Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2002-11. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

A note on hypergraph connectivity augmentation

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December 2002

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Abstract

We prove an abstract version of an edge-splitting theorem for directed hypergraphs that appeared in [1], and use this result to obtain min-max theorems on hypergraph augmentation problems that are linked to orientations. These problems include (k,l)-edge-connectivity augmentation of directed hypergraphs, and (k,l)-partition-connectivity augmentation of undirected hypergraphs by uniform hyperedges.

1 Introduction

In [1], Berg, Jackson and Jordán proved an interesting egde-splitting theorem for directed hypergraphs, which led to a solution for the problem of directed hypergraph edge-connectivity augmentation by uniform hyperarcs. In this note, we show that their edge-splitting result can be formulated in a more general form (using essentially the same proof). The result gives a method for solving a broader class of undirected and directed augmentation problems where the new hyperedges have the same prescribed size. In Section 3 we study problems where the aim is to obtain a directed hypergraph that covers a given crossing supermodular set function; this includes the problem of (k,l)-edge-connectivity augmentation. In Section 4 the objective is to obtain an undirected hypergraph that has an orientation covering a non-negative crossing supermodular set function. A notable special case is the (k,l)-partition-connectivity augmentation of undirected hypergraphs.

Let V be a finite ground set. For a function $m: V \to \mathbb{R}$ and a set $X \subseteq V$, we use the notation $m(X) := \sum_{v \in X} m(v)$. Hyperedges are considered to be multisets, so a *hyperedge* can be defined as a function $e: V \to \mathbb{Z}_+$, but we use the notations $v \in e$ for $e(v) > 0$ and $|e \cap X|$ for $e(X)$. A *hyperarc* a is a hyperedge with a designated *head node* $h(a) \in a$; the rest of its nodes are called *tail nodes* ($t(a) = a - h(a)$). An *orientation* of a hypergraph $H = (V, \mathcal{E})$ is a directed hypergraph obtained by

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designating a head node $h(e)$ for every $e \in \mathcal{E}$. A *ν-hyperedge* is a hyperedge e with $|e| = \nu$, while an $(r,1)$-hyperarc is a hyperarc a with $|t(a)| = r$. A hyperedge e *enters* a set X if $e \cap X \neq \emptyset$ and $e \cap (V - X) \neq \emptyset$, while a hyperarc a enters X if $h(a) \in X$ and $t(a) \cap (V - X) \neq \emptyset$. For a hypergraph $H = (V, \mathcal{E})$ and a directed hypergraph $D = (V, \mathcal{A})$ we define $d_H(X) := |\{e \in \mathcal{E} \mid e \text{ enters } X\}|$, $\varrho_D(X) := |\{a \in \mathcal{A} \mid a \text{ enters } X\}|$ and $\delta_D(X) = \varrho_D(V - X)$, which have the following properties:

$$d_H(X) + d_H(Y) \geq d_H(X \cap Y) + d_H(X \cup Y) \quad \text{for every } X, Y \subseteq V, \tag{1}$$
$$\varrho_D(X) + \varrho_D(Y) \geq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \quad \text{for every } X, Y \subseteq V, \tag{2}$$
$$\delta_D(X) + \delta_D(Y) \geq \delta_D(X \cap Y) + \delta_D(X \cup Y) \quad \text{for every } X, Y \subseteq V. \tag{3}$$

For a family $\mathcal{F}$ of subsets of V, we use the notation $\text{co}(\mathcal{F}) := \{V - X \mid X \in \mathcal{F}\}$. Two sets X and Y are crossing if all of $X - Y, Y - X, X \cap Y, V - (X \cup Y)$ are nonempty. A family of sets is *cross-free* if it does not contain two crossing members. Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a set function (we always assume that $p(\emptyset) = 0$). A hypergraph H (a directed hypergraph D) is said to *cover* p if $d_H(X) \geq p(X)$ ($\varrho_D(X) \geq p(X)$) for every $X \subseteq V$. The set function p is *crossing supermodular* if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \tag{4}$$

holds whenever $X \cap Y \neq \emptyset$ and $V - (X \cup Y) \neq \emptyset$.

2 Directed splitting off

A special case of the following theorem (when $p(X) = k$ for every $\emptyset \neq X \subset V$ for some positive integfer k) was proved in [1]. Here we show that a more general result can be proved with essentially the same techniques.

Theorem 2.1. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular set function, $m_i : V \to \mathbb{Z}_+$ and $m_o : V \to \mathbb{Z}_+$ degree specifications such that $m_o(V) = rm_i(V)$ for some positive integer r, and*

$$m_i(X) \geq p(X) \text{ for every } X \subseteq V, \tag{5}$$
$$m_o(V - X) \geq p(X) \text{ for every } X \subseteq V. \tag{6}$$

Then there is a directed $(r,1)$-hypergraph D such that $\delta_D(v) = m_o(v)$ and $\varrho_D(v) = m_i(v)$ for every $v \in V$, and

$$\varrho_D(X) \geq p(X) \text{ for every } X \subseteq V.$$

Proof. Consider a hyperarc a for which $m_i(h(a)) > 0$, and $m_o(v) \geq |t(a) \cap \{v\}|$ for every $v \in t(a)$. We define vectors $m_i^a : V \to \mathbb{Z}_+$, $m_o^a : V \to \mathbb{Z}_+$, and a set function $p^a : 2^V \to \mathbb{Z} \cup \{-\infty\}$ the following way: m_i^a is obtained from m_i by decreasing it by 1 on $h(a)$, m_o^a is obtained from m_o by decreasing it on the nodes of $t(a)$ by their multiplicities in $t(a)$, and p^a is obtained by decreasing p by 1 on every set entered by a. The hyperarc a can be *split off* if $m_i^a(X) \geq p^a(X)$ and $m_o^a(V - X) \geq p^a(X)$ for every $X \subseteq V$. The operation is called a *feasible $(l,1)$-splitting* if $|t(a)| = l$. Note that p^a is crossing supermodular by (2). The following lemma describes conditions when a feasible splitting is available.

