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The Generalized Kaneko Theorem

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Abstract

The following problem was first studied by Kaneko: which simple, connected graphs have vertex disjoint paths of length at least 2, that altogether cover all the vertices of the graph. He gave a good characterization to the problem. Hell and Kirkpatrick observed that a possible generalization of the problem of Kaneko is to characterize those graphs which have a spanning forest such that each tree component of this forest has highest degree k for a fixed integer $k \geq 1$. In this paper we prove a generalization of the theorem of Kaneko in this sense, show an algorithm solving the generalized problem and prove a Berge-Tutte-type theorem on the minimum number of nodes which are missed by a tree packing.

We can solve the following problem too: given two bounds l, $u: V(G) \to \mathbb{N}$, does G have a spanning subgraph having degrees at most u such that each component of this subgraph covers a vertex w the degree of which in this component is at least $l(w)$. Many known and new results follow from this formulation.

Keywords: factor-critical graph, factor, packing

1 Introduction

The factorization problem is the following: given a graph G and a family $\mathcal{H}$ of graphs, does G have an *$\mathcal{H}$-factor*, i.e. a spanning subgraph every connected component of which is isomorphic to a member of $\mathcal{H}$. All results stating the polynomiality of a factorization problem assumed that $K_2 \in \mathcal{H}$ (we may assume $K_1 \notin \mathcal{H}$). The breakthrough was due to Kaneko [4], who gave a sufficient and necessary condition for the existence of an $\mathcal{H}$-factor, $\mathcal{H}$ consisting of the paths of length at least 2, shortly *long paths*. (Previously Wang [11] proved this theorem for bipartite graphs.) Kano, Katona and Király [5] showed a simple proof for this theorem, moreover, they gave a

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simple proof for a formula on the maximum number of vertices which can be covered by a long path packing.

No polynomial algorithm was known to the long path factorization problem.

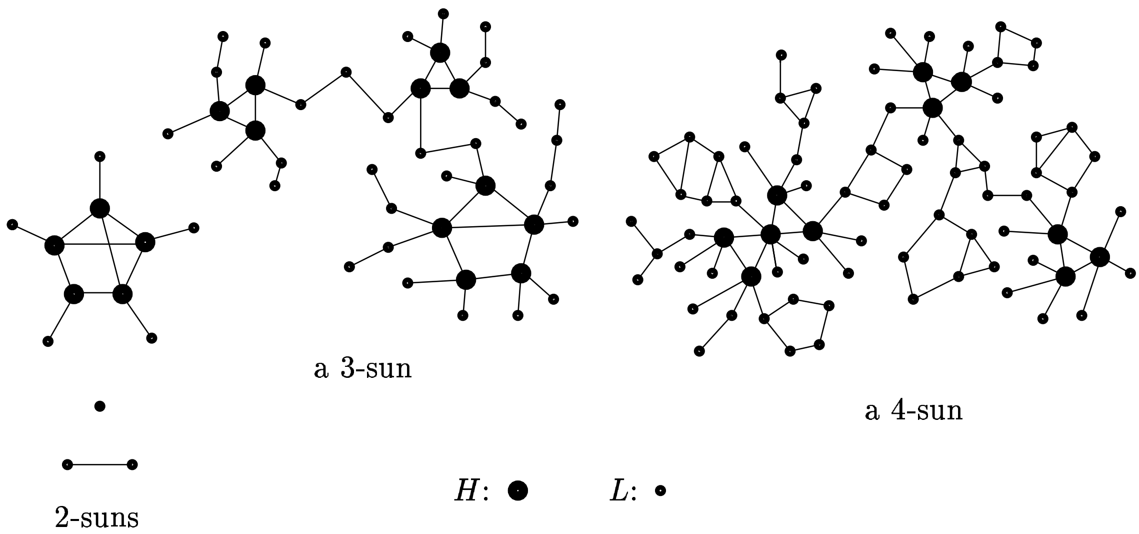
Hell and Kirkpatrick [3] observed, calling a tree a *k-tree* if its highest degree is k, that the k-tree factorization problem in the case $k = 1$ is just the matching problem, while in the case $k = 2$ it is the long path factorization problem.

In this paper we generalize the theorems of Kaneko and of Kano, Katona and Király in this sense and we show an algorithm to this generalized problem. In Section 6 we mention that all the results remain true in a more general setting, namely when two bounds l, $u\colon V(G) \to \mathbb{N}$ are given and we ask for a spanning forest having degrees at most u such that each tree component of this forest covers a vertex w the degree of which in this component is at least $l(w)$.

All the graphs occurring in the paper are unoriented, simple graphs. The trees in Sections 2 – 5 are defined by their edge sets and for an edge set E let $V(E)$ denote the set of nodes covered by E. The maximal degree of a graph G is denoted by $\Delta(G)$. For $X \subseteq V(G)$ the subgraph of G induced by X is denoted by $G[X]$. Recall, that a connected graph is *factor-critical* if deleting any vertex the remaining graph has a perfect matching. K_1 is called *trivial*, all other factor-critical graphs are called *non-trivial.*

2 Main result

Definition 2.1. A tree is a *k-tree* if its highest degree is k. The edge set $E' \subseteq E(G)$ of a graph G is a *k-tree packing* if all the connected components of E' is a k-tree. A k-tree packing covering all the vertices of G is a *k-tree factor*.



Definition 2.2. For an integer $k \geq 1$ the simple, connected graph S is a *k-sun* if it satisfies the followings:

- $\deg_S(v) \neq k$ for $v \in V(S)$,

- denoting by H the set of nodes of degree at least $k+1$, the induced subgraph $S[H]$ is a (possibly empty) union of non-trivial factor-critical graphs,

- for $v \in H$ it holds that $\deg_S(v) = \deg_{S[H]}(v) + k - 1$ and all these $k-1$ edges are cutting edges in S.

Let $L = \{v \in V(S) : \deg_S(v) \leq k-1\} = V(S) - H$.

It is easy to see that the 1-suns are exactly the factor-critical graphs. The 2-suns were defined by Kaneko: the 2-suns are K_1, K_2 and for all non-trivial factor-critical graphs F and a disjoint copy $V(F)'$ of $V(F)$ the graph with vertex set $V(F) \cup V(F)'$ and edge set $E(F) \cup \{vv' : v \in V(F)\}$. Hence the name *sun* and the forthcoming term *hair*.

Now we can state our main result, the generalization of the theorem of Kaneko:

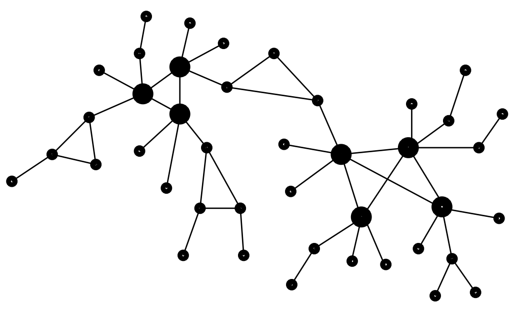
Theorem 2.3. *For the integer $k \geq 1$ the graph G has a k-tree factor if and only if for all $Y \subseteq V(G)$ the number of the k-sun components of $G - Y$ is at most $k|Y|$.*

The case $k = 1$ is the Tutte theorem, since the 1-suns are the factor-critical graphs and the only 1-tree is K_2. The case $k = 2$ is due to Kaneko [4]:

