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Minimally k-edge-connected directed graphs of maximal size

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Abstract

Let $D = (V, E)$ be a minimally k-edge-connected simple directed graph. We prove that there is a function $f(k)$ such that $|V| \geq f(k)$ implies $|E| \leq 2k(|V| - k)$. We also determine the extremal graphs whose size attains this upper bound.

1 Introduction

A number of extremal problems related to graph connectivity have been studied in recent years. One of the central problems in this area is to determine the maximum possible size (i.e. number of edges) of a minimally k-(edge)-connected (multi)graph or directed (multi)graph on n vertices. (Graphs and digraphs in this paper are assumed to be simple. When multiple edges may be present, we use the terms multigraph or multidigraph.)

It is easy to show that a minimally k-edge-connected multigraph on n vertices has at most $k(n-1)$ edges, and that this value is best possibe for all values of n and k. Mader [5] proved that this can be improved for graphs: in this case the size is at most $k(n-k)$, provided $n \geq 3k-2$. The complete bipartite graph $K_{k,n-k}$ shows that this bound is tight. Mader [6] verified that the latter bound is valid for minimally k-connected graphs as well, if $n \geq 3k-2$ holds (see also Cai [1]).

Dalmazzo [3] proved that a minimally k-edge-connected multidigraph on n vertices has at most $2k(n-1)$ edges, and that this is tight for all values of n and k. Mader [8] showed that a minimally k-connected digraph with $n \geq 4k+3$ contains at most $2k(n-k)$ edges. The complete bipartite digraph $DK_{k,n-k}$ shows that this upper bound is also best possible.

One item is missing from this list, as far as the asymptotic extremal value is concerned: the case of minimally k-edge-connected digraphs appears to be open. In the

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present paper we determine the missing extremal value by showing that if multiple edges are not allowed then Dalmazzo's upper bound can be improved to $2k(n-k)$, provided n is sufficiently large compared to k. Again, $DK_{k,n-k}$ shows that our bound is best possible. We shall also prove that this digraph is the unique extremal digraph, for every given (and large enough) value of n. As in most of the other related problems, there exist 'small' digraphs for which this improved upper bound does not hold. For example, consider the digraph H obtained from a bidirected circuit of length $2k-1$ by adding $k-2$ independent vertices and connecting each of them to the vertices of the circuit in both directions. This digraph has $n = 3k-3$ vertices and $4k^2-6k+2 > 2k(2k-3)$ edges. This shows that we need a lower bound on n in terms of k in order to guarantee the required upper bound. Since our methods are unlikely to yield the best function of k, we shall not try to improve the (exponential) function $f(k)$ that follows from our proofs, although a linear function of k might suffice.

Now we introduce some basic definitions and notation. Let $D = (V, E)$ be a multidigraph. We use $d^+_D(X)$ ($d^-_D(X)$) to denote the number of edges entering (leaving, respectively) a set $X \subseteq V$. If $X = \{v\}$ is a singleton, we write $d^+_D(v)$ ($d^-_D(v)$). We omit the subscript D if the digraph considered is clear from the context. In what follows $\subset$ means proper inclusion and $\subseteq$ means $\subset$ or $=$. For a set $X \subseteq V$ we use $N^+(X)$ to denote the 'out-neighbours' of X, i.e. the set of vertices v in $V - X$ for which there is a vertex $u \in X$ with $uv \in E$. The definition of $N^-(X)$ is similar. For some $X \subseteq V$ the subdigraph induced by X is denoted by $D[X]$. A set X of vertices is *independent* in D if $|E(D[X])| = 0$.

A multidigraph $D = (V, E)$ is *k-edge-connected* if $|V| \geq 2$ and $d^-(X) \geq k$ holds for every $\emptyset \neq X \subset V$. We call D *minimally k-edge-connected* if D is k-edge-connected but $G - e$ is no longer k-edge-connected for any $e \in E$. A set $X \subset V$ is an *in-set* (*out-set*) if $d^-(X) = k$ ($d^+(X) = k$, resp.) holds. It is easy to see that if D is minimally k-edge-connected then every edge $e \in E$ enters an in-set (and hence leaves an out-set). A vertex v with $d^+(v) = d^-(v) = k$ will be called an *atom*.

2 Preliminaries

Two sets $X, Y \subseteq V$ are *crossing* if $X - Y, X \cap Y, Y - X$, and $V - (X \cup Y)$ are all non-empty. A family of sets is *cross-free* if it contains no two crossing sets. A family $\mathcal{F}$ of in-sets of V is a *witness family (of in-sets)* of D if every edge $e \in E$ enters a member of $\mathcal{F}$. As we noted, the family of all in-sets of D is a witness family in a minimally k-edge-connected multidigraph.

The next lemma can be proved by using the so-called "uncrossing method".