Lemma 2.2. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular set function, $m_i : V \to \mathbb{Z}_+$ and $m_o : V \to \mathbb{Z}_+$ degree specifications such that $m_i(V) \leq m_o(V) \leq rm_i(V)$ for some integer r, and*

$$m_i(X) \geq p(X) \text{ for every } X \subseteq V, \tag{7}$$
$$m_o(V - X) \geq p(X) \text{ for every } X \subseteq V. \tag{8}$$

Let $u \in V$ be such that $m_i(u) > 0$. Then there is a hyperarc a with $h(a) = u$ and $|t(a)| \leq r$ that can be split off.

Proof. We can assume that $m_i(V) \geq 2$. A set X is called *in-critical* if $u \in X$ and $p(X) = m_i(X)$. The maximal in-critical sets are pairwise co-disjoint, since they intersect, and by the crossing supermodularity of p, the union of two crossing in-critical sets is in-critical. The complement of a maximal in-critical set is called a *petal*. Let $\mathcal{F}$ denote the family of maximal in-critical sets, and let $\alpha := |\mathcal{F}|$; $\mathcal{F}$ is called an *α-flower*.

Claim 2.3. $\alpha \leq r$.

Proof. Otherwise we would have

$$\sum_{X \in \mathcal{F}} p(X) = \sum_{X \in \mathcal{F}} m_i(X) > rm_i(V) \geq m_o(V) \geq \sum_{X \in \mathcal{F}} m_o(V - X),$$

which contradicts (8). □

First, suppose that $\alpha = 1$ ($a = \{u\}$ is obviously good for $\alpha = 0$), and let P be the single petal; $m_o(P) \geq m_i(V - P) > 0$. A set X is called *out-critical* if $u \notin X$ and $m_o(V - X) = p(X) > 0$; if there are no such sets, then for any $v \in P$ with $m_o(v) > 0$ the arc $a = vu$ can be split off. By the crossing supermodularity of p, the non-empty intersection of two out-critical sets is also out-critical. Since $u \notin X$ and $m_i(X) \geq m_o(V - X)$ for any out-critical set, there are no two disjoint out-critical sets, so there is a unique minimal out-critical Y. One of $P - Y$ and $Y - P$ is empty, otherwise $m_o(P - Y) + m_i(Y - P) < m_o(V - Y) + m_i(V - P) = p(Y) + p(V - P) \leq p(Y - P) + p(V - (P - Y)) \leq m_i(Y - P) + m_o(P - Y)$ would be a contradiction. Also, $m_o(Y) = m_o(V) - m_o(V - Y) \geq m_i(V) - m_o(V - Y) \geq m_i(V) - m_i(Y) > 0$, hence $m_o(Y \cap P) > 0$. Let $v \in Y \cap P$ be a node with $m_o(v) > 0$. Then the arc $a = vu$ can be split off.

If $\alpha \geq 2$, we define a by selecting as tail nodes one arbitrary node v with $m_o(v) > 0$ from each petal. We prove that a can be split off. By the construction of a, (7) holds after the splitting.

Suppose that there is a set X which violates (8) after the splitting, i.e. $m_o^a(V - X) < p^a(X)$. This means that if a enters X, then $p(X) > m_o(V - X) - |(t(a)) \cap (V - X)| + 1$ and if a does not enter X, then $u \notin X$ and $p(X) > m_o(V - X) - |(t(a)) \cap (V - X)|$. In both cases $p(X) > m_o(V - X) - |a \cap (V - X)| + 1$.

There is a petal P such that $P - X \neq \emptyset$ and $X - P \neq \emptyset$ (this is trivial if X is subset of a petal; if it is not, then any petal P is good for which $P \cap a \notin X$, and such a petal exists otherwise $m_o(V-X) = m_o^a(V-X) < p^a(X) \le p(X)$ contradicting (8)). The crossing supermodularity of p implies that

$$m_i(V-P)+m_o(V-X)-|a\cap(V-X)|+1 < p(V-P)+p(X) \le p(X-P)+p(V-(P-X)),$$

so

$$\begin{aligned} m_i(X-P)+m_o(P-X)-|a\cap(P-X)|+1 &< p(X-P)+p(V-(P-X)),\\ m_o(P-X)-|a\cap(P-X)|+1 &< p(V-(P-X)), \end{aligned}$$

which would imply that $V-(P-X)$ violates (8), since $|a\cap(P-X)| \le 1$. □

Proof of Theorem 2.1: According to Lemma 2.2 we can obtain a directed hypergraph D^* by successive feasible splitting off operations such that $\delta_{D^*}(v) = m_o(v)$, $\varrho_{D^*}(v) = m_i(v)$ for every $v \in V$, and $\varrho_{D^*}(X) \ge p(X)$ for every $X \subseteq V$. Since $m_o(V) = rm_i(V)$, $|a| \ge r+1$ holds for at least one hyperarc a of D^*. So there is a feasible $(r_1,1)$-splitting with head $h(a)$ for some $r_1 \ge r$; moreover, by Lemma 2.2 there is also a feasible $(r_2,1)$-splitting with head $h(a)$ for some $r_2 \le r$.

