'$-matching of $(G, \mathcal{O})$ . □

Our next observation is that Corollary 5.7 has a direct generalization to nonbipartite models.

**Theorem 6.8.** *Let $(G, \mathcal{O})$ be a preference system and $b : V(G) \to \mathbb{N}$ be a quota function. Then for each vertex $v$ of $G$, there is a partition of $D(v)$ into $b(v)$ parts $D_1(v), D_2(v), \ldots, D_{b(v)}(v)$ so that $|S \cap D_i(v)| \leq 1$ for any stable $b$-matching $S$ and any integer $i$ with $1 \leq i \leq b(v)$.*

The proof is the reduction to Corollary 5.7 via a third construction. The *duplicated preference system of* $(G, \mathcal{O})$ is $(G^d, \mathcal{O}^d)$, where

$$\begin{aligned} V(G^d) &:= \{\overline{v} : v \in V(G)\} \cup \{\underline{v} : v \in V(G)\} \\ E(G^d) &:= \{\overline{u}\underline{v}, \underline{u}\overline{v} : uv \in E(G)\} \\ \mathcal{O}^d &:= \{<_{\overline{v}}, <_{\underline{v}} : v \in V(G)\}, \text{ where} \end{aligned}$$

$$\underline{u}\overline{v} <_{\underline{u}} \underline{u}\overline{w} \iff uv <_u uw \iff \overline{u}\underline{v} <_{\overline{u}} \overline{u}\underline{w} \quad \text{for any } uv \in E(G) \ .$$

That is, we take two disjoint copies of $V(G)$, and the edges go along the original edges between the two copies. Preference orders are induced naturally by the original preference orders. Note that $(G^d, \mathcal{O}^d)$ is a bipartite preference system. Define $b^d : V(G^d) \to \mathbb{N}$ by $b^d(\overline{v}) := b^d(\underline{v}) := b(v)$ for any vertex $v$ of $G$.

*Proof of Theorem 6.8.* Apply Theorem 5.7 to quota function $b^d$ and to bipartite preference system $(G^d, \mathcal{O}^d)$. For each vertex $v$ of $G$, we get a partition of $D_{G^d}(\overline{v})$ into $b^d(\overline{v}) = b(v)$ parts in such a way that any stable $b^d$-matching of $(G^d, \mathcal{O}^d)$ contains at most one edge from each part. This partition induces a partition on $D_G(v)$ that satisfies the property of Theorem 6.8. This is true because $S^d := \{\overline{u}\underline{v}, \underline{u}\overline{v} : uv \in S\}$ is a stable $b$-matching of $(G^d, \mathcal{O}^d)$ for any stable $b$-matching $S$ of $(G, \mathcal{O})$. □

The proof of Theorem 6.3 is an application of the following lemma of Scarf to vector $b$, the extended incidence matrix and the extended domination matrix of the preference system. Notation $[n]$ stands for the set of the first $n$ positive integers.

**Theorem 6.9 (Scarf 1967 [37]).** *Let $n < m$ be positive integers, $b$ be a vector in $\mathbb{R}^n_+$ and $B = (b_{i,j})$, $C = (c_{i,j})$ be matrices of dimensions $n \times m$, satisfying the following three properties: the first $n$ columns of $B$ form an $n \times n$ identity matrix (i.e. $b_{i,j} = \delta_{i,j}$ for $i, j \in [n]$), the set $\{x \in \mathbb{R}^n_+ : Bx = b\}$ is bounded, and $c_{i,i} \le c_{i,k} \le c_{i,j}$ for any $i \in [n]$, $i \ne j \in [n]$ and $k \in [m] \setminus [n]$.*

*Then there is a nonnegative vector $x$ of $\mathbb{R}^m_+$ such that $Bx = b$ and the columns of $C$ that correspond to* supp $(x)$ *form a dominating set, that is, for any column $i \in [m]$ there is a row $k \in [n]$ of $C$ such that $c_{k,i} \ge c_{k,j}$ for any $j \in$* supp $(x)$.