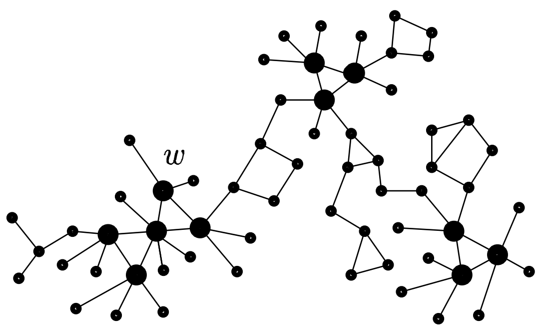
Theorem 2.4 (Kaneko). *G has a long path factor if and only if for all $Y \subseteq V(G)$ the number of the 2-sun components of $G - Y$ is at most $2|Y|$.*

3 Preparations

This section consists of some definitions and lemmas which we need for proving the validity of our algorithm discussed in Section 4.



an almost 4-sun of type 1



an almost 4-sun of type 2

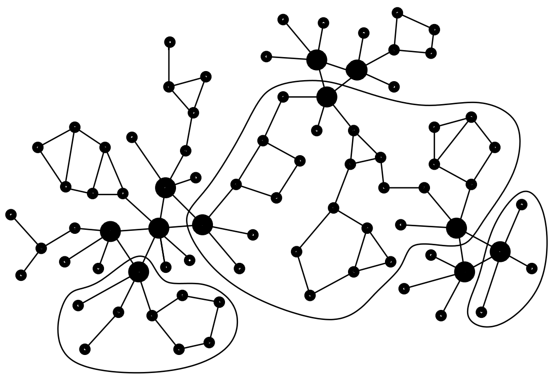
Definition 3.1. For the integer $k \geq 1$ the connected graph G is an *almost k-sun of type 1* if, denoting by H the nodes of degree at least k, $G[H]$ is a union of non-trivial factor-critical graphs and exactly one connected perfectly matchable graph; $\deg_G(v) = \deg_{G[H]}(v) + k - 1$ for $v \in H$ and all these $k-1$ edges are cutting edges in G.

Definition 3.2. For the integer $k \geq 1$ the connected graph G is an *almost k-sun of type 2* if, denoting by H the nodes of degree at least k, $G[H]$ is a union of non-trivial factor-critical graphs; there is a distinguished vertex $w \in H$ such that $\deg_G(w) = \deg_{G[H]}(w) + k - 2$ and all these $k-2$ edges are cutting edges in G and $\deg_G(v) = \deg_{G[H]}(v) + k - 1$ for $v \in H - w$ and all these $k-1$ edges are cutting edges in G.

Definition 3.3. For a graph G and $v \in V(G)$ the *k-hair* or shortly the *hair* of v is the following set $O_v \subseteq V(G)$. Starting from $\{v\}$ we construct the dynamic set O: if $x \in O$ then

- if $\deg_G(x) \geq k$ then let $y \in O$ for all the neighbors y of x such that $\deg_G(y) \leq k-1$ and
- if $\deg_G(x) \leq k-1$ then let $y \in O$ for all the neighbors y of x,

in both cases if y is already an element of O then it remains there. O_v is the final set O.



some hairs of a 4-sun

Remark 3.4. If $u \in O_v$ then $O_u = O_v$ holds, hence the hairs are partitioning the vertex set. Owing to the construction O_v induce a connected graph and contains v. If G is a k-sun or an almost k-sun then $\Delta(G[O]) \leq k-1$ for each k-hair O, furthermore, for all two distinct connected components $P_1,\ P_2$ of $G[H]$ and for $u, v \in V(P_1),\ \ u \neq v$ it holds that $O_u \cap O_v = \emptyset$ and $|(O_u \cup O_v) \cap V(P_2)| \leq 1$. If S is a k-sun then $\deg_{S[O_v]}(v) = k-1$ for $v \in H$ and $\deg_{S[O_v]}(v) = \deg_S(v)$ for $v \in L$.

For brevity we introduce the following term.

Definition 3.5. The graph G is called *k-good* (or simply good if the value of k is unambiguous) if it has a k-tree factor.

Lemma 3.6. *The almost k-suns of type 1 and 2 are k-good.*

Proof. For the almost sun G let O_v be the k-hair of $v \in V(G)$ and $T_v \subseteq E(G)$ be a spanning tree of $G[O_v]$.

Let G be an almost k-sun of type 1. We proceed by induction on the number of nodes. Let P denote the perfectly matchable component of $G[H]$. For all the edges

uv of a perfect matching of P we consider $T_u \cup \{uv\} \cup T_v$, which is a k-tree. Besides, $G - \bigcup\{O_v : v \in V(P)\}$ is either the empty graph or all of its connected components are almost k-suns of type 1 so this graph is good by our inductive hypothesis.

Let G be an almost k-sun of type 2. Denote by F the non-trivial factor-critical component of $G[H]$ containing the distinguished vertex w. Let w' and w'' be two neighbors of w in F such that $F - \{w', w, w''\}$ has a perfect matching N. For all the edges $uv \in N$ we consider $T_u \cup \{uv\} \cup T_v$, which is a k-tree. $T := T_{w'} \cup \{w'w\} \cup T_w \cup \{ww''\} \cup T_{w''}$ is a k-tree as well, since $\deg_{T_w}(w) = k - 2$ so $\deg_T(w) = k$ and all the other nodes have degree at most k. Besides, $G - \bigcup\{O_v : v \in V(F)\}$ is either the empty graph or all of its connected components are almost k-suns of type 1 so this graph is good due to the first part of the lemma. □

From now on the term hair will only be applied to suns.

Definition 3.7. From an algorithmic point of view the *decomposition* of a k-sun S consists of the odd ear decompositions of the factor-critical components of $S[H]$ and of spanning trees of the graphs induced by the hairs.

Remark 3.8. For a non-trivial factor-critical graph F and $w \in V(F)$ the followings can be obtained in linear time if the odd ear decomposition of F is given:

- a perfect matching of $F - w$ and
- a perfect matching of $F - \{w', w, w''\}$ for two suitable neighbors w' and w'' of w.

Claim 3.9. *If O is a hair of the k-sun S then $S - O$ is k-good. A k-tree factor of $S - O$ can be obtained in $O(|V(S)|)$ time if the decomposition of S is given.*

Proof. $S - O$ is either the empty graph or all of its connected components are almost k-suns of type 1 which are good by Lemma 3.6. The statement concerning the running time follows from Remark 3.8 and the construction of Lemma 3.6. □

Lemma 3.10. *If S is a k-sun and $x, y \in H$ are vertices in distinct components of $S[H]$ then there exist trees $T^x, T^y \subseteq E(S)$ such that $V(T^x) \cap V(T^y) = \emptyset$, $\Delta(T^y) \leq k-1$, $\Delta(T^x) = k-1$, $\deg_{T^x}(x) = k-1$ and $S - V(T^x) - V(T^y)$ is k-good. T^x, T^y and a k-tree factor of $S - V(T^x) - V(T^y)$ can be obtained in $O(|V(S)|)$ time if the decomposition of S is given.*

Proof. Denote by O_v the k-hair of $v \in V(S)$ and by F the component of $S[H]$ containing y. It is clear from the tree-like structure of the suns that all the $y-x$ paths leave F in a particular edge wz. $S - wz$ has two connected components, let D_y resp. D_x denote the one containing y and w resp. x and z. D_x is a k-sun, let O' be the hair of x in it and T^x be a spanning tree of $D_x[O']$. Let T_v denote a spanning tree of $S[O_v \cap V(D_y)]$ for $v \in V(F)$. Let $T^y = T_y$, now $V(T^x) \cap V(T^y) = \emptyset$, $\Delta(T^y)$, $\Delta(T^x) \leq k - 1$ and $\deg_{T^x}(x) = k - 1$ so it remains to show that $S - V(T^x) - V(T^y)$ is good. First, $D_x - O'$ is good by Claim 3.9. For all the edges uv of a perfect matching of $F - y$ consider the tree $T_u \cup \{uv\} \cup T_v$. This is a k-tree because in F there is only one vertex, namely w, for which $O_w \not\subseteq V(D_y)$. So for at least one of u and v, say u, it holds that $\deg_{T_u}(u) = k-1$. Moreover, $D_y - \bigcup\{V(T_v) : v \in V(F)\}$ is good by Lemma 3.6 because all of its connected components are almost k-suns of type 1.