Lemma 2.4. *If for some $r_1 > r > r_2$ there is a feasible $(r_1,1)$-spitting and a feasible $(r_2,1)$-splitting with head u , then there is a feasible $(r,1)$-splitting with head u.*

Proof. Let a be the hyperarc obtained by the $(r_1,1)$-splitting. By induction, it suffices to show that for some $v \in t(a)$, the hyperarc a' defined by $h(a') = u$, $t(a') = t(a) - v$ gives a feasible splitting. If $a(v) \ge 2$ for some v we are ready, so suppose $a \le 1$. If $r_1 > 2$, then suppose indirectly that for every $v \in t(a)$ there is an in-critical set X_v such that $a - v \subseteq X_v$ and $v \notin X_v$. We can assume that these are maximal in-critical sets. Thus the sets $\{X_v \mid v \in t(a)\}$ form a flower with r_1 petals centered on u; but this contradicts the fact that there is a feasible $(r_2,1)$-splitting with head u.

If $r_1 = 2$, then $r_2 = 0$, so there are no in-critical sets. As we have seen, there is a unique minimal out-critical set Y with $u \notin Y$. Then $t(a) - Y = \emptyset$, otherwise the $(2,1)$-splitting would not be feasible; thus both $(1,1)$-splittings are feasible. □

We prove Theorem 2.1 by induction on $m_i(V)$. According to Lemma 2.4, there is a node $u \in V$ with $m_i(u) > 0$ for which there exists a feasible $(r,1)$-splitting at u; let a be the resulting $(r,1)$-hyperarc. By induction, there is a directed $(r,1)$-hypergraph D' that satisfies the conditions given by m_i^a, m_o^a and p^a. The directed hypergraph obtained by adding a to D' satisfies the conditions of Theorem 2.1. □

3 Directed hypergraph augmentation

As in [1], one can obtain an edge-connectivity augmentation result from Theorem 2.1, using the following theorem of Fujishige:

Theorem 3.1 (Fujishige). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular function and let*

$$B(p) := \left\{x \in \mathbb{R}^V \mid x(Z) \ge p(Z)\ \forall Z \subseteq V;\ x(V) = p(V)\right\}. \tag{9}$$

Then $B(p)$ is nonempty if and only if

$$\sum_{i=1}^{t} p(X_i) \leq p(V), \quad \sum_{i=1}^{t} p(V - X_i) \leq (t-1)p(V)$$

both hold for every partition $\{X_1, X_2, \ldots, X_t\}$ of V. Furthermore, if $B(p)$ is nonempty, then it is a base polyhedron, thus its vertices are integral. □

Theorem 3.2. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular set function. There exists a directed $(r, 1)$-hypergraph with γ hyperarcs that covers p if and only if*

$$\gamma \geq \sum_{X \in \mathcal{F}} p(X), \tag{10}$$

$$r\gamma \geq \sum_{X \in \mathcal{F}} p(V - X), \tag{11}$$

$$(|\mathcal{G}| - 1)\,\gamma \geq \sum_{X \in \mathcal{G}} p(V - X) \tag{12}$$

hold for every sub-partition $\mathcal{F}$ and for every partition $\mathcal{G}$ of V.

Proof. The necessity of the conditions can be seen easily. To prove sufficiency, one can construct degree specifications m_i and m_o that satisfy the conditions of Theorem 2.1. Let us define the set function $p' : 2^V \to \mathbb{Z}_+$ by $p'(V) = \gamma$, $p'(x) = \max\{0, p(x)\}$ for singletons and $p'(X) = p(X)$ otherwise. Note that p' is crossing supermodular. If $\mathcal{F}$ is a partition of V, then (10) implies that

$$\sum_{X \in \mathcal{F}} p'(X) \leq \sum_{X \in \mathcal{F},\ p(X) > 0} p(X) \leq \gamma,$$

and either (12), or (10) applied to sub-partitions with one class, implies that

$$\sum_{X \in \mathrm{co}(\mathcal{F})} p'(X) \leq \sum_{X \in \mathrm{co}(\mathcal{F}),\ p(X) > 0} p(X) \leq (|\mathcal{F}| - 1)\,\gamma.$$

Thus, by applying Theorem 3.1 to p', we get a nonnegative integer vector m_i s.t. $m_i(X) \geq p(X)$ for all $X \subseteq V$, and $m_i(V) = \gamma$.

To construct m_o consider a nonnegative vector satisfying $m_o(V - X) \geq p(X)$ for all $X \subseteq V$, which is minimal in the sense that for every $v \in V$ with $m_o(v) > 0$, there exists a set X for which $v \notin X$ and $m_o(V - X) = p(X)$. Choose a family $\mathcal{G} = \{X_1, X_2, \ldots, X_l\}$ of such sets with l minimal. Hence, two sets cannot cross, since we could replace them by their intersection. If the family is composed of co-disjoint sets, then

$$m_o(V) = \sum_{i=1}^{l} m_o(V - X_i) = \sum_{i=1}^{l} p(X_i) \leq r\gamma.$$

by (11). If there are two disjoint sets, X_i and X_j, then

$$m_o(V) \leq m_o(V - X_i) + m_o(V - X_j) = p(X_i) + p(X_j) \leq \gamma.$$

Now we can increase m_o on an arbitrary node to obtain $m_o(V) = r\gamma$, and apply Theorem 2.1 to construct a directed $(r,1)$-hypergraph with degrees m_i and m_o that covers p. $\square$

The following example demonstrates that condition (12) cannot be left out. Let $V = \{v_1, v_2, v_3\}$, $p(\{v_1, v_2\}) = p(\{v_1, v_3\}) = p(\{v_2, v_3\}) = 2$ and $p(X) = 0$ for the other sets, $r = 3$, $\gamma = 2$. Conditions (10) and (11) are satisfied, but (12) is not, and there is no directed $(3,1)$-hypergraph of 2 hyperarcs covering p. In the graph case (12) easily follows from (11).