Lemma 2.1. *[2, Lemma 2],[4, Section 5] Let D be a minimally k-edge-connected multidigraph. Then D has a cross-free witness family of in-sets.*

A family $\mathcal{L}$ of non-empty subsets of a groundset M is called *laminar* if for any pair $X, Y \in \mathcal{L}$ either $X \cap Y = \emptyset$ or $X \subset Y$ or $Y \subset X$ holds. For a set $X \in \mathcal{L}$ we define the *core* of X, denoted by $C(X)$, as follows:

$$C(X) = X - \bigcup \{Y : Y \subset X, Y \in \mathcal{L}\} \tag{1}$$

Let $c(X) = |C(X)|$. A laminar family $\mathcal{L}$ is *strongly laminar* if $M = \cup_{X \in \mathcal{L}} X$, the members of $\mathcal{L}$ are pairwise distinct, and $C(X) \neq \emptyset$ for every $X \in \mathcal{L}$. Given a strongly laminar family $\mathcal{L}$ on M, let $s(\mathcal{L}) = \sum_{X \in \mathcal{L}} (c(X) - 1)$ be the *surplus* of $\mathcal{L}$. It is easy to see that a strongly laminar family $\mathcal{L}$ on M satisfies

$$|\mathcal{L}| = |M| - s(\mathcal{L}) \leq |M| \tag{2}$$

Let $D = (V, E)$ be a minimally k-edge-connected multidigraph and let $r \in V$ be a designated vertex, called the *root*. Let $W = V - r$. We say that a pair $(\mathcal{L}_i, \mathcal{L}_o)$ is a *witness pair of D (with root r)* if

(a) $\mathcal{L}_i$ is a strongly laminar family of in-sets of D on groundset W,
(b) $\mathcal{L}_o$ is a strongly laminar family of out-sets of D on groundset W,
(c) $\mathcal{L}_i \cup \mathcal{L}_o$ is laminar,
(d) every edge of D enters a member of $\mathcal{L}_i$ or leaves a member of $\mathcal{L}_o$.

The next lemma is easy to verify (see also the proof of Lemma 2.5).

Lemma 2.2. *Let $D = (V, E)$ be a minimally k-edge-connected multidigraph and let $r \in V$. Then D has a witness pair with root r.*

Lemma 2.2 gives rise to a short proof of Dalmazzo's result on the maximal size of minimally k-edge-connected multidigraphs and illustrates one of the proof techniques we shall use later.

Theorem 2.3. *[3] Let $D = (V, E)$ be minimally k-edge-connected multidigraph. Then $|E| \leq 2k(|V| - 1)$.*

Proof: Let $r \in V$ be a designated root vertex. By Lemma 2.2 there exists a witness pair $(\mathcal{L}_i, \mathcal{L}_o)$ of D on groundset $V - \{r\}$. By property (d) every edge of D enters an in-set in $\mathcal{L}_i$ or leaves an out-set in $\mathcal{L}_i$. Thus, by using (2), we get

$$|E| \leq k|\mathcal{L}_i| + k|\mathcal{L}_o| \leq 2k(|V| - 1). \tag{3}$$

This proves the theorem. •

The bound in Theorem 2.3 is best possible for all values of $k \geq 1$ and $|V| \geq 2$. It is also known [3] that the size of D attains the upper bound if and only if D is obtained from a tree by replacing every edge uv by k parallel edges from u to v and k parallel edges from v to u. Theorem 2.3 solves our extremal problem for digraphs as well, when $k = 1$. Thus in what follows we shall always assume that $k \geq 2$.

It will be convenient to work with *strong witness pairs*, i.e. witness pairs satisfying the following two additional properties:

(e) all singleton in-sets in W belong to $\mathcal{L}_i$ and all singleton out-sets in W belong $\mathcal{L}_o$,
(f) the root r is an atom and $W = \cup_{\mathcal{L}_i} X = \cup_{\mathcal{L}_o} Y$.

To show that a strong witness pair exists we need the following theorem of Mader.

Theorem 2.4. *[7] Every minimally k-edge-connected multidigraph has a vertex v with $d^+(v) = d^-(v) = k$.*

Lemma 2.5. *Let $D = (V, E)$ be a minimally k-edge-connected multidigraph. Then D has a strong witness pair.*

Proof: (sketch) By Lemma 2.1 we can pick a cross-free witness family $\mathcal{L}$ of in-sets of D. We can assume that all singleton in-sets and all in-sets whose complement is a singleton belong to $\mathcal{L}$. Let the root vertex r be an atom. This choice is possible by Theorem 2.4. Let $W = V - r$. Since r is an atom, we have $\{r\}, W \in \mathcal{L}$. Define two families as follows:

$$\mathcal{L}'_i = \{X \in \mathcal{F} : r \notin X\}$$

$$\mathcal{L}'_o = \{X : V - X \in \mathcal{F} : r \notin X\}$$

Since $\mathcal{L}$ is a cross-free witness family of in-sets, $\mathcal{L}'_i$ and $\mathcal{L}'_o$ are laminar families that satisfy conditions (c) and (d). By deleting sets from $\mathcal{L}'_i$ (resp. $\mathcal{L}'_o$) whose core is empty, we obtain a strong witness pair $(\mathcal{L}_i, \mathcal{L}_o)$ of D with root r. Note that by deleting a set whose core is empty we cannot not violate condition (d). Properties (e) and (f) follow from the choice of r and $\mathcal{L}$. •

For a strong witness pair $(\mathcal{L}_i, \mathcal{L}_o)$ we call $s(\mathcal{L}_i)$ and $s(\mathcal{L}_o)$ the *in-surplus* and the *out-surplus* of this pair, respectively.