Theorem 6.9 can be interpreted such that in any weighted hypergraphic preference system there always exists a fractional stable $b$-matching. It turned out that Theorem 6.9 is a close relative of the topological fixed point theorem of Kakutani (see also [7, 6]). A *simplicial complex* is a non-empty family $\mathcal{C}$ of subsets of a finite ground set such that $A \subset B \in \mathcal{C}$ implies $A \in \mathcal{C}$. Members of $\mathcal{C}$ are called *simplices* or *faces*. Let us call simplicial complex $\mathcal{C}$ *manifold-like* if, denoting its rank by $n$ (that is, the maximum cardinality of a simplex in it is $n+1$), every face of cardinality $n$ of $\mathcal{C}$ is contained in two faces of cardinality $n+1$. The *dual* $\mathcal{C}^*$ of a complex $\mathcal{C}$ is the set of complements of its simplices. Just like in the case of complexes, members of a dual complex are also called *faces*.

**Lemma 6.10 (Aharoni 2001 [6]).** *If $\mathcal{C}$ and $\mathcal{C}'$ are two manifold-like complexes on the same ground set, then the number of maximum cardinality faces of $\mathcal{C}$ that are also minimum cardinality faces of $\mathcal{C}'^*$ is even.*

What examples are there of manifold-like complexes? Of course, a triangulation of a closed manifold is of this sort. (We call this complex a *manifold-complex*.) Another well known example of a dual manifold-like complex is the *cone complex*: let $X$ be a set of vectors in $\mathbb{R}^n$, and $b$ a vector not lying in the positive cone spanned by any $n-1$ elements of $X$. A third example of a manifold-like complex is the *domination complex*. Let $C$ be a matrix as in Theorem 6.9 with the additional property that in each row of $C$ all entries are different. Then the family of dominating column sets together with the extra member $[n]$ is a manifold-like complex. (For the details, see [7].)

If we plug the cone complex and the domination complex in Lemma 6.10 and apply a general position argument then we can deduce Theorem 6.9. The application of Lemma 6.10 to the cone complex and the manifold complex yields the following discrete version of Kakutani's fixed point theorem.

**Theorem 6.11 (see [6]).** *Let us label the vertices of an $n$-dimensional simplex $S$ by the unit vectors of $\mathbb{R}^{n+1}$, and label the other vertices of a triangulation $T$ of $S$ by vectors of $\mathbb{R}^{n+1}$ in such a way that the label of any vertex $v$ of $T$ is in the positive hull of the labels of those vertices of $S$ that lie on the minimal face of $S$ that contain $v$. For any vector $b \in \mathbb{R}^{n+1}_+$, there is a elementary simplex in the triangulation of $S$ whose vertex labels contain $b$ in their cone.*

Theorem 6.11 and a continuity argument implies Kakutani's fixed point theorem.

**Theorem 6.12 (Kakutani 1941 [22]).** *Let $S \subseteq \mathbb{R}^n$ be an $n$-dimensional simplex and $K(S)$ be the family of closed convex subsets of $S$. Let $\Phi : S \to K(S)$ be upper semicontinuous, that is, whenever $y_i \to y$, $x_i \to x$ and $y_i \in \Phi(x_i)$ then $y \in \Phi(x)$. Then there is a point $x$ of $S$ so that $x \in \Phi(x)$ .*

Brouwer's fixed point theorem is the special case of Theorem 6.12 where $\Phi(x)$ is one point for all $x$. The discrete version of Brouwer's fixed point theorem is Sperner's lemma and Theorem 6.11, the discrete version of Kakutani's theorem is a genuine generalization of Sperner's lemma. Note that a result of Shapley in [38] that generalize Sperner's lemma is formally a special case of Theorem 6.11. Still, Shapley's method in [38] proves Theorem 6.11.

It is interesting to see the close relationship of the bipartite stable matching theorem to the lattice theoretical fixed point theorem of Tarski and of the nonbipartite version to the topological fixed point theorem of Kakutani. One might ask himself whether there is some fixed-point theorem that implies a generalization of Tan's results on the nonbipartite preference model.

# 7 Stable matchings and graph paths

In this section, we discuss two consequences of the stable marriage theorem on paths of graphs. Recall Corollary 3.4 that roughly said that if we have two partial orders on a ground set then there is a common antichain that dominates every element of the ground set. If we consider the Hasse diagrams of the posets (that is, the directed graph in which arcs go along covering elements) then we can formulate Corollary 3.4 as follows. For any two acyclic directed graphs on a common ground set there is a subset $S$ of the ground set such that there is no directed path between two vertices of $S$ in any of the graphs but from any vertex outside $S$ there is a directed path into $S$ in one or both of the graphs.