A special case where (12) follows from (10) is the (k,l)-edge-connectivity augmentation of directed hypergraphs (for $l \leq k$), which is a generalization of the k-edge-connectivity augmentation problem studied in [1]. For a directed hypergraph D and a fixed $s \in V$, let $p(X) = k - \varrho_D(X)$ if $s \notin X \neq \emptyset$, $p(X) = l - \varrho_D(X)$ if $s \in X \neq V$ and $p(X) = 0$ otherwise. Let $\mathcal{F} = \{X_1, X_2, \ldots, X_t\}$ be a partition of V $(t \geq 2)$. If (10) holds, then $\sum_{X \in \text{co}(\mathcal{F})} p(X) = (t-1)l + k - \sum_{X \in \text{co}(\mathcal{F})} \varrho_D(X) \leq l + (t-1)k - \sum_{X \in \text{co}(\mathcal{F})} \delta_D(X) = l + (t-1)k - \sum_{X \in \mathcal{F}} \varrho_D(X) = \sum_{X \in \mathcal{F}} p(X) \leq \gamma$.

Corollary 3.3. *A directed hypergraph $D = (V, \mathcal{A})$ can be made (k,l)-edge-connected with γ new $(r,1)$-hyperarcs if and only if*

$$\gamma \geq \sum_{X \in \mathcal{F}} p(X),$$
$$r\gamma \geq \sum_{X \in \mathcal{F}} p(V - X)$$

hold for every subpartition $\mathcal{F}$ of V, where p is the above-defined set-function.

4 Augmentation and orientation

In this section we consider only non-negative crossing supermodular set functions. We are interested in the problem of adding ν-hyperedges to an initial undirected hypergraph, so that the resulting hypergraph has an orientation covering a given set function p. Similar problems for graphs were studied in [2]. As in that case, we first solve the degree specified problem, and then obtain a min-max formula for minimum cardinality augmentation. Some new notations are introduced to facilitate the formulation of the min-max results.

A family $\mathcal{F}$ of sets is a *composition* of a set $X \subseteq V$ if the value $\sum_{Z \in \mathcal{F}} \chi_Z(v) - \chi_X(v)$ is the same for every $v \in V$. A composition of V is called a *regular family*. For a set X and a family $\mathcal{F}$ that is a composition of X, let

$$\alpha_X(\mathcal{F}) := \sum_{Z \in \mathcal{F}} \chi_Z(v) - \chi_X(v) \quad \text{for an arbitrary } v \in V.$$

A composition of X is a *tree-composition* if it is cross-free and it contains no proper subfamily that is a partition or a co-partition of V. Tree-compositions have the following properties:

Claim 4.1. *If $\mathcal{F} \neq \emptyset$ is a tree-composition of X that is not a partition of X, then it contains a subfamily $\{Z_1, \ldots, Z_t\}$ ($t \geq 2$) of pairwise co-disjoint sets such that $\cap Z_i \subseteq X$. If $X \neq V$, then $Z_i - X \neq \emptyset$ ($i = 1, \ldots, t$).* □

For a hypergraph $H = (V, \mathcal{E})$ and a set $X \subseteq V$, let $i_H(X)$ denote the number of hyperedges $e \in \mathcal{E}$ with $e \cap (V - X) = \emptyset$. For a regular family $\mathcal{F}$ let

$$e_H(\mathcal{F}) := \alpha_\emptyset(\mathcal{F})|\mathcal{E}| - \sum_{X \in \mathcal{F}} i_H(X). \tag{13}$$

More intuitively, $e_H(\mathcal{F}) = \max\{\sum_{X \in \mathcal{F}} \varrho_{\vec{H}}(X) \mid \vec{H} \text{ is an orientation of } H\}$.

Theorem 4.2. *Let $H = (V, \mathcal{E})$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a non-negative crossing supermodular set function, and $m : V \to \mathbb{Z}_+$ a degree specification where $m(V)$ is divisible by a fixed integer $\nu \geq 2$. There exists a ν-uniform hypergraph I such that $H + I$ has an orientation covering p and $d_I(v) = m(v)$ for every $v \in V$ if and only if the following hold for every partition $\mathcal{F}$ of V:*

$$\frac{m(V)}{\nu} \geq \sum_{Z \in \mathcal{F}} p(Z) - e_H(\mathcal{F}), \tag{14}$$

$$\min_{X \in \mathcal{F}} m(V - X) \geq \sum_{Z \in \mathcal{F}} p(Z) - e_H(\mathcal{F}), \tag{15}$$

$$\min_{\mathcal{F}' \subseteq \mathcal{F},\ X = \cup \mathcal{F}'} \left(m(V - X) + (|\mathcal{F}'| - 1)\frac{m(V)}{\nu} \right) \geq \sum_{Z \in \mathcal{F}} p(V - Z) - e_H(\mathrm{co}(\mathcal{F})). \tag{16}$$

Proof. The right hand side of the inequalities is the deficiency of the hyperedges of H. The necessity of the conditions follows from the observation that the left hand side is always an upper bound on the contribution of the new hyperarcs. In (14): every new hyperarc can enter at most one set of $\mathcal{F}$; in (15): every hyperarc that enters a set of $\mathcal{F}$ must have a node in $V - X$; in (16): the number of sets of $\mathrm{co}(\mathcal{F})$ that a new hyperarc enters is at most $|\mathcal{F}'| - 1$ plus the number of nodes it has in $V - X$.