3 Finding a large independent set of atoms

In this section we shall consider a minimally k-edge-connected digraph $D = (V, E)$ and a strong witness pair $\mathcal{L} = (\mathcal{L}_i, \mathcal{L}_o)$ of D with root r. Our goal is to show that if the size of D is large then there is a small subset $S \subset V$ such that $V - S$ is an independent set of atoms. To show this we shall improve on our count in (3) by taking into account the in-surplus and the out-surplus of the strong witness pair as well as edges which are counted several times.

An edge $e \in E$ is a *multiedge* in D (with respect to the given strong witness pair), if e enters at least two in-sets of $\mathcal{L}_i$, or leaves at least two out-sets of $\mathcal{L}_o$, or leaves an out-set of $\mathcal{L}_o$ as well as enters an in-set of $\mathcal{L}_i$. We denote the number of multiedges by $m(\mathcal{L})$. The next inequality is a sharper version of (3) that we obtain by using the equality of (2) and the fact that multiedges are counted more than once.

$$|E| \leq 2k(|V| - 1) - k(s(\mathcal{L}_i) + s(\mathcal{L}_o)) - m(\mathcal{L}) \tag{4}$$

The *in-multiplicity* of an edge $e \in E$ (denoted by $im(e)$) is the number of in-sets of $\mathcal{L}_i$ entered by e. The *out-multiplicity* is defined in a similar way and is denoted by $om(e)$. A similar counting argument shows that for any $e \in E$ we have

$$|E| \leq 2k(|V| - 1) - k(s(\mathcal{L}_i) + s(\mathcal{L}_o)) - (im(e) - 1) \tag{5}$$

Consider a laminar family $\mathcal{F}$ and $X, Y \in \mathcal{F}$. We say that Y is a *child* of X if $Y \subset X$ holds and there is no $Z \in \mathcal{F}$ with $Y \subset Z \subset X$. We say that X is a *leaf* if it has no children and we call X a *semi-leaf* if X is not a leaf but every child of X is a leaf. A *strong semi-leaf* is a semi-leaf X whose children are all singleton leaves and for which $c(X) = 1$ holds. It is easy to see that if X is an in-set in D then, since there are no multiple edges, either $|X| = 1$ or $|X| \geq k$ holds. Moreover, if $|X| = k$ then each vertex v in X has $d^-(v) = k$. These observations and the fact that $\mathcal{L}_i$ is strongly laminar imply the next lemma.

Lemma 3.1. *Let X be a leaf of $\mathcal{L}_i$. Then either $|X| = 1$ or $|X| \geq k + 1$.*

Lemma 3.2. *Let X be a strong semi-leaf in $\mathcal{L}_i$. Then at least one of the following holds:*
(a) there is an edge which enters X as well as one of the children of X,
(b) $D[X - C(X)]$ contains a circuit.

Proof: Let v be a singleton leaf in X. Since X is strong, we have $c(X) = 1 \leq k - 1$. Hence, since D is simple, either there is an edge with tail in $V - X$ and head v (in which case (a) holds) or there is an edge with tail in $X - C(X)$ and head v. If the latter holds for all leaves in X then (b) must hold. •

Lemma 3.3. *Let K be a circuit in D with $V(K) \subseteq W$. Then either*
(a) $V(K) \subseteq C(Y)$ for some $Y \in \mathcal{L}_o$, or
(b) there is an edge $e \in E(K)$ which leaves a set $Y' \in \mathcal{L}_o$.

Proof: Let $u \in V(K)$. Since $\{C(Y) : Y \in \mathcal{L}_o\}$ partitions W (by property (f)), we have that $u \in C(Y)$ for some $Y \in \mathcal{L}_o$. If $V(C)$ intersects $W - Y$ (or Y', for some child Y' of Y), then there is an edge of C which leaves Y (or Y'), and hence (b) holds. Otherwise (a) holds. •

3.1 Semi-leaves and strong semi-chains

It follows from (4) and (5) that a digraph with $|E| \geq 2k(|V| - k)$ must have $k(s(\mathcal{L}_i) + s(\mathcal{L}_o)) + m(\mathcal{L}) \leq 2k(k-1)$ and also $im(e), om(e) \leq 2k(k-1) + 1$ for all $e \in E$. This motivates the assumptions of the following lemmas.

Lemma 3.4. *Suppose that $k(s(\mathcal{L}_i) + s(\mathcal{L}_o)) + m(\mathcal{L}) \leq 2k(k-1)$. Then $\mathcal{L}_i$ has at most $2k(k-1)$ semi-leaves.*

Proof: Let p denote the number of semi-leaves in $\mathcal{L}_i$. By definition, every non-strong semi-leaf X' either contains a non-singleton leaf or has $c(X') \geq 2$. Therefore X' or a subset of X' contributes to $s(\mathcal{L}_i)$ by at least one.