The statement concerning the running time follows from Remark 3.8 and the construction of Lemma 3.6. □

Claim 3.11. *A k-sun cannot be k-tree factorized.*

Proof. We proceed by induction on the number of nodes. Suppose that E' is a k-tree factor of the k-sun S. If $w \in H$ and wz is one of the $k-1$ cutting edges of w such that $wz \notin E'$ then the component of $S - wz$ containing z is k-tree factorized as well, but this component is a k-sun with fewer nodes than S, a contradiction. So for all the vertices of H the $k-1$ cutting edges incident with that vertex are all in E'.

For a hair O let $w_O \in O \cap H$ be a vertex such that $\deg_{E'}(w_O) = k$. w_O exists, otherwise no edge in E' would leave O, but $S[O]$ does not have a k-tree factor because $\Delta(S[O]) \leq k-1$. It can be easily seen from the tree-like structure of S that there exists a factor-critical component F of $S[H]$ such that for all $v \in V(F)$ if v is a node of the hair O then $v = w_O$. This means that $E' \cap E(F)$ is a perfect matching, which is impossible. □

We prove the easy direction of Theorem 2.3: suppose that for $Y \subseteq V(G)$ the number of the k-sun components of $G - Y$ is at least $k|Y| + 1$ and E' is a k-tree factor of G. If a tree in E' intersects Y in t nodes then this tree can intersect at most kt of the k-sun components of $G - Y$. So at least one of the k-sun components is factorized in itself, which is impossible by Claim 3.11.

The other part will be proved by the validity of our algorithm.

In the $\mathcal{H}$-factorization problem, which we mentioned in the introduction, we call a graph *$\mathcal{H}$-critical* if it is not $\mathcal{H}$-factorizable but deleting any node of it the remaining graph is. After many previous paper in this area (e. g. [10, 7, 1, 2, 8, 9]) Király and Szabó [6] show that if $K_2 \in \mathcal{H}$ and a certain condition holds then every $\mathcal{H}$-critical graph is factor-critical, moreover, a graph G has an $\mathcal{H}$-factor if and only if there is no $Y \subseteq V(G)$ such that $G - Y$ has too many critical components.

In the k-tree factorization problem the k-suns play the role of the critical graphs. They are not k-tree factorizable by Claim 3.11 and though in the case $k \geq 2$ it is not true that deleting a node from a k-sun the remaining graph is k-tree factorizable, there are other connected subgraphs which we can delete. These subgraphs are the hairs by Lemma 3.9 and the connected components of $L(S)$ because it is easy to see that now the remaining graph is a union of almost k-suns of type 2 which are good by Lemma 3.6.

Only a few lemmas remained in this section.

Definition 3.12. Let S be a k-sun. The pair $x, y \in V(S)$ is *unlucky* if either

- $x = y \in L$, $\deg_S(x) = k-1$, all the neighbors of x are in L and all the $k-1$ edges of x are cutting or

- $x, y \in H$ and they are in the same factor-critical component of $S[H]$.

Otherwise the pair x, y is *lucky.*

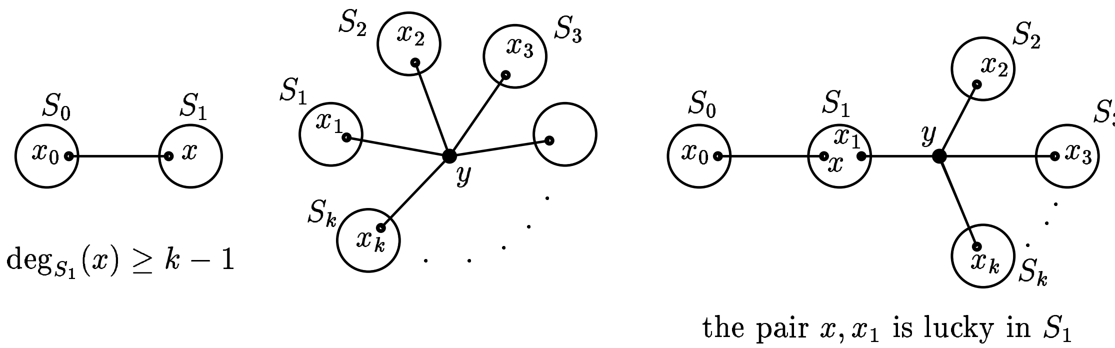
Lemma 3.13. *Let $S_0, S_1, \dots, S_k$ be k-suns of disjoint node sets, $x_i \in V(S_i)$ for $0 \leq i \leq k$ and $x \in V(S_1)$.*

(a) *G is k-good if $V(G) = V(S_0) \cup V(S_1)$, $E(G) = E(S_0) \cup \{x_0x\} \cup E(S_1)$ and $\deg_{S_1}(x) \geq k-1$.*

(b) *G is k-good if $V(G) = \bigcup\{V(S_i) : 1 \leq i \leq k\} \cup \{y\}$ and $E(G) = \bigcup\{E(S_i) : 1 \leq i \leq k\} \cup \{yx_i : 1 \leq i \leq k\}$.*

(c) *G is k-good if $V(G) = \bigcup\{V(S_i) : 0 \leq i \leq k\} \cup \{y\}$, $E(G) = \bigcup\{E(S_i) : 0 \leq i \leq k\} \cup \{yx_i : 1 \leq i \leq k\} \cup \{x_0x\}$ and the pair x, x_1 is lucky in S_1.*

A k-tree factor of G can be obtained in $O(|V(G)|)$ time if the decompositions of the suns are given.



(a) (b) (c)

Proof. Suppose that for $0 \leq i \leq k$ the vertex x_i is in the hair O_i in S_i and x is in the hair O in S_1.

(a) Claim 3.9 yields that $S_0 - O_0$ and $S_1 - O$ are good, furthermore, the simple graph $G[O_0 \cup O]$ has a k-tree spanning tree because $\Delta(G[O_0]) \leq k-1$, $\Delta(G[O]) \leq k-1$ and $\deg_{S_1[O]}(x) = k-1$ holds apart from $x \in L(S_1)$ or $x \in H(S_1)$ by Remark 3.4.

(b) Claim 3.9 yields that $S_i - O_i$ is good for $1 \leq i \leq k$, moreover, the simple graph $G[\bigcup\{O_i : 1 \leq i \leq k\} \cup \{y\}]$ has a k-tree spanning tree because its highest degree is k.

(c) We distinguish the following cases:

Case 1: $x \neq x_1$, $x \in L(S_1)$. $G[\bigcup\{V(S_i) : 1 \leq i \leq k\} \cup \{y\}]$ is good due to (b), let E_1 be a k-tree factor of it and let E_2 be a spanning tree of $G[O_0 + x]$. $E_1 \cup E_2$ is a disjoint union of k-trees because $\deg_{E_1}(x) \leq k-1$ and $\deg_{E_2}(x) = 1$. $S_0 - O_0$ is good by Claim 3.9 so G is good.