Actually, by dropping the condition of acyclicity in the above formulation, we get the following well-known result of Sands *et al.*

**Theorem 7.1 (Sands *et al.* 1982 [36]).** *If $A_1$ and $A_2$ are arc-sets on vertex-set $V$, such that there is no infinite path in any of the $A_i$'s that starts at some vertex then there is a subset $S$ of $V$ such that*

*for each element $v \in V$ there is a simple path in $A_1$ or in $A_2$ from $v$ to $S$, and* (23)

*there is neither a simple $A_1$-, nor a simple $A_2$-path between different vertices of $S$.* (24)

The following result of Pym is an unexpected application of the stable marriage theorem. One proof of Pym actually implies the stable marriage theorem, but it seems that the author was unaware of this application.

**Theorem 7.2 (Pym 1969 [26, 27]).** *Let $D = (V, A)$ be a directed graph and $X, Y$ be subsets of $V$. Let moreover $\mathcal{P}$ and $\mathcal{Q}$ be families of vertex-disjoint simple $XY$-paths. Then there exists a family $\mathcal{R}$ of vertex-disjoint simple $XY$-paths, such that*

*any path of $\mathcal{R}$ consists of a (possibly empty) initial segment of a path of $\mathcal{P}$ and of a (possibly empty) end segment of a path of $\mathcal{Q}$, moreover* (25)

$$In(\mathcal{P}) \subseteq In(\mathcal{R}) \subseteq In(\mathcal{P} \cup \mathcal{Q}) \quad (26)$$

$$End(\mathcal{Q}) \subseteq End(\mathcal{R}) \subseteq End(\mathcal{P} \cup \mathcal{Q}). \quad (27)$$

*Sketch of the proof.* Define bipartite preference system in such a way that men and women correspond to paths of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Each common vertex $v$ of a path $P$ of $\mathcal{P}$ and $Q$ of $\mathcal{Q}$ yields an edge $e_v$ between $P$ and $Q$. Let path $P$ of $\mathcal{P}$ prefer edge $e_v$ to $e_u$ if $v$ is closer to the initial vertex of $P$ on $P$ then $u$. Similarly, path $W$ of $\mathcal{Q}$ prefers edge $e_v$ to $e_u$ if $v$ is closer to the terminal vertex of $Q$ on $Q$. Let $S$ be a stable matching in the bipartite preference model, that corresponds to vertex set $R$ in $D$. Merge initial segments of $\mathcal{P}$-paths to end segments of $\mathcal{Q}$-paths along the vertices of $R$. This results in a collection $\mathcal{R}$ of paths with the desired property. □

Note that Theorem 7.2 is essentially the proof of that gammoids are matroids.

# 8 Stable matching polyhedra

A recent development is that a linear programming approach has been worked out to the theory of bipartite and nonbipartite stable matchings by Abeledo, Blum, Roth, Rothblum, Sethuraman, Teo, Vande Vate and others (see [3, 4, 1, 2, 32, 42]). In this section, we survey linear descriptions of bipartite stable matching polyhedra. The earliest such result is that of Vande Vate.

We denote by $P^b(G, \mathcal{O})$ the convex hull of characteristic vectors in $\mathbb{R}^E$ of stable $b$-matchings of bipartite preference system $(G, \mathcal{O})$. (So $P^{\mathbf{1}}(G, \mathcal{O})$ is the polytope of ordinary stable matchings.) As usual in linear programming, we define $x(S) := \sum\{x(e) : e \in S\}$ for a vector $x \in \mathbb{R}^E$ and subset $S$ of $E$.

**Theorem 8.1 (Vande Vate 1989 [43]).** *Let $(G, \mathcal{O})$ be a bipartite preference system with colour classes $M$ and $W$, $|M| = |W|$ and $E = M \times W$. Then*

$$P^{\mathbf{1}}(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(D(v)) = 1\ \forall\ v \in M \cup W,\ x(\psi(e)) \leq 1\ \forall\ e \in E\}$$

*where $\psi(mw) := \{f \in E : f \geq_m mv \text{ or } f \geq_w mw\}$ .*

Rothblum gave a shorter proof of a modified description for a more general problem in [35], and his proof was further simplified by Roth *et al.* in [32].

**Theorem 8.2 (Rothblum 1992 [35]).** *Let $(G, \mathcal{O})$ be a bipartite preference system with colour classes $M$ and $W$. Then*

$$P^{\mathbf{1}}(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(D(v)) \leq 1\ \forall\ v \in M \cup W,\ x(\phi(e)) \geq 1\ \forall\ e \in E\}$$

*where $\phi(mw) := \{f \in E : f \leq_m mw \text{ or } f \leq_w mw\}$ .*

Based on these results, standard tools of linear programming allow us to find a maximum weight stable matching in polynomial time.