Now focus on a strong semi-leaf X. By Lemma 3.2 either X is entered by a multi-edge or $D[X - C(X)]$ contains a circuit K. In the latter case it follows from Lemma 3.3 that either X contains at least two vertices from $C(Y)$ for some $Y \in \mathcal{L}_o$ or some edge in K is a multiedge (since every edge of $D[X - C(X)]$ enters a singleton in-set). In each of these cases X contributes to $s(\mathcal{L}_o) + m(\mathcal{L})$ by at least one. Since the semi-leaves are pairwise disjoint, we can add up these contributions and conclude that $p \le (s(\mathcal{L}_i)+s(\mathcal{L}_o))+m(\mathcal{L}) \le k(s(\mathcal{L}_i)+s(\mathcal{L}_o))+m(\mathcal{L}) \le 2k(k-1)$, as required. •

A decreasing sequence $X_1 \supset X_2 \supset ... \supset X_t$ of members of $\mathcal{L}_i$ is a *strong semi-chain* of $\mathcal{L}_i$ if
(i) X_{j+1} is a child of X_j, for $1 \le j \le t-1$,
(ii) $c(X_j) = 1$ for all $1 \le j \le t$, and
(iii) every member of $\mathcal{L}_i$ in $X_j - X_{j+1}$ is a singleton leaf, for all $1 \le j \le t-1$.

Lemma 3.5. *Suppose that $k(s(\mathcal{L}_i) + s(\mathcal{L}_o)) + m(\mathcal{L}) \le 2k(k-1)$ and for every edge $e \in E$ we have $im(e) \le 2k(k-1)+1$. Then the length of a strong semi-chain in $\mathcal{L}_i$ is less than $8k^4$.*

Proof: For a contradiction suppose that there is a strong semi-chain $\mathcal{X} = X_1 \supset X_2 \supset ... \supset X_r$ of $\mathcal{L}_i$ with $r \ge 8k^4$. Let $Z_j = X_j - X_{j+1}$ denote the *cell* of X_j, $1 \le j \le r-1$. Since the cells are pairwise disjoint and $m \le 2k(k-1)$, it follows that at most $4k(k-1)$ cells are incident to multiedges. Let C^* denote the union of non-singleton cores of sets of $\mathcal{L}_o$. It is easy to see that $s(\mathcal{L}_o) \ge |C^*|/2$. Since $ks(\mathcal{L}_o) \le 2k(k-1)$, this implies $|C^*| \le 4(k-1)$. Therefore at most $4(k-1)$ cells intersect C^*. Let us partition $\mathcal{X}$ into smaller chains by cutting it at every member X_i whose cell is either incident to multiedges or intersects C^*. This way we get at most $4k(k-1) + 4(k-1) + 1 = 4(k+1)(k-1)+1$ subchains.

Since $\frac{8k^4}{4(k+1)(k-1)+1} \ge 2k^2 \ge 2k(k-1)+2$, it follows that one of these subchains has length at least $2k(k-1)+2$. Thus $\mathcal{L}_i$ contains a strong semi-chain $X_l \supset X_{l+1} \supset ... \supset X_{l+2k(k-1)+1}$ of length $2k(k-1)+2$ whose cells are all disjoint from the multiedges as well as from C^*.

Claim 3.6. $D[Z_j - C(X_j)]$ *is acyclic for all $l \le j \le l+2k(k-1)$.*

Proof: Suppose that K is a circuit in $D[Z_j - C(X_j)]$. By Lemma 3.3 either there is an edge $e \in E(K)$ which leaves a member of $\mathcal{L}_o$ (in which case e is a multiedge, since it enters a singleton leaf of $\mathcal{L}_i$) or $V(K) \subseteq C(Y)$ for some $Y \in \mathcal{L}_o$ with $c(Y) \ge 2$. This is a contradiction, since Z_j is disjoint from multiedges as well as from C^*. •

Claim 3.7. *If $Z_j - C(X_j) \ne \emptyset$ then $Z_j - C(X_j)$ is an independent set of atoms for all $l+1 \le j \le l+2k(k-1)$.*

Proof: Let $\{z\} = C(X_j)$ and let $v \in Z_j - z$. It follows from the definition of strong semi-chain that v is a singleton leaf of $\mathcal{L}_i$, and hence $d^-(v) = k$ holds. We shall prove that v is a singleton leaf in $\mathcal{L}_o$ as well.