Case 2: $x \neq x_1$, $x \in H(S_1)$, $x_1 \in L(S_1)$. $G[V(S_0) \cup V(S_1)]$ is good by (a), let E_1 be one of its k-tree factors. $G[\{x_1, y\} \cup \bigcup\{V(S_i) : 2 \leq i \leq k\}]$ is good by (b) as well, let E_2 be one of its k-tree factors. $E_1 \cup E_2$ is a k-tree factor of G because $\deg_{E_1}(x_1) \leq k-1$ and $\deg_{E_2}(x_1) = 1$.

Case 3: $x, x_1 \in H(S_1)$. Now they are in distinct factor-critical components of $H(S_1)$. Lemma 3.10 yields trees $T^x, T^{x_1} \subseteq E(S_1)$ such that $V(T^x) \cap V(T^{x_1}) = \emptyset$, $\Delta(T^{x_1}) \le k-1$, $\Delta(T^x) = k-1$, $\deg_{T^x}(x) = k-1$ and $S_1 - V(T^x) - V(T^{x_1})$ is good. Let T_i be a spanning tree of $S_i[O_i]$ for $i = 0, 2, \dots, k$. $S_i - O_i$ is good for $i = 0, 2, \dots, k$ by Claim 3.9, moreover, $T_0 \cup \{x_0 x\} \cup T^x$ and $T^{x_1} \cup \bigcup\{T_i : 2 \le i \le k\} \cup \{yx_1, \dots, yx_k\}$ are k-trees.

Case 4: $x = x_1$ and $\deg_{S_1}(x) \le k-2$. For $0 \le i \le k$ the graph $S_i - O_i$ is good due to Claim 3.9, furthermore, the simple graph $G[\bigcup\{O_i : 0 \le i \le k\} \cup \{y\}]$ has a k-tree spanning tree because its highest degree is k.

Case 5: $x = x_1$ and $\deg_{S_1}(x) = k-1$. If x is not adjacent to $H(S_1)$ then x has a non-cutting edge e because the pair x, x_1 is lucky in S_1. $S_1 - e$ is a k-sun as well and $\deg_{S_1-e}(x) = k-2$ so we can apply **Case 4**. Suppose that x is adjacent to a vertex $w \in H(S_1)$. The edge xw is cutting in S_1 so is in G. The component S' of $S_1 - xw$ containing x is a k-sun and $\deg_{S'}(x) = k-2$ so we can apply **Case 4** to the component of $G - xw$ containing x. Furthermore, the component containing w is an almost sun of type 2 which is good by Lemma 3.6.

The statement concerning the running time follows from the construction. □

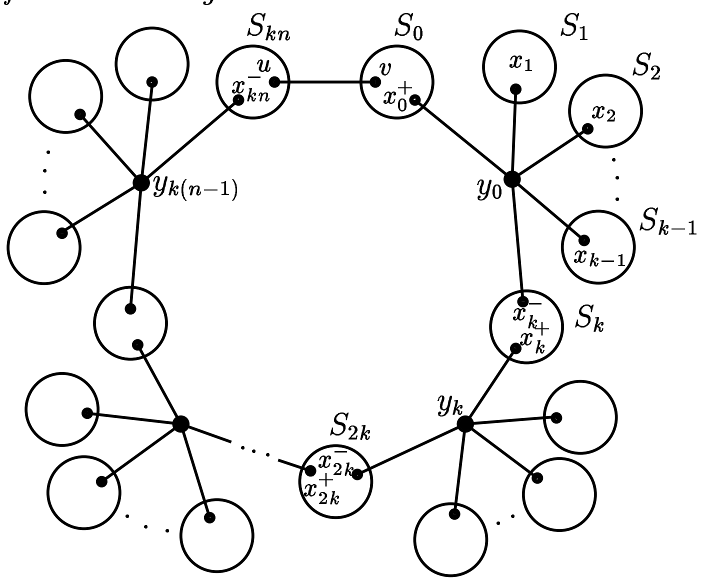
Lemma 3.14. *Let $n \ge 1$ be an integer and $S_0, \dots, S_{kn}$ be k-suns of disjoint node sets. For $1 \le i \le n-1$ let $x_{ki}^+, x_{ki}^- \in V(S_{ki})$, $x_0^+, v \in V(S_0)$, $x_{kn}^-, u \in V(S_{kn})$ and for $0 \le i \le n-1$, $1 \le j \le k-1$ let $x_{ki+j} \in V(S_{ki+j})$. If*

$$V(G) = \bigcup\{V(S_h) : 0 \le h \le kn\} \cup \{y_{ki} : 0 \le i \le n-1\} \text{ and}$$

$$E(G) = \bigcup\{E(S_h) : 0 \le h \le kn\} \cup$$

$$\cup \left\{y_{ki}x_{ki}^+, y_{ki}x_{k(i+1)}^-, y_{ki}x_{ki+j} : 0 \le i \le n-1,\ 1 \le j \le k-1\right\} \cup \{uv\}$$

then G is either a k-sun or it is k-good. We can decide which case arises in $O(|V(G)|)$ time and if G is good a k-tree factor of it can be determined in $O(|V(G)|)$ time if the decompositions of the suns are given.



Proof. The notations (a) – (c) of the proof refer to Lemma 3.13.

If the pair x_0^+, v is lucky in S_0 then

$$G\left[V(S_{kn}) \cup V(S_0) \cup \{y_0\} \cup \bigcup \{V(S_j) : 1 \leq j \leq k-1\}\right]$$

is good by (c) and the rest of G decompose to graphs occurring in (b). The same applies to S_{kn} so we can assume that the pair x_0^+, v is unlucky in S_0 and the pair x_{kn}^-, u is unlucky in S_{kn}. On the basis of this assumption $G[V(S_{kn}) \cup V(S_0)]$ is good by (a).

Denote the hair of x_h in S_h by O_h. If $\deg_{S_{kf+g}}(x_{kf+g}) \geq k-1$ for some $0 \leq f \leq n-1$, $1 \leq g \leq k-1$ then G is good because $S_{kf+j} - O_{kf+j}$ is good for $1 \leq j \leq k-1$ by Claim 3.9, the simple graph $G[\bigcup\{O_{kf+j} : 1 \leq j \leq k-1\} \cup \{y_{kf}\}]$ has a k-tree spanning tree because its highest degree is k, we already know that $G[V(S_{kn}) \cup V(S_0)]$ is good, moreover, the rest of G decompose to graphs which are good by (b). So we can assume that $\deg_{S_{kf+g}}(x_{kf+g}) \leq k-2$ for $0 \leq f \leq n-1$, $1 \leq g \leq k-1$. Now it can be easily seen that $G[\{y_{kf}\} \cup \bigcup\{V(S_{kf+j}) : 1 \leq j \leq k-1\}]$ is a k-sun.