The proof of sufficiency is similar to that of the augmentation theorem in [2]. We add a new node z to the set of nodes, and for every $v \in V$ we add $m(v)$ parallel edges between v and z; the resulting hypergraph is denoted by $H' = (V', \mathcal{E}')$. Our first aim is to find an orientation $\vec{H}'$ of H' that has the following properties:

$$\varrho_{\vec{H}'}(V) = \frac{m(V)}{\nu}, \tag{17}$$

$$\varrho_{\vec{H}'}(X) \geq p(X) \quad \text{if } \emptyset \neq X \subset V, \tag{18}$$

$$\varrho_{\vec{H}'}(X + z) \geq p(X) \quad \text{if } \emptyset \neq X \subset V. \tag{19}$$

To find such an orientation, we use the following lemma (see e.g. [3]):

Lemma 4.3. *Given a hypergraph H' and a vector $x' : V' \to \mathbb{Z}_+$, there is an orientation $\vec{H}'$ of H' such that $\varrho_{\vec{H}'}(v) = x'(v)$ for every $v \in V'$ if and only if $x'(V') = |\mathcal{E}'|$ and $x'(Y) \geq i_{H'}(Y)$ for every $Y \subseteq V'$.* □

We call a vector $x : V \to \mathbb{Z}_+$ *feasible* if it is the vector of in-degrees (restricted to V) of an orientation satisfying (17)–(19). It is easy to see using Lemma 4.3 that x is feasible if and only if $x(V) = |\mathcal{E}| + \frac{m(V)}{\nu}$ and $x(Z) \geq p_m(Z)$ for every $Z \subseteq V$, where

$$p_m(X) := p(X) + i_H(X) + \left(m(X) - \frac{\nu - 1}{\nu} m(V)\right)^+ \qquad (X \subseteq V). \tag{20}$$

The set function p_m is crossing supermodular. A vector x is feasible if and only if it is an integral element of $B(p_m)$ (as defined in (9)).

Claim 4.4. *If conditions (14)-(16) are satisfied, then $B(p_m)$ is non-empty.*

Proof. By Theorem 3.1, it suffices to show that

$$\sum_{X \in \mathcal{F}} p_m(X) \leq |\mathcal{E}| + \frac{m(V)}{\nu}, \tag{21}$$

$$\sum_{X \in \mathcal{F}} p_m(V - X) \leq (|\mathcal{F}| - 1)\left(|\mathcal{E}| + \frac{m(V)}{\nu}\right) \tag{22}$$

for every partition $\mathcal{F}$. Note that $m(X) - \frac{\nu-1}{\nu} m(V)$ can be positive for at most one member of a partition. Thus (21) follows from (13), and either (15) or (14), depending on whether $\mathcal{F}$ has such a member or not. The inequality (22) follows from (13) and (16). □

By Theorem 3.1, $B(p_m)$ is a base polyhedron with integral vertices, and any such vertex x is the vector of in-degrees (restricted to V) of an orientation $\vec{H}'$ satisfying (17)–(19).

Let $m_i(v)$ be the multiplicity of the arc zv in $\vec{H}'$, $m_o(v)$ be the multiplicity of the arc vz in $\vec{H}'$, and let $\vec{H}$ denote the directed hypergraph obtained from $\vec{H}'$ by deleting the node z. Then $m_i(X) \geq p(X) - \varrho_{\vec{H}}(X)$ and $m_o(V - X) \geq p(X) - \varrho_{\vec{H}}(X)$ for every $X \subseteq V$. By (2) and the crossing supermodularity of p, the set function $q(X) := p(X) - \varrho_{\vec{H}}(X)$ is crossing supermodular. Theorem 2.1 asserts the existence of a directed $(\nu-1, 1)$-hypergraph D that covers q, and satisfies the degree specifications m_i and m_o. This means that $\vec{H} + D$ covers p, and the undirected hypergraph I that underlies D satisfies the degree specification m. Since $\vec{H} + D$ is an orientation of $H + I$, this completes the proof of Theorem 4.2. □

Given a characterization of the degree specifications that allow a good augmentation, it is often possible to deduce a characterization of the minimum number of hyperedges needed. In the present case we obtain the following theorem:

Theorem 4.5. *Let $H = (V, \mathcal{E})$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a non-negative crossing supermodular set function, and $\nu \geq 2$ an integer. There exists a ν-uniform hypergraph I with γ hyperedges such that $H + I$ has an orientation covering p if and only if*

$$\gamma\left(\nu + \alpha_X(\mathcal{F}_1) + (\nu - 1)\alpha_X(\mathcal{F}_2)\right) \geq \sum_{Z \in \mathcal{F}_1 + \mathrm{co}(\mathcal{F}_2)} p(Z) - e_H\left(\mathcal{F}_1 + \mathrm{co}(\mathcal{F}_2)\right) \tag{23}$$

whenever $\mathcal{F}_1$ and $\mathcal{F}_2$ are tree-compositions of some set $X \subseteq V$, $\mathcal{F}_1 + \mathcal{F}_2$ is cross-free, and $\alpha_X(\mathcal{F}_2) \le 0$ (i.e. either $\mathcal{F}_2$ is a partition of X, or $X = V$ and $\mathcal{F}_2 = \emptyset$).