$D[Z_j - z]$ is acyclic by Claim 3.6. First suppose that v is a source in $D[Z_j - z]$. Since v is a source, D is simple, $k \ge 2$, and $c(X_j) = 1$, there is an edge uv with $u \notin Z_j$. Since Z_j is disjoint from multiedges, we must have $u \in X_{j+1}$ (otherwise uv enters two in-sets: X_j and $\{v\}$). It follows from property (f) of strong witness pairs that every edge either enters or leaves a set of $\mathcal{L}_o$ or is contained by a core of some set in $\mathcal{L}_o$. If edge uv leaves a set in $\mathcal{L}_o$ or is in the core of some set of $\mathcal{L}_o$ then Z_j intersects a multiedge or C^*, contradicting our assumption. Thus uv enters an out-set $Y \in \mathcal{L}_o$. Since $\mathcal{L}_i \cup \mathcal{L}_o$ is laminar (by property (c)), it follows that $Y \subseteq Z_j$. Since $Z_j \cap C^* = \emptyset$, there exists a vertex $y \in Y$ such that $\{y\}$ is a singleton leaf of $\mathcal{L}_o$. If $y = v$ then v is a singleton leaf in $\mathcal{L}_o$, and $d^+(v) = k$ follows, as claimed. Otherwise $y \in Y - v$. Since $k \geq 2$, at least two edges leave y. Since D is simple, there is an edge yw with $w \neq z$. Since $y \in Z_j$ and Z_j is disjoint from multiedges, yw cannot leave Y and cannot enter any vertex in $Y - z$, a contradiction. This shows that all sources in $D[Z_j - z]$ are atoms.

Hence, since Z_j is disjoint from multiedges, it follows that all vertices in $D[Z_j - C(X_j)]$ are sources, and hence all of them are atoms and there are no edges in $D[Z_j - C(X_j)]$, as required. •

Now consider the first two sets $X' = X_l$ and $X = X_{l+1}$ in the subchain. Let $B = X' - X$, $A = X - X_{l+2}$, $\{a\} = C(X)$, $\{b\} = C(X')$. Focus on the $k \geq 2$ edges entering X. These edges cannot enter $X - A$, since $X - X_{l+2k(k-1)+2}$ is disjoint from multiedges and if it enters X_j for $j \geq l + 2k(k-1) + 2$ then it has in-multiplicity at least $2k(k-1) + 2$, which would contradict our assumption. These edges cannot enter $A - a$ either, since $A - a$ consists of singleton leaves of $\mathcal{L}_i$ and $A - a$ is disjoint from multiedges. Thus these edges enter a. Now there is no edge from $B - b$ to a, since it would be a multiedge incident to A, because $B - b$ consists of atoms. So, since D is simple, at most one edge (from b to a) can come from B and at least one edge must come from $V - X'$. However, this contradicts the fact that A is disjoint from multiedges. This proves the lemma. •

Lemma 3.8. *Suppose that $k(s(\mathcal{L}_i)+s(\mathcal{L}_o))+m \leq 2k(k-1)$ and for every edge $e \in E$ we have $im(e) \leq 2k(k-1)+1$ and $om(e) \leq 2k(k-1)+1$. Then there is a set $S \subset V$ with $|S| \leq 130k^7$ such that $V - S$ is an independent set of atoms.*

Proof: Let $\mathcal{L}_i^* = \{X : X \text{ is a leaf in } \mathcal{L}_i\}$ and let $\mathcal{L}_i' = \mathcal{L} - \mathcal{L}_i^*$.

Claim 3.9. $|\mathcal{L}_i'| \leq 64k^7$.

Proof: Suppose, for a contradiction, that $|\mathcal{L}_i'| \geq 64k^7 + 1$. Clearly, $\mathcal{L}_i'$ is a laminar family and the leaves of $\mathcal{L}_i'$ are precisely the semi-leaves of $\mathcal{L}$. Thus, by Lemma 3.4, $\mathcal{L}_i'$ has at most $2k(k-1)$ leaves. Thus, by considering the natural rooted tree structure of the laminar family $\mathcal{L}_i'$, it is easy to see that $\mathcal{L}_i'$ has at most $2k(k-1)-1$ members with at least two children. Thus at least $|\mathcal{L}_i'| - 4k(k-1) + 1 \geq 64k^7 - 4k^2$ members of $\mathcal{L}_i'$ have precisely one child. By deleting those nodes of this rooted tree that correspond to leaves or sets with at least two children, we obtain a set of disjoint paths. Thus there must be a chain $\mathcal{X}$ of $\mathcal{L}_i'$ whose length is at least $\frac{64k^7-4k^2}{4k^2} \geq 16k^5 - 1$. By the hypothesis of the lemma we have $s(\mathcal{L}_i) \leq 2(k-1)$, and hence at most $2(k-1)$ members X of the chain $\mathcal{X}$ of $\mathcal{L}_i'$ can have $c(X) \geq 2$ (in $\mathcal{L}_i$) or can contain a non-singleton leaf (in $\mathcal{L}_i$). Thus there is a subchain of $\mathcal{X}$ of length at least $\frac{16k^5-1}{2(k-1)+1} \geq 8k^4$ which corresponds to a strong semi-chain in $\mathcal{L}_i$. This contradicts Lemma 3.5. •