Let $1 \leq f \leq n-1$. We assumed that $G[\{y_{kf}\} \cup \bigcup\{V(S_{kf+j}) : 1 \leq j \leq k-1\}]$ is a k-sun so if the pair x_{kf}^-, x_{kf}^+ is lucky in S_{kf} then

$$G\left[\{y_{kf}\} \cup \bigcup \{V(S_{kf+j}) : 1 \leq j \leq k-1\} \cup V(S_{kf}) \cup \right.$$
$$\left. \cup \left\{y_{k(f-1)}\right\} \cup \bigcup \left\{V(S_{k(f-1)+j}) : 1 \leq j \leq k-1\right\}\right]$$

is good due to (c). We already know that $G[V(S_{kn}) \cup V(S_0)]$ is good and the rest of G decompose to graphs which are good by (b). So we can assume that the pair x_{kf}^-, x_{kf}^+ is unlucky in S_{kf}.

Summarizing, if G is not good then the following pairs are unlucky: x_0^+, v in S_0, x_{kn}^-, u in S_{kn} and x_{kf}^-, x_{kf}^+ in S_{kf} for $1 \leq f \leq n-1$. Furthermore, $\deg_{S_{kf+g}}(x_{kf+g}) \leq k-2$ for $0 \leq f \leq n-1$, $1 \leq g \leq k-1$. This means that G is a k-sun.

The statements concerning the running times follow from the construction. ☐

Lemma 3.15. *If S is a k-sun, $u, v \in V(S)$ and the simple graph $S + uv$ is not a k-sun then $S + uv$ is k-good. We can decide in constant time whether $S + uv$ is a k-sun. If not then we can obtain a k-tree factor of it in $O(|V(S)|)$ time and if it is a k-sun then the decomposition of $S + uv$ can be determined in constant time, provided that the decomposition of S is given.*

Proof. Suppose that u is in the hair O_u and v is in O_v. If $S + uv$ is not a k-sun then we can distinguish the following cases:

Case 1: $O := O_u = O_v$ and the degree of one of them, say u, in $S[O]$ is $k-1$. $S - O$ is good by Claim 3.9, furthermore, the simple graph $S[O] + uv$ has a k-tree spanning tree because its highest degree is k.

Case 2: $O := O_u = O_v$ and the degree of u, v in $S[O]$ is $\leq k-2$. $S + uv$ is not a k-sun so it contains a circuit which passes through a vertex $w \in H$. Let the edge wz

be in this circuit. $S + uv - wz$ is an almost sun of type 2 which is good by Lemma 3.6.

Case 3: $O_u \neq O_v$ and one of the nodes, say u, is in L. Let T_v be a spanning tree of $S[O_v]$. $S - O_v$ is good due to Claim 3.9, let E_1 denote a k-tree factor of it. $E' := E_1 \cup \{uv\} \cup T_v$. $\deg_{E_1}(u) \leq k-1$, $\deg_{T_v}(v) \leq k-1$ so E' is a k-tree factor of $S + uv$.

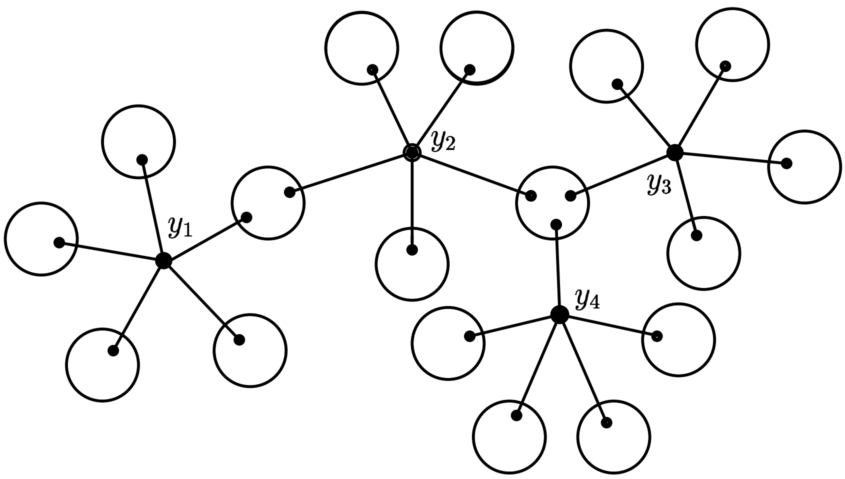
Case 4: $O_u \neq O_v$ and $u, v \in H$. u and v are in distinct components of $S[H]$ so Lemma 3.10 yields that $S - V(T^u) - V(T^v)$ is good. $T^u \cup \{uv\} \cup T^v$ is a k-tree so we are done.

$S + uv$ is a k-sun if and only if either u and v are in the same factor-critical component of H or they are in the same hair O in S, their degree in $S[O]$ are at most $k-2$ and there is no circuit in $S + uv$ passing through a node of H. We can decide in constant time whether this is the case, in the second case knowing a spanning tree of $S[O]$ will suffice. If $S + uv$ is a k-sun then its decomposition can be obtained from the decomposition of S by a constant time change. If $S + uv$ is not a k-sun then our construction can be performed in $O(|V(S)|)$ time. □

4 The algorithm

In this section we present a variant of the matching algorithm of Edmonds, which solves the k-tree factorization problem in the following way: if G has a k-tree factor the algorithm shows one, otherwise it constructs a set $Y \subseteq V(G)$ such that the number of the k-sun components of $G - Y$ is more than $k|Y|$ indicating that G cannot have a k-tree factor, as it was proved in page 6.

Let G be a simple, connected graph. In every step the algorithm records a special subgraph of G:



a galaxy (B, Y), $Y = \{y_1, y_2, y_3, y_4\}$, suns: ◯, $k = 4$

Definition 4.1. (B, Y) is a *galaxy* if B is a simple, connected graph, $Y \subseteq V(B)$ is a possibly empty set such that $B - Y$ consists of $k|Y| + 1$ k-suns, there is no edge between the nodes of Y, and for all $y \in Y$ it holds that $\deg_B(y) = k + 1$ and these $k + 1$ edges are cutting edges in B.

Let $\mathcal{S} \neq \emptyset$ denote the set of the components of $B - Y$.

Note that for $S \in \mathcal{S}$ the graph $B - V(S)$ is k-good because it can be decomposed into graphs occurring in Lemma 3.13 (b).

The algorithm records a galaxy (B, Y) where B is a subgraph of G, the decompositions of the suns in $\mathcal{S}$ and a k-tree packing E'. In the beginning let $E' = \emptyset$. It is a fundamental property that in every step E' misses at least one node of $V(B)$ and the edges of E' do not leave $V(B)$. In every step either the algorithm increases $|E(B)|$ or it finds a k-tree packing covering more nodes, which is called an *augmentation.* When stopping it gives either a k-tree factor or for the set Y of the galaxy the number of the k-sun components of $G - Y$ is more than $k|Y|$ showing that G is not k-tree factorizable.

Start: If E' covers all the nodes of G then we are ready. Otherwise let $z \in V(G)$ be an uncovered vertex, $B := (\{z\}, \emptyset)$, $Y := \emptyset$. Go to step **1.**

1. If there exists an edge $uv \in E(G) \setminus E(B)$ such that $u, v \in V(B) - Y$ then go to **2.** If $S \in \mathcal{S}$ is adjacent to a vertex in $V(G) \setminus V(B)$ which is uncovered by E' then go to **3.** If $S \in \mathcal{S}$ is adjacent to a vertex in $V(G) \setminus V(B)$ which is covered by E' then go to **4.** Otherwise in the graph $G - Y$ the number of the k-sun components is at least $k|Y| + 1$ showing that G is not good by the proved part of Theorem 2.3.