Proof. The right hand side of (23) is the deficiency of H relative to the family $\mathcal{F}_1 + \mathrm{co}(\mathcal{F}_2)$. The number of sets of $\mathcal{F}_1$ that a new hyperarc enters is at most $\alpha_X(\mathcal{F}_1)$, plus 1 if its head is in X. The number of sets of $\mathrm{co}(\mathcal{F}_2)$ that a new hyperarc enters is at most $(\nu - 1)\alpha_X(\mathcal{F}_2)$ plus the number of tail nodes it has in X. This shows the necessity of (23). To prove sufficiency, we define for every $X \subseteq V$ and compositions $\mathcal{F}_1, \mathcal{F}_2$ of X:

$$\begin{aligned} Q_X(\mathcal{F}_1,\mathcal{F}_2) &:= \sum_{Z\in\mathcal{F}_1+\mathrm{co}(\mathcal{F}_2)} p(Z) - e_H\left(\mathcal{F}_1 + \mathrm{co}(\mathcal{F}_2)\right) - \gamma(\alpha_X(\mathcal{F}_1) + (\nu-1)\alpha_X(\mathcal{F}_2)),\\ q(X) &:= \max\{Q_X(\mathcal{F}_1,\mathcal{F}_2) :\ \mathcal{F}_1 \text{ and } \mathcal{F}_2 \text{ are tree-compositions of } X,\\ &\qquad \mathcal{F}_1 + \mathcal{F}_2 \text{ is cross-free, } \alpha_X(\mathcal{F}_2) \le 0\}. \end{aligned}$$

Condition (23) is equivalent to the inequality $\max_{X\subseteq V} q(X) \le \nu\gamma$; let us assume that this holds. We can observe that if $m : V \to \mathbb{Z}_+$ satisfies $m(X') \ge q(X')$ for every $X' \subseteq V$ and $m(V) = \nu\gamma$, then m satisfies (14)–(16). The choice where $X' = V$, $\mathcal{F}_1 = \mathcal{F}$ is a partition of V, $\mathcal{F}_2 = \emptyset$, $(\alpha_{X'}(\mathcal{F}_1) = 0,\ \alpha_{X'}(\mathcal{F}_2) = -1)$ easily yields (14), by $\gamma = \frac{m(V)}{\nu}$. With $X' = V - X$, where $\mathcal{F}_1 = \mathcal{F} - \{X\}$ is a partition of X' and $\mathcal{F}_2 = \{X'\}$, $(\alpha_{X'}(\mathcal{F}_1) = 0,\ \alpha_{X'}(\mathcal{F}_2) = 0)$, (15) follows. To obtain (16), we set $X' = V - X$, $\mathcal{F}_1 = \mathrm{co}(\mathcal{F}')$, $\mathcal{F}_2 = \mathcal{F} - \mathcal{F}'$, $(\alpha_{X'}(\mathcal{F}_1) = |\mathcal{F}_1| - 1,\ \alpha_{X'}(\mathcal{F}_2) = 0)$. Thus by Theorem 4.2 the existence of such an m implies the existence of a hypergraph I that satisfies the requirements. To prove that such an m exists, we use the properties of a set function slightly different from q:

$$q'(X) := \max\{Q_X(\mathcal{F}_1,\mathcal{F}_2) :\ \mathcal{F}_1 \text{ and } \mathcal{F}_2 \text{ are tree-compositions of } X\}.$$

Claim 4.6. *The value $Q_X(\mathcal{F}_1,\mathcal{F}_2)$ does not decrease if we remove a partition or a co-partition of V from $\mathcal{F}_1$ or $\mathcal{F}_2$.*

Proof. It is easy to see that if $X \cap Y = \emptyset$, $\mathcal{F}_1^X, \mathcal{F}_2^X$ are compositions of X, and $\mathcal{F}_1^Y, \mathcal{F}_2^Y$ are compositions of Y, then

$$Q_X(\mathcal{F}_1^X,\mathcal{F}_2^X) + Q_Y(\mathcal{F}_1^Y,\mathcal{F}_2^Y) = Q_{X\cup Y}(\mathcal{F}_1^X + \mathcal{F}_1^Y, \mathcal{F}_2^X + \mathcal{F}_2^Y). \tag{24}$$

The case $Y = \emptyset$ proves the claim, since $q(V) \le \nu\gamma$ implies that $q(\emptyset) \le 0$. □

Claim 4.7. *The set function q' is fully supermodular.*

Proof. Let $X, Y \subseteq V$, and suppose that the maximum in the definition of q' is reached on families $\mathcal{F}_1^X, \mathcal{F}_2^X$, and $\mathcal{F}_1^Y, \mathcal{F}_2^Y$, respectively. Let $\mathcal{F}_1 := \mathcal{F}_1^X + \mathcal{F}_1^Y$, $\mathcal{F}_2 := \mathcal{F}_2^X + \mathcal{F}_2^Y$. We apply the following operations, as long as any of them is possible:

- If $Z_1, Z_2 \in \mathcal{F}_1$ are crossing, then replace them in $\mathcal{F}_1$ by $Z_1 \cap Z_2$ and $Z_1 \cup Z_2$.
- If $Z_1, Z_2 \in \mathcal{F}_2$ are crossing, then replace them in $\mathcal{F}_2$ by $Z_1 \cap Z_2$ and $Z_1 \cup Z_2$.