Since $s(\mathcal{L}_i) \leq 2(k-1)$, it follows from (2) that $|\mathcal{L}_i| \geq |V|-1-2(k-1) \geq |V|-2k+1$. By Claim 3.9 we have $|\mathcal{L}'_i| \leq 64k^7$, and hence $|\mathcal{L}^*_i| \geq |V| - 64k^7 - 2k + 1$. Moreover, since $s(\mathcal{L}_i) \leq 2(k-1)$, it follows from Lemma 3.1 that $\mathcal{L}_i$ has at most one non-singleton leaf. Thus $\mathcal{L}_i$ has at least $|V| - 64k^7 - 2k$ singleton leaves. By symmetry, the same argument implies that $\mathcal{L}_o$ has at least $|V| - 64k^7 - 2k$ singleton leaves. Therefore there exist at least $|V| - 128k^7 - 4k$ atoms in D. Since an edge connecting two atoms is a multiedge, and $m(\mathcal{L}) \leq 2k(k-1)$, at most $4k(k-1)$ atoms can be connected to other atoms. So we can conclude that there is a set of independent atoms of size at least $|V| - 128k^7 - 4k - 4k(k-1) \geq |V| - 130k^7$. This proves the lemma. •

4 The upper bound and the extremal digraphs

In this section we complete the proof of our main result. To this end we first prove a lemma which can be used to extend a subgraph of D to a k-edge-connected spanning subgraph by adding a sufficiently small set of new edges.

We also need the following well-known inequality, which is easy to check by counting the contribution of an edge to the two sides.

Proposition 4.1. *Let $H = (V, E)$ be a multidigraph and let $X, Y \subseteq V$. Then*

$$d^-(X) + d^-(Y) \geq d^-(X \cap Y) + d^-(X \cup Y). \tag{6}$$

Let $D = (V, E)$ be a multidigraph and let $u, v \in V$. We use $\lambda_D(u, v)$ to denote the maximum number of pairwise edge-disjoint directed paths from u to v in D. Let $r \in V$ be a designated root vertex in D and let k be a positive integer. Let

$$od^k_{D,r}(v) = \max\{k - \lambda_D(r, v); 0\}$$

denote the "out-deficiency" of vertex v with respect to k. Let $P^k_{out}(D, r) = \sum(od^k_{D,r}(v) : v \in V - r)$. We say that D is *k-out-connected from r* if $\lambda_D(r, v) \geq k$ holds for all $v \in V - r$ (or equivalently, if $od^k_{D,r}(v) = 0$ for all $v \in V - r$). Similarly, we define $id^k_{D,r}(v) = \max\{k - \lambda_D(v, r); 0\}$ and $P^k_{in}(D, r) = \sum(id^k_{D,r}(v) : v \in V - r)$, and call D *k-in-connected from r* if $\lambda_D(v, r) \geq k$ holds for all $v \in V - r$. We shall omit some of the indices when they are clear from the context. Note that if D is simultaneously k-out-connected and k-in-connected from r then D is k-edge-connected.

Lemma 4.2. *Let $D = (V, E)$ be a k-edge-connected multidigraph, let $r \in V$ be a designated root vertex, and let $D' = (V, E')$ be a spanning subgraph of D. Then*
(a) there is a set of edges $\bar{E} \subseteq E - E'$ with $|\bar{E}| \leq P^k_{out}(D', r)$ such that $\bar{D} = (V, E' \cup \bar{E})$ is k-out-connected from r,
(b) there is a set of edges $\tilde{E} \subseteq E - E'$ with $|\tilde{E}| \leq P^k_{in}(D', r)$ such that $\tilde{D} = (V, E' \cup \tilde{E})$ is k-in-connected from r.

Proof: Consider part (a) first. Our proof is by induction on $P_{out}(D')$. If $P_{out}(D') = 0$ then D' is k-out-connected from r, and the lemma trivially holds. Now suppose $P_{out}(D') > 0$. Let $M = \max\{od_{D'}(v) : v \in V - r\}$ and let $w \in V - r$ be a vertex with $od_{D'}(w) = M > 0$. By Menger's theorem there is a set, and hence there is a maximal set $X \subseteq V - r$ with $w \in X$ and $d^-_{D'}(X) = \lambda_{D'}(r, w) = k - M$. Since D is k-edge-connected, there is an edge $e = uz \in E - E'$ with $u \in V - X$ and $z \in X$. By the maximality of M, and since X separates z and r, we must have $od_{D'}(z) = M$. Let $D'' = D' + e$. Clearly, $od_{D''}(v) \leq od_{D'}(v)$ for all $v \in V - r$.

We claim that $od_{D''}(z) = od_{D'}(z) - 1$. For a contradiction suppose that $\lambda_{D''}(r, z) = k - M$, and hence there is a set $Y \subseteq V - r$ with $z \in Y$ and $d^-_{D''}(Y) = k - M$. Since $d^-_{D'}(Y) \geq k - M$, we have $u \in Y$ and $d^-_{D'}(Y) = k - M$. By applying (6) to the pair X, Y in D', we obtain

$$k - M + k - M = d^-_{D'}(X) + d^-_{D'}(Y) \geq d^-_{D'}(X \cap Y) + d^-_{D'}(X \cup Y) \geq k - M + k - M,$$

and hence $d^-_{D'}(X \cup Y) = k - M$ follows. Since $u \in Y - X$, this contradicts the maximality of X. This proves that $od_{D''}(z) = od_{D'}(z) - 1$ holds.