2. Suppose that $u, v \in V(S)$ for $S \in \mathcal{S}$. If $S + uv$ is a k-sun then let $B' = B + uv$ and go to step **1.** If $S + uv$ is not a k-sun then by Lemma 3.15 it is good, $B - V(S)$ is good too, moreover, E' misses at least one node of $V(B)$ and the edges of E' do not leave $V(B)$ so we can augment E', go to **Start.**

Suppose that $u \in V(S')$ and $v \in V(S'')$ for the distinct suns $S', S'' \in \mathcal{S}$. Let J be the following subgraph of G. Due to the tree-like structure of B all the $u - v$ paths of B go through the same set of vertices of Y, let $\emptyset \neq X \subseteq Y$ denote this set. Let $\mathcal{R} \subseteq \mathcal{S}$ be the set of those suns which are adjacent to X in B. $V(J) := \bigcup\{V(R) : R \in \mathcal{R}\} \cup X$ and $J := B[V(J)] + uv$.

It is clear that J satisfies the conditions of Lemma 3.14 so if J is not a k-sun then it is good, moreover, $B - V(J)$ is good because it can be decomposed into graphs occurring in Lemma 3.13 (b) so we can augment E', go to **Start.** If J is a k-sun then we *contract*, i.e. let $B' = B + uv$ and $Y' = Y \setminus X$. (Now $\mathcal{S}' = \mathcal{S} \setminus \mathcal{R} + J$.) Note that the new galaxy (B', Y') satisfies the conditions. E' remained the same, $|E(B)|$ increased, go to step **1.**

3. Let $u \in V(S)$ be adjacent to $v \notin V(B)$ where v is left uncovered by E'. If $k = 1$ or $\deg_S(u) \geq k - 1$ then by Lemma 3.13 (a) $S + uv$ is good and $B - V(S)$ is good so we can augment E', go to **Start**. Otherwise it is clear that $S + uv$ is a k-sun so let $V(B') = V(B) + v$, $E(B') = E(B) + uv$ and $Y' = Y$. Go to step **1.**

4. Let $u \in V(S)$ be adjacent to $v \notin V(B)$ where v is covered by a k-tree component T of E'. Suppose that the hair of u in S is O and let T_u be a spanning tree of $S[O]$.

If $\deg_T(v) \leq k-1$ then $T_u \cup \{uv\} \cup T$ is a k-tree, $S - O$ is good due to Claim 3.9, $B - V(S)$ is good too so we can augment E', go to **Start.**

Assume that $\deg_T(v) = k$. Omitting v from T we get k components: $D_1, \ldots, D_k$. Suppose that, say $\Delta(D_1) = k$. Let T' denote the tree we get when deleting D_1 from T. D_1 is a k-tree, $T_u \cup \{uv\} \cup T'$ is a k-tree too, $S - O$ is good by Claim 3.9, $B - V(S)$ is good too so we can augment E', go to **Start.** If $\Delta(D_i) \leq k-1$ for all $1 \leq i \leq k$ then let $V(B') = V(B) \cup V(T)$, $E(B') = E(B) \cup \{uv\} \cup T$ and $Y' = Y + v$. Now $\mathcal{S}' = \mathcal{S} \cup \{D_i : 1 \leq i \leq k\}$. Note that the new galaxy (B', Y') satisfies the conditions. E' remained the same, $|E(B)|$ increased, go to step **1.**

The running time of the algorithm can be bounded by $O(en^2)$. Let those steps belong to the same phase when E' covers the same number of nodes. The number of the phases is at most n and in a phase we increase $|E(B)|$ at most e times. A suitable data structure is needed to scan the edges of $E(G) \setminus E(B)$. For the first part of **2.** if $S + uv$ is a k-sun then only constant time is needed and if it is not then $O(n)$ time is needed for a k-tree factor of $S + uv$ and of $B - V(S)$ by Lemmas 3.13 and 3.15 when the phase ends. For the second part of **2.** we need $O(n)$ time to determine J and to decide if it is a k-sun or not and if not to augment E' by Lemmas 3.13 and 3.14. The same applies to **3.**, here we need constant time to decide if $S + uv$ is a k-sun and if not we need $O(n)$ time to augment E' when the phase ends. **4.** can be arranged in $O(n)$ time.

On account of the validity of the algorithm, we have a proof of Theorem 2.3.

Note that the algorithm in the case $k = 1$ is precisely the well-known algorithm of Edmonds on maximum matchings.

5 A Berge-Tutte-type theorem

We prove a Berge-Tutte-type theorem for the k-tree packing problem concerning the minimum number of nodes which can be missed by a k-tree packing.

Definition 5.1. Let S be a k-sun. In the case $k \geq 2$ let $p(S)$ denote the minimum number of nodes occurring in the connected components of $S[L]$, while in the case $k = 1$ let $p(S) = 1$.

A k-tree packing is *optimal* if it covers a maximum number of nodes. Let $d_k(G)$ denote the number of nodes which are missed by an optimal k-tree packing of G.

Lemma 5.2. *$d_k(S) = p(S)$ for a k-sun S. For $k \geq 2$ an optimal k-tree packing of S is a k-tree factor of $S - D$ for a connected component D of $S[L]$ having $p(S)$ nodes.*

Proof. The lemma is known to be true for $k = 1$ so let $k \geq 2$.

Let E' be an optimal k-tree packing and suppose that E' covers at least one node in every connected component of $S[L]$. Every vertex in H has a neighbor in L and by Claim 3.11 E' misses at least one node of S so there is an edge $uv \in E(S)$ such that $u \in L$ and u is covered by E', while v is not. Now $E' + uv$ is a k-tree packing because $\deg_S(u) \leq k-1$ and it covers more nodes than E', a contradiction.

So there is at least one component D of $S[L]$ such that E' misses all of its vertices. For all the components D' of $S[L]$ it holds that $S - D'$ is a (possibly empty) union of almost k-suns of type 2 so $S - D'$ is good by Lemma 3.6. E' is optimal, thus it is a k-tree factor of $S - D$ and $|V(D)|$ is minimal, i.e $|V(D)| = p(S)$. Finally, E' shows that $d_k(S) = p(S)$. □

Definition 5.3. For a graph G and $X \subseteq V(G)$ let $c_i(X)$ denote the number of those k-sun components S of $G - X$ for which $p(S) \geq i$.

Theorem 5.4. *For a simple, connected graph G*

$$d_k(G) = \max \left\{ \sum_{i=1}^{\infty} c_i(X_i) - k|X_i| \; : \; \ldots \subseteq X_i \subseteq \ldots \subseteq X_1 \subseteq V(G) \right\}.$$

Proof. $\geq$: Let $\ldots \subseteq X_i \subseteq \ldots \subseteq X_1 \subseteq V(G)$ be a set-system where the maximum is attained and E' be an optimal k-tree packing. First observe that for $v \in X_1$ the number of edges between v and $V(G) \backslash X_1$ is at least k, otherwise setting $X'_i := X_i \backslash \{v\}$ we get $\sum_{i=1}^{\infty} c_i(X_i) - k|X_i| < \sum_{i=1}^{\infty} c_i(X'_i) - k|X'_i|$, a contradiction. We construct disjoint sets W_i, $i \in \mathbb{N}$, such that $|W_i| \geq c_i(X_i) - k|X_i|$ and E' misses $\bigcup_{i=1}^{\infty} W_i$, thus proving the $\geq$ part. W_i will contain at most one vertex from each connected component of $G - X_i$.