But these results handle only the stable matching problem and do not say much about stable $b$-matchings. The following theorem of Baïou and Balinski [10] is an exception as it gives a linear description of the stable admissions polytope and generalizes Theorem 8.2.

**Theorem 8.3 (Baïou and Balinski 1999 [10]).** *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : M \cup W \to \mathbb{N}$ be a quota function so that $b(w) = 1$ for all nodes $w$ of $W$. Then*

$$\begin{aligned} P^b(G, \mathcal{O}) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0}, \\ & x(D(w)) \leq 1\ \forall\ w \in W,\ x(D(m)) \leq b(m)\ \forall\ m \in M, \\ & x(C(m, w_1, w_2, \dots, w_{b(m)})) \geq b(m) \\ & \text{for all combs } C(m, w_1, w_2, \dots, w_{b(m)})\} , \end{aligned}$$

*where a comb is defined for $m \in M$ and $mw_1 >_m mw_2 >_m \dots >_m mw_{b(m)}$ as*

$$\begin{aligned} C(m, w_1, w_2, \dots, w_{b(m)}) = & \{mw \in E : mw \leq_m mw_1\} \cup \\ & \{mw_i' \in E : m'w_i \leq_{w_i} mw_i \text{ for some } i = 1, 2, \dots, b(m)\} \end{aligned}$$

Because of the comb constraints, the above characterization can consist of $\Omega(n^B)$ linear inequalities, where $n$ is the number of "colleges" and $B$ is the maximum of all quotas. But in spite of the exponential number of constraints, it is still possible to find an optimum weight stable admission by the ellipsoid method, using the separation algorithm of Baïou and Balinski. Note however, that the main significance of Theorem 8.3 lies in the the description of the polytope itself and not in the fact that we can optimize over stable admissions. This is because already Theorem 8.2 is enough to find a maximum weight stable admission scheme of $(G, \mathcal{O})$ by finding a maximum weight stable matching of $(G^s, \mathcal{O}^s)$ (see Section 6 for the definition.) In the $b$-splitted matching model, Rothblum's description characterizes a stable matching polytope $P$ so that the stable admissions polytope is a projection of $P$. Moreover, the related LP needs only $O(n + mB)$ constraints, where $n$ is the number of agents, $m$ is the number of possible admissions and $B$ is the maximum of the quotas. Note also that the $b$-splitting and subdivision construction (explained in Section 6) is also sufficient to optimize over stable $b$-matchings with Theorem 8.2. Namely, a maximum weight stable matching in $((G^s)^b, (\mathcal{O}^s)^b)$ induces a maximum weight stable $b$-matching of $(G, \mathcal{O})$. (Note that $(G^s)^b$ is bipartite.) So the significance of a linear description of the bipartite stable $b$-matching polytope is not that we can optimize over stable $b$-matchings (as we can do it already with Theorem 8.2), but it is rather theoretical (namely, that we do have an explicit description of the object).

In [17, 14], with the help of the theory of blocking polyhedra and lattice polyhedra, Fleiner gave a linear description of certain polyhedra that are related to $\mathcal{C}_M\mathcal{C}_W$-stable sets. Fix functions $\mathcal{C}_M\mathcal{C}_W : 2^E \to 2^E$ and let us denote by

$$\begin{aligned}
\mathcal{S} &:= \{S \subseteq W : S \text{ is an } \mathcal{C}_M\mathcal{C}_W\text{-stable set}\} \\
\mathcal{B} &:= \{B \subseteq E : B \cap S \neq \emptyset \ \text{ for any } S \in \mathcal{S}\} \\
\mathcal{A} &:= \{A \subseteq E : |A \cap S| \leq 1 \text{ for any member } S \text{ of } \mathcal{S}\} \\
K &:= E \setminus \bigcup \mathcal{S}
\end{aligned}$$

family of $\mathcal{C}_M\mathcal{C}_W$-stable sets, the *blocker*, the *antiblocker* of $\mathcal{S}$ the set of nonstable elements, respectively. Define further the $\mathcal{C}_M\mathcal{C}_W$-stable set polytope, its dominant and submissive polyhedra by