Thus $P_{out}(D'') \leq P_{out}(D') - 1$. By the induction hypothesis there is a set of edges F with $|F| \leq P_{out}(D'')$ for which $(V, E' \cup \{e\} \cup F)$ is k-out-connected from r. Thus $\bar{E} = F + e$ is the required set of edges for D'.

Part (b) follows from (a) after reversing the directions of all edges in D. •

Note that Lemma 4.2 gives rise to another short proof of Theorem 2.3. To see this consider the subgraph $D' = (V, \emptyset)$ of a minimally k-edge-connected multidigraph $D = (V, E)$. By applying Lemma 4.2(a) and (b) we obtain two subsets of edges $\bar{E}$ and $\tilde{E}$ of E with $|\bar{E}|, |\tilde{E}| \leq k(|V| - 1)$ such that $D'' = (V, \bar{E} \cup \tilde{E})$ is k-edge-connected. Thus $D = D''$ must hold, and hence $|E| \leq 2k(|V| - 1)$ follows.

Theorem 4.3. *Let $D = (V, E)$ be a minimally k-edge-connected digraph with $|V| \geq (k-1)\binom{130k^7}{k}^2 + 130k^7 + 1$, where $k \geq 2$. Then $|E| \leq 2k(|V| - k)$, and equality holds if and only if $D = DK_{k,|V|-k}$.*

Proof: By Lemma 3.8 there is a set $S \subset V$ with $|S| = 130k^7$ such that $T = V - S$ is an independent set of atoms. Since $|V| \geq (k-1)\binom{|S|}{k}^2 + 130k^7 + 1$, we have $|T| = |V| - |S| = |V| - 130k^7 \geq (k-1)\binom{130k^7}{k}^2 + 1$. Since T is an independent set of atoms, and D is simple, we have $N^+(v), N^-(v) \subseteq S$ and $|N^+(v)| = |N^-(v)| = k$ for all $v \in T$. The pigeon-hole principle and our bounds on $|S|$ and $|T|$ imply that there is a set $T' \subseteq T$ with $|T'| = k$ such that $N^+(v) = N^+(w)$ and $N^-(v) = N^-(w)$ for all pairs $v, w \in T'$. Let us fix such a set T' and let us denote the common sets of out- and in-neighbours by $A = N^+(v)$ and $B = N^-(v)$, for $v \in T'$.

Let $H = D/T'$ denote the multidigraph obtained from D by contracting the set T' into a new vertex r. H is k-edge-connected, since D is k-edge-connected. We shall denote the vertex set and edge set of H by V' and E', respectively. Note that $|V'| = |V| - k + 1$. Since T' is an independent set, there is a natural bijection between the edge sets of D and H.

Claim 4.4. *Let $\bar{H} = (V', \bar{E})$ be a k-edge-connected subgraph of H. Then $\bar{D} = (V, \bar{E})$ is also k-edge-connected.*

Proof: For a contradiction suppose that $\bar{D}$ is not k-edge-connected and let $\emptyset \neq X \subset V$ be a set of vertices with $d^+(X) < k$. Since $\bar{H}$ is k-edge-connected, and by the construction of $\bar{H}$, we must have $X \cap T' \neq \emptyset$ and $T' - X \neq \emptyset$. Moreover, there is an edge from each vertex of $T' \cap X$ to each vertex of $A - X$, and there is an edge from each vertex of $B \cap X$ to each vertex of $T' - X$. Thus

$$d^+(X) \geq |T' \cap X| \cdot |A - X| + |T' - X| \cdot |B \cap X|.$$

Since $d^+(X) < k$ and $|T'| = k$, this implies that either $A \subseteq X$ or $B \cap X = \emptyset$. Consider the case when $A \subseteq X$. (The other case is similar.) If $|B \cap X| = k$ then we also have $d^+(X) \geq k$, thus we can also assume that $B - X \neq \emptyset$. Let $b \in B - X$.

Since $\bar{H}$ is k-edge-connected, there exist k edge-disjoint paths in $\bar{D}$ from T' to b. Observe that for every edge tu in $\bar{D}$ with $t \in T'$ we have $u \in A$. Thus there exist k edge-disjoint paths from A to b in $\bar{D}$. Since $A \subseteq X$ and $b \notin X$, this implies $d^+(X) \geq k$, a contradiction. This completes the proof of the claim. •

We shall consider two cases.

Case 1. $A \neq B$.

In this case we define a k-edge-connected spanning subgraph of D by constructing a k-edge-connected spanning subgraph of H, with the help of Lemma 4.2. Let F denote the set of edges incident to T' in D. This set corresponds to the set of edges incident to r in H. Since T' is a set of independent atoms, we have $|F| = 2k^2$. Consider the subgraph $H' = (V', F)$ of H.