It is easy to see that after a finite number of steps, the resulting families $\mathcal{F}_1'$ and $\mathcal{F}_2'$ become cross-free. Then $\mathcal{F}_i'$ decomposes into a composition $\mathcal{F}_i^{X\cap Y}$ of $X\cap Y$ and a composition $\mathcal{F}_i^{X\cup Y}$ of $X\cup Y$ $(i=1,2)$; and all of these families are cross-free. The crossing supermodularity of p implies that $\sum_{Z\in\mathcal{F}_1^X+\mathcal{F}_1^Y} p(Z) \leq \sum_{Z\in\mathcal{F}_1^{X\cap Y}+\mathcal{F}_1^{X\cup Y}} p(Z)$ and $\sum_{Z\in\mathcal{F}_2^X+\mathcal{F}_2^Y} p(V-Z) \leq \sum_{Z\in\mathcal{F}_2^{X\cap Y}+\mathcal{F}_2^{X\cup Y}} p(V-Z)$. It is easy to check that $e_H(\mathcal{F}_1^X + \mathrm{co}(\mathcal{F}_2^X)) + e_H(\mathcal{F}_1^Y + \mathrm{co}(\mathcal{F}_2^Y)) \geq e_H(\mathcal{F}_1^{X\cap Y} + \mathrm{co}(\mathcal{F}_2^{X\cap Y})) + e_H(\mathcal{F}_1^{X\cup Y} + \mathrm{co}(\mathcal{F}_2^{X\cup Y}))$, and that $\alpha_X(\mathcal{F}_i^X) + \alpha_Y(\mathcal{F}_i^Y) = \alpha_{X\cap Y}(\mathcal{F}_i^{X\cap Y}) + \alpha_{X\cup Y}(\mathcal{F}_i^{X\cup Y})$ $(i=1,2)$. Hence $Q(\mathcal{F}_1^X,\mathcal{F}_2^X) + Q(\mathcal{F}_1^Y,\mathcal{F}_2^Y) \leq Q(\mathcal{F}_1^{X\cap Y},\mathcal{F}_2^{X\cap Y}) + Q(\mathcal{F}_1^{X\cup Y},\mathcal{F}_2^{X\cup Y})$; using Claim 4.6, we obtain that $q'(X)+q'(Y) \leq q'(X\cap Y) + q'(X\cup Y)$. □

Claim 4.7 and Theorem 3.1 imply that there exists a vector $m: V\to\mathbb{Z}_+$ with $m(V) = \nu\gamma$ that satisfies $m(X)\geq q'(X)$ for every $X\subseteq V$ if and only if $\max_{X\subseteq V} q'(X) \leq \nu\gamma$.

Claim 4.8. *If condition (23) holds, then* $\max_{X\subseteq V} q'(X) = \max_{X\subseteq V} q(X) \leq \nu\gamma$.

Proof. Let X be the set where the maximum is reached for q', and let $\mathcal{F}_1,\mathcal{F}_2$ be tree-compositions of X for which $q'(X) = Q_X(\mathcal{F}_1,\mathcal{F}_2)$. We transform $\mathcal{F}_1$ and $\mathcal{F}_2$ using the following operations until none of them is applicable:

- If $Z_1,Z_2\in\mathcal{F}_1$ are crossing, then replace Z_1,Z_2 by $Z_1\cap Z_2, Z_1\cup Z_2$ in $\mathcal{F}_1$.
- If $Z_1,Z_2\in\mathcal{F}_2$ are crossing, then replace Z_1,Z_2 by $Z_1\cap Z_2, Z_1\cup Z_2$ in $\mathcal{F}_2$.
- If $\mathcal{F}_2$ is a partition of some $Z\subseteq V$, and $Z_1\in\mathcal{F}_1$ and $Z_2\in\mathcal{F}_2$ are crossing, then replace Z_1 by Z_1-Z_2 in $\mathcal{F}_1$, and replace Z_2 by Z_2-Z_1 in $\mathcal{F}_2$.
- If $\{Z_1,\dots,Z_t\}\subset\mathcal{F}_1$ or $\{Z_1,\dots,Z_t\}\subset\mathcal{F}_2$ is a partition or a co-partition of V, then remove $Z_1,\dots,Z_t$ from that family.
- If $\mathcal{F}_2$ is a composition of $Z\subseteq V$ and it contains a subfamily $\{Z_1,\dots,Z_t\}$ $(t\geq 2)$ of pairwise co-disjoint sets such that $\emptyset\neq\cap Z_i\subseteq Z$, then remove $Z_1,\dots,Z_t$ from $\mathcal{F}_2$, and add $V-Z_1,\dots,V-Z_t$ to $\mathcal{F}_1$.

It is easy to see that this terminates after a finite number of steps. We denote by $\mathcal{F}_1'$ and $\mathcal{F}_2'$ the families obtained at the end of the process. Then, by Claim 4.1, $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are tree-compositions of some $X'\subseteq X$, $\alpha_{X'}(\mathcal{F}_2')\leq 0$, and $\mathcal{F}_1'+\mathcal{F}_2'$ is cross-free. Moreover, $Q_X(\mathcal{F}_1,\mathcal{F}_2)$ does not decrease in any of the steps (in the first 3 cases this follows from the supermodularity of p, in the 4th it is a consequence of Claim 4.6, and in the 5th it is obvious from the definition of $Q_X(\mathcal{F}_1,\mathcal{F}_2)$). This proves that $\max_{X\subseteq V} q'(X) = \max_{X\subseteq V} q(X)$. □

By Claim 4.8, $\nu\gamma\geq\max_{X\subseteq V} q'(X)$. Thus there exists a vector $m: V\to\mathbb{Z}_+$ with $m(V)=\nu\gamma$ that satisfies (14)–(16), therefore by Theorem 4.2 there exists a ν-uniform hypergraph I with γ hyperedges such that $H+I$ has an orientation covering p. This concludes the proof of Theorem 4.5. □

If the requirement function is monotone decreasing (i.e. $p(X)\geq p(Y)$ if $\emptyset\neq X\subseteq Y$), or symmetric, then the conditions of Theorem 4.2 and Theorem 4.5 can be simplified.