First let $k \geq 2$. Suppose we have constructed $W_1, \ldots, W_{i-1}$, $i \geq 1$. The k-trees of E' meeting X_i can intersect altogether at most $k|X_i|$ suns of $G - X_i$ so the number of the suns S of $G - X_i$ such that $p(S) \geq i$ and E' does not enter S is at least $c_i(X_i) - k|X_i|$. We will take one node from all such suns to W_i, thus constructing W_i. Let S be such a sun. E' covers a maximum number of nodes in G and does not enter S so by Lemma 5.2 there is a connected component D of $S[L]$ such that $|V(D)| = p(S)$ and $E' \cap E(S)$ is a k-tree factor of $S - D$. $V(D) \cap X_1 = \emptyset$ because for $v \in V(D)$ the number of the edges between v and $G - X_1$ is at most as between v and $G - X_i$ which is at most $k - 1$ so $v \notin X_1$. Hence the connected graph D is in a connected component of $G - X_j$ for $j < i$ too so $|W_j \cap V(D)| \leq 1$ for $j < i$. $|V(D)| = p(S) \geq i$, thus we can take a node from the nonempty set $V(D) \setminus \bigcup_{j=1}^{i-1} W_j$ to W_i.

It is known that we do the same in the case $k = 1$, i.e. in the matching problem, but only for $i = 1$ because for $i \geq 2$ it holds that $c_i = 0$ so $X_i = \emptyset$.

$\leq$: Let (B_j, Y_j), $1 \leq j \leq t$, $(t \geq 0)$ be galaxies such that the B_j-s are subgraphs of G of disjoint node sets. We denote the set of the components of $B_j - Y_j$ by $\mathcal{S}_j$, by definition $\mathcal{S}_j$ consists of $k|Y_j| + 1$ k-suns. Let $p_j = \min\{p(S) : S \in \mathcal{S}_j\}$. Choose this galaxy system in such a way that the suns of $\bigcup_{j=1}^{t} \mathcal{S}_j$ are connected components in the graph $G - \bigcup_{j=1}^{t} Y_j$, moreover,

(1) t is *maximum*,

(2) $\sum_{j=1}^{t} p_j$ is *minimum* subject to (1) and

(3) $\sum_{j=1}^{t} p_j |Y_j|$ is *maximum* subject to (1) and (2).

There exists such a galaxy-system because we allow $t = 0$. $G' := G - \bigcup_{j=1}^{t} V(B_j)$ is good, since otherwise with the input G' our above algorithm when stopping would give a galaxy (B, Y) with sun-set $\mathcal{S}$ such that B is a subgraph of G' and the suns of $\mathcal{S}$ are connected components in the graph $G' - Y$, contradicting the maximality of t. So $d_k(G) \leq \sum_{j=1}^{t} d_k(B_j)$. Note that for a galaxy (B_j, Y_j) we have $d_k(B_j) \leq p_j$ because $B_j - V(S)$ is good by Lemma 3.13 (b) for the sun S for which $p(S) = p_j$ holds and by Lemma 5.2 $d_k(S) = p(S) = p_j$.

Suppose that a vertex $u \in S \in \mathcal{S}_h$ has a neighbor $v \in Y_i$, where $p_i < p_h$. Denote the components of $B_i - v$ by $D^0, \ldots, D^k$. At least one of these components, say D^0, has a sun S' such that $p(S') = p_i < p_h$. Let $B'_i = D^0$, $Y'_i = Y_i \cap V(D^0)$ and B'_h be the galaxy what we get when gluing $B_i - D^0$ to B_h via the edge uv, i.e. $B'_h := B_h + uv + (B_i - D^0)$, $Y'_h := Y_h \cup (Y_i \setminus V(D^0))$. Let $(B'_j, Y'_j) = (B_j, Y_j)$ for $j \neq h, i$ and let p'_j be defined as was p_j above.

Now if $p(S) \geq p_h$ for all the suns S in the components D^l, $l \neq 0$, then $p_j = p'_j$ for all j and $\sum_{j=1}^{t} p_j |Y_j| < \sum_{j=1}^{t} p'_j |Y'_j|$ because we have put the nodes in $Y_i \setminus V(D^0) \neq \emptyset$ from Y_i to Y_h and $p'_h = p_h > p_i$, a contradiction. On the other hand if $p(S'') < p_h$ for a sun S'' in a component D^l, $l \neq 0$, then $p_j = p'_j$ for $j \neq h$ and $p_h > p'_h$, a contradiction too.

Thus if $S \in \mathcal{S}_h$ is adjacent to Y_i then $p_i \geq p_h$. Let $X_i = \bigcup\{Y_j : p_j \geq i\}$. Denote by $c'_i(X_i)$ the number of the k-sun components S in $G - X_i$ for which $p(S) \geq i$ and which occur among the connected components of $G - X_1$ as well. It is easy to see that

$$\sum_{j=1}^{t} p_j = \sum_{i=1}^{\infty} c'_i(X_i) - k|X_i|.$$

Finally,

$$d_k(G) \leq \sum_{j=1}^{t} d_k(B_j) \leq \sum_{j=1}^{t} p_j = \sum_{i=1}^{\infty} c'_i(X_i) - k|X_i| \leq \sum_{i=1}^{\infty} c_i(X_i) - k|X_i|.$$

□

The case $k = 1$ is the Berge-Tutte theorem, the case $k = 2$ was proved by Kano, Katona and Király [5]. It is interesting that the right hand side of the minimax formula contains a sequence of node sets instead of only one. The reason for this is the difference between the k-tree packing problem and other graph packing problems where we allow K_2-components in the packing as we saw in page 6, namely now an optimal packing of a "critical" graph can miss an arbitrary number of nodes instead of only one.

Actually, the $\leq$ direction of the above proof is algorithmic. We remark that this direction could be proved in a less algorithmic way: choose a set-system $\ldots \subseteq X_i \subseteq \ldots \subseteq X_1 \subseteq V(G)$ where the maximum is attained and then using a sequence of the Mendelsohn-Dulmage theorem construct a k-tree packing missing $\sum_{i=1}^{\infty} c_i(X_i) - k|X_i|$ nodes.

A Gallai-Edmonds-type theorem for the problem would be interesting.

6 A more general form

In this section we state a further generalization which can be gained from Theorem 2.3 and mention some corollaries. From now on a *tree* is meant as a graph not just an edge set, so e.g. the graph consisting of a single vertex is a tree.

Definition 6.1. Given two bounds $l,\ u\colon V(G)\to\mathbb{N},\ l\le u$, a tree subgraph T of G is an (l,u)*-tree* if

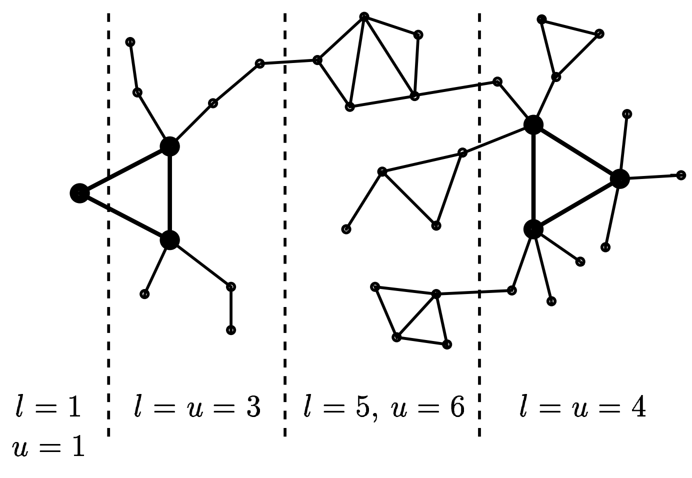
- $\deg_T(v)\le u(v)$ for $v\in V(T)$ and
- there exists a vertex $w\in V(T)$ such that $\deg_T(w)\ge l(w)$.