Clearly, $\lambda_{H'}(r, v) = k$ for all $v \in A$. Thus $P_{out}(H', r) = k(|V'| - 1 - |A|) = k(|V| - 2k)$. By applying Lemma 4.2(a) to H and its subgraph H', we obtain that there is a set $\bar{E} \subseteq E' - F$ with $|\bar{E}| \leq k(|V| - 2k)$ for which $\bar{H} = (V', F \cup \bar{E})$ is k-out-connected from r.

Let $p = |B - A| \geq 1$ and let $B - A = \{b_1, b_2, ..., b_p\}$. Since $\bar{H}$ is k-out-connected from r, we have $d_{\bar{H}}(b_i) \geq k$ for $1 \leq i \leq p$. Since D is simple, $|B| = k$, and there are no edges from r to $B - A$ in H, it follows that there is an edge $e_i = w_i b_i$ in $\bar{H}$ with $w_i \in V' - B - r$ for all $1 \leq i \leq p$. Note that $e_i \in \bar{E}$ for $1 \leq i \leq p$. Let $H^* = H' + \{e_1, e_2, ..., e_p\}$. Clearly, we have $\lambda_{H^*}(v, r) = k$ for all $v \in B$, and $\lambda_{H^*}(y, r) = d^+_{H^*}(y)$ for all $y \in V' - B - r$. By the choice of the edges e_i, we have $\sum(d^+_{H^*}(y) : y \in V' - B - r) = p$. Thus $P_{in}(H^*, r) = k(|V| - 2k) - p$. By Lemma 4.2(b) this implies that there is a set $\tilde{E} \subseteq E' - (F \cup \{e_1, e_2, ..., e_p\})$ of edges with $|\tilde{E}| \leq k(|V| - 2k) - p$ such that $H^* + \tilde{E}$ is k-in-connected from r. Therefore $\hat{H} = (V', F \cup \bar{E} \cup \tilde{E})$ is a k-edge-connected subgraph of H with $|E(\hat{H})| \leq |F| + |\bar{E}| + |\tilde{E}| \leq 2k^2 + k(|V| - 2k) + k(|V| - 2k) - p = 2k(|V| - k) - p$. By Claim 4.4 it follows that D has a k-edge-connected spanning subgraph $\hat{D}$ with at most $2k(|V| - k) - p$ edges. Since D is minimally k-edge-connected, we must have $D = \hat{D}$, and hence $|E| \leq 2k(|V| - k) - p < 2k(|V| - k)$. This proves the theorem in Case 1.

Case 2. $A = B$.

As before, let F denote the set of edges incident to T'. In this case the subgraph of D induced by F is k-edge-connected. Thus, since D is minimally k-edge-connected, this implies that A is an independent set in D. Since $|E(D_{k,|V|-k})| = 2k(|V|-k)$, we may assume that $D \neq D_{k,|V|-k}$. Now we must have an edge au for some $a \in A$ for which either $N^+(u) \neq A$ or $N^+(u) = A$ and $N^-(u) \neq A$ holds. Thus either there is a path au, uz or a path zu, ua with $z, u \notin A$. By symmetry we may assume that the first alternative holds. Since D is k-edge-connected, and $k \geq 2$, there exists a path P in D from z to A not using the edge zu (this edge may or may not be an edge of D). Let W be the subgraph of D induced by the edges $E(P) \cup \{au, uz\}$. It is easy to see that $|E(W)| \leq 2|V(W) - A| - 1$ and that the subgraph $D[T' \cup A \cup V(W)]$ is strongly connected.

Let $D' = (V, F \cup E(W))$ and let $r \in T'$ be a designated root vertex. Clearly, we have $\lambda_{D'}(r, a) = \lambda_{D'}(a, r) = k$ for all $a \in A \cup (T' - r)$ and $\lambda_{D'}(r, w) = \lambda_{D'}(w, r) = 1$ for all $w \in V(W) - A$. Therefore we can deduce that $P_{out}(D') = P_{in}(D') = k(|V| - 2k) - |V(W) - A|$. By Lemma 4.2(a) and (b) this implies that there exist edge sets $\bar{E}, \tilde{E} \subseteq E - (F \cup E(W))$ with $|\bar{E}|, |\tilde{E}| \leq k(|V| - 2k) - |V(W) - A|$ such that $\bar{D} = (V, F \cup E(W) \cup \bar{E})$ is k-out-connected from r and $\tilde{D} = (V, F \cup E(W) \cup \tilde{E})$ is k-in-connected from r.

Thus $\hat{D} = (V, F \cup E(W) \cup \bar{E} \cup \tilde{E})$ is a k-edge-connected spanning subgraph of D with $|E(\hat{D})| \leq 2k^2 + |E(W)| + 2k(|V| - 2k) - 2|V(W) - A| \leq 2k(|V| - k) - 1$. Since D is minimally k-edge-connected, we must have $D = \hat{D}$, and hence $|E| < 2k(|V| - k)$ follows. This proves the theorem. •

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