$$P(\mathcal{C}_M, \mathcal{C}_W) := \text{conv}\{\chi^S : S \in \mathcal{S}\} \tag{28}$$

$$P(\mathcal{C}_M, \mathcal{C}_W)^{\uparrow} := P(\mathcal{C}_M, \mathcal{C}_W) + \mathbb{R}^E_+ = \{x + y : x \in P(\mathcal{C}_M, \mathcal{C}_W), y \geq 0\} \tag{29}$$

$$P(\mathcal{C}_M, \mathcal{C}_W)^{\downarrow} := \{x - y : x \in P(\mathcal{C}_M, \mathcal{C}_W), y \geq 0\} \cap \mathbb{R}^E_+ . \tag{30}$$

**Theorem 8.4 (Fleiner 2000 [17, 14]).** *If $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are increasing comonotone functions then*

$$P(\mathcal{C}_M, \mathcal{C}_W)^{\uparrow} = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(B) \geq 1 \ \ \textit{for } B \in \mathcal{B}\} , \tag{31}$$

$$P(\mathcal{C}_M, \mathcal{C}_W)^{\downarrow} = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(K) = 0 \textit{ and } x(A) \leq 1 \textit{ for any } A \in \mathcal{A}\} , \tag{32}$$

$$P(\mathcal{C}_M, \mathcal{C}_W) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(B) \geq 1 \ \ \textit{for } B \in \mathcal{B} \textit{ and } x(A) \leq 1 \ \ \textit{for } A \in \mathcal{A}\} . \tag{33}$$

If Theorem 8.4 is applied to the bipartite stable $b$-matching problem then it gives the following linear description of the stable $b$-matching polytope.

**Theorem 8.5 (Fleiner 2000 [17, 15, 14]).** *Let $(G,\mathcal{O})$ be a bipartite preference system and $b : V(G) \to \mathbb{N}$ be a quota function. Then*

$$P^b(G,\mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(A) \leq 1\ \forall A \in \mathcal{A},\ x(B) \geq 1\ \forall\ B \in \mathcal{B}\}$$

*where*

$$\mathcal{A} := \{A \subseteq E : |A \cap S| \leq 1 \text{ for any stable } b\text{-matching } S\} \text{ and}$$
$$\mathcal{B} := \{B \subseteq E : B \cap S \neq \emptyset \text{ for any stable } b\text{-matching } S\}\ .$$

Note that the constraints in Theorem 8.2 are special cases of the ones in Theorem 8.5. However, there are two important differences between Theorem 8.5 and the above earlier results. A shortage of Theorem 8.5 is that it uses implicit constraints, hence if it is specialized to the stable marriage problem, it might require more constraints than Rothblum's explicit description. (This is why Theorem 8.5 is rather an extension than a generalization of Theorem 8.2.) A positive feature of Theorem 8.5 is that unlike Theorem 8.3, both the matrix and the right hand side vector in the description contains only 0 and 1 entries.

The following result is a strenghtening of Theorem 8.5 for the stable $b$-matching polytope and it is a genuine generalization of Theorem 8.2.

**Theorem 8.6 (Fleiner 2002 [16]).** *Let $(G,\mathcal{O})$ be a bipartite preference system and $b : M \cup W \to \mathbb{N}$ be a quota function. Then the partitions in Corollary 5.7 satisfy*

$$\begin{aligned} P^b(G,\mathcal{O}) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0}, \\ & x(D_i(v)) \leq 1\ \forall\ v \in M \cup W, 1 \leq i \leq b(v), \\ & x(\phi_{i,j}(mw)) \geq 1\ \forall mw \in E, 1 \leq i \leq b(m), 1 \leq j \leq b(w)\}\ , \\ \textit{where}\quad & \phi_{i,j}(mw) := \{mw\} \cup \{mw' : mw' <_m mw, w' \in D_i(m)\} \cup \\ & \{m'w : m'w <_w mw, m' \in D_j(w)\}\ . \end{aligned}$$

Note that star-partitions in Corollary 5.7 can be found with $m$ deferred acceptance algorithms, where $m$ is the number of edges of $G$ (see [16]). Still, the linear description of the stable $b$-matching polytope needs $O(n + mB^2)$ constraints ($n$ is the number of vertices of $G$ and $B$ is the maximum of $B$). The previously mentioned construction of splitting nodes and subdividing edges allows us to construct a linear program in $O(mB)$ dimensions to optimize a stable $b$-matching with $O((n + m)B)$ constraints.

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