A forest F is an (l,u)*-tree packing* if each connected component of F is an (l,u)-tree. A spanning (l,u)-tree packing is an (l,u)*-tree factor.*

Given the bounds $l,\ u\colon V(G)\to\mathbb{N},\ l\le u$, does G have an (l,u)-tree factor? Setting $l\equiv u\equiv k$ we see that this problem generalizes the k-tree factorization problem. Actually, our main Theorem 2.3 implies this generalization. Let $k=1+\max\{u(x): x\in V(G)\}$ and $A_x,\ B_x$ be disjoint sets of extra vertices for $x\in V(G)$ such that $|A_x|=u(x)-l(x)+1$ and $|B_x|=k-u(x)-1$. Let the vertices in $A_x\cup B_x$ be adjacent to x and let the nodes in A_x be pairwise adjacent. It is easy to see that the new graph has a k-tree factor if and only if G has an (l,u)-tree factor. Hence we get Theorem 6.3, a generalization of Theorem 2.3.

Definition 6.2. Given the bounds $l,u\colon V(S)\to\mathbb{N}$ the simple, connected graph S is an (l,u)*-sun* if it satisfies the followings:

- $\deg_S(v)\ne l(v)$ for $v\in V(S)$,
- denoting by H the set of nodes v of degree at least $l(v)+1$, the induced subgraph $S[H]$ is a (possibly empty) union of non-trivial factor-critical graphs,
- $l(v)=u(v)$ for $v\in H$,
- $\deg_S(v)=\deg_{S[H]}(v)+l(v)-1$ and all these $l(v)-1$ edges are cutting edges in S.



an (l,u)-sun, H : ●

Theorem 6.3. *Given the bounds $l, u\colon V(G) \to \mathbb{N}$ the simple graph G has an (l,u)-tree factor if and only if for all $Y \subseteq V(G)$ the number of the (l,u)-sun components of $G - Y$ is at most $u(Y) = \sum\{u(y) : y \in Y\}$.*

Observe that the existence of an (l,u)-tree factor is equivalent to the existence of a spanning subgraph each component T of which satisfying the two conditions in Definition 6.1.

We mention some corollaries of Theorem 6.3.

If l and u are constant functions then we look for a spanning subgraph F the highest degree of each connected component of which is bounded from below and above. The case $l = u = k$ is the k-tree factorization problem. Observe that in the case $l < u$ the (l,u)-suns are the graphs having highest degree at most $l-1$ so the following specialization of Theorem 6.3 holds:

Theorem 6.4. *For the integers $l < u$ a simple graph G has a spanning subgraph F such that for each component T of F it holds that $l \leq \Delta(T) \leq u$ if and only if for all $Y \subseteq V(G)$ the number of the components of $G - Y$ having highest degree at most $l-1$ is at most $u|Y|$.*

If $l \equiv 1$ and $u(v) \geq 2$ for $v \in V(G)$ then the only (l,u)-trees are the nodes of G, hence we get the theorem of Las Vergnas [7]:

Definition 6.5. The i-star R_i is a connected graph such that it has a specified vertex, the center c such that $R_i - c$ is the union of i isolated vertices.

For $u\colon V(G) \to \mathbb{N},\ u \geq 2,$ an $\leq u$*-star factor* of G is a spanning subgraph of G the components of which are stars such that a star with center c is R_i for $1 \leq i \leq u(c)$.

Theorem 6.6 (Las Vergnas). *For $u\colon V(G) \to \mathbb{N},\ u \geq 2$, the graph G has an $\leq u$-star factor if and only if there is no $Y \subseteq V(G)$ such that $G - Y$ has more than $u(Y)$ isolated vertices.*

Proof. The existence of an (l,u)-tree factor for $l \equiv 1$ is equivalent to the existence of an $\leq u$-star factor. ☐

Suppose that $l > 0$ exactly for the nodes of $W \subseteq V(G)$. For $v \notin W$ the graph consisting of the single vertex v is an (l,u)-tree of G, hence we get a theorem characterizing when a vertex set W can be covered by an (l,u)-tree packing of G. E.g. if $l = \chi_W$ and $u \equiv 1$ we get the following well-known theorem of Tutte [10]:

Theorem 6.7 (Tutte). *G has a matching covering $W \subseteq V(G)$ if and only if for all $Y \subseteq V(G)$ the number of the factor-critical components the vertex set of which is a subset of W is at most $|Y|$.*

Setting $l = \chi_W$ and $u \geq 2$ we get a variant of Las Vergnas' theorem characterizing the graphs having an $\leq u$-star packing covering $W \subseteq V(G)$.

We mention that Theorem 6.3 can be proved directly by a straightforward variant of our above algorithm.

We state the generalization of our Berge-Tutte-type theorem, its proof is similar to the proof of Theorem 5.4.

Definition 6.8. For an (l,u)-sun S let $p(S)=1$ if $u=1$ somewhere in $V(S)$, otherwise let $p(S)$ be the minimum number of nodes occurring in the connected components of $V(S)-H$.

Theorem 6.9. *For a simple, connected graph G with two bounds $l,\ u\colon V(G)\to\mathbb{N}$ an (l,u)-tree packing covering a maximum number of nodes misses*

$$\max\left\{\sum_{i=1}^{\infty} c_i(X_i)-u(X_i)\ :\ \ldots\subseteq X_i\subseteq\ldots\subseteq X_1\subseteq V(G)\right\}$$

nodes, where $c_i(X)$ denotes the number of those (l,u)-sun components S of $G-X$ for which $p(S)\geq i$.

We mention that we can solve the following more general problem too: given two bounds $l,\ u\colon V(G)\to\mathbb{N}$, $l\leq u$, and a set $\mathcal{F}$ consisting of factor-critical subgraphs of G, decide if G has a spanning subgraph each component of which is either an (l,u)-tree or an (l,u)-sun S such that $H(S)$ is connected and belongs to $\mathcal{F}$.

Remark. While writing this paper I became aware that Hartvigsen and Hell obtained analogous results on this topic.

References

[1] G. Cornuéjols, D. Hartvigsen, An Extension of Matching Theory *J. Combin. Theory Ser. B* (1986) **40**, no. 3, 285–296.

[2] P. Hell, D. G. Kirkpatrick, Packings by complete bipartite graphs *SIAM J. Algebraic Discrete Methods* (1986) **7**, no. 2, 199–209.

[3] P. Hell, D. G. Kirkpatrick, personal communication

[4] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, submitted

[5] M. Kano, G. Y. Katona, Z. Király, Packing paths of length at least two. Discrete Math., accepted

[6] Z. Király, J. Szabó, Generalized Induced Factor Problems. EGRES Technical Reports, 2002-07.

[7] M. Las Vergnas, An extension of Tutte's 1-factor theorem *Discrete Math.* (1978) **23**, no. 3, 241–255.

[8] M. Loebl, S. Poljak, Good Family Packing *Annals of Discrete Math* (1992) **51**, 181–186.

[9] M. Loebl, S. Poljak, Efficient Subgraph Packing *J. Combin. Theory Ser. B* (1993) **59**, no. 1, 106–121.

[10] W. T. Tutte, The factorization of linear graphs. *J. London Math. Soc.* (1947) **22**, 107–111.

[11] H. Wang, Path factors of bipartite graphs. *J. Graph Theory* (1994) **18**, no. 2, 161–167.