Theorem 4.9. *Let $H = (V, \mathcal{E})$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a monotone decreasing or symmetric non-negative crossing supermodular set function, and $m : V \to \mathbb{Z}_+$ a degree specification where $m(V)$ is divisible by a fixed integer $\nu \geq 2$. There exists a ν-uniform hypergraph I with degree-specification m such that $H + I$ has an orientation covering p if and only if the following hold for every partition $\mathcal{F}$ of V:*

$$\frac{m(V)}{\nu} \geq \sum_{Z \in \mathcal{F}} p(Z) - e_H(\mathcal{F}), \tag{25}$$

$$\min_{X \in \mathcal{F}} m(V - X) \geq \sum_{Z \in \mathcal{F}} p(Z) - e_H(\mathcal{F}). \tag{26}$$

Proof. By definition, $e_H(\mathcal{F}) \leq e_H(\text{co}(\mathcal{F}))$ for every partition $\mathcal{F}$ of V, and the monotonicity or symmetry of p implies that $\sum_{Z \in \text{co}(\mathcal{F})} p(Z) \leq \sum_{Z \in \mathcal{F}} p(Z)$ also holds. It is easy to see from this that (16) is implied by (14) if $|\mathcal{F}'| = 0$ or $|\mathcal{F}'| \geq 2$, and it is implied by (15) if $|\mathcal{F}'| = 1$. □

Theorem 4.10. *Let $H = (V, \mathcal{E})$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a monotone decreasing or symmetric non-negative crossing supermodular set function, and $\nu \geq 2$ an integer. There exists a ν-uniform hypergraph I with γ hyperedges such that $H + I$ has an orientation covering p if and only if the following hold:*

$$\gamma \geq \sum_{Z \in \mathcal{F}} p(Z) - e_H(\mathcal{F}) \text{ for every partition } \mathcal{F}, \tag{27}$$

$$\nu\gamma \geq \sum_{Z \in \mathcal{F}_1 + \text{co}(\mathcal{F}_2)} p(Z) - e_H(\mathcal{F}_1 + \text{co}(\mathcal{F}_2)) \tag{28}$$

whenever $\mathcal{F}_1$ and $\mathcal{F}_2$ are partitions of some $X \subseteq V$ and $\mathcal{F}_1$ is a refinement of $\mathcal{F}_2$.

Proof. It suffices to show that if condition (23) is violated for some pair $(\mathcal{F}_1, \mathcal{F}_2)$, then it is also violated by a pair $(\mathcal{F}'_1, \mathcal{F}'_2)$ that has the additional properties that $\mathcal{F}'_1$ is a partition of some $X' \subseteq V$, and $Y_2 \not\subseteq Y_1$ for every $Y_1 \in \mathcal{F}'_1$, $Y_2 \in \mathcal{F}'_2$. Such families can be obtained from $\mathcal{F}_1$ and $\mathcal{F}_2$ by repeating the following operations as long as any of them is possible:

- If $\mathcal{F}_1$ is a composition of $Z \subseteq V$, it contains a subfamily $\{W_1, \dots, W_s\}$ $(s \geq 2)$ of pairwise co-disjoint sets such that $W := \cap W_i \subseteq Z$, and $\mathcal{F}_2$ contains a partition $\{Z_1, \dots, Z_t\}$ of W, then remove $W_1, \dots, W_s$ from $\mathcal{F}_1$ and $Z_1, \dots, Z_t$ from $\mathcal{F}_2$.
- If $\mathcal{F}_1$ is a composition of $Z \subseteq V$ and it contains a set $W \subseteq Z$ such that $\mathcal{F}_2$ contains a partition $\{Z_1, \dots, Z_t\}$ of W, then replace W in $\mathcal{F}_1$ by the sets $Z_1, \dots, Z_t$, and replace $Z_1, \dots, Z_t$ in $\mathcal{F}_2$ by W.

After a finite number of steps, none of the above operations are applicable; let $(\mathcal{F}'_1, \mathcal{F}'_2)$ be the pair obtained at that point. Then it folllows from Claim 4.1 that $\mathcal{F}'_1$ must be a partition of some $X' \subseteq V$, and $\mathcal{F}'_1$ is a refinement of $\mathcal{F}'_2$ if $\mathcal{F}'_2 \neq \emptyset$. If

(27) holds, then the value $Q_X(\mathcal{F}_1, \mathcal{F}_2)$ does not decrease during the above two operations: in the first case this follows from (24), since $Q_W(\{W_1, \dots, W_s\}, \{Z_1, \dots, Z_t\}) \leq Q_\emptyset(\{V - W_1, \dots, V - W_s, Z_1, \dots, Z_t\}, \emptyset) \leq 0$; in the second case, it follows because $p(W) + \sum_{i=1}^t p(V - Z_i) \leq p(V - W) + \sum_{i=1}^t p(Z_i)$. □

The above results have a straightforward application concerning the partition-connectivity augmentation of undirected hypergraphs. A hypergraph H is called (k,l)-*partition-connected* for non-negative integers $k \geq l$ if $e_H(\mathcal{F}) \geq (|\mathcal{F}| - 1)k + l$ for every partition $\mathcal{F}$. These hypergraphs have the following characterization:

Lemma 4.11 ([3]). *A hypergraph is (k,l)-partition-connected if and only if it has a (k,l)-edge-connected orientation.* □

From Lemma 4.11 and Theorems 4.9 and 4.10 we obtain the following corollaries:

Corollary 4.12. *Let $H = (V, \mathcal{E})$ be a hypergraph, $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ divisible by a fixed integer $\nu \geq 2$, and $k \geq l$ non-negative integers. There exists a ν-uniform hypergraph I such that $H + I$ is (k,l)-partition-connected and $d_H(v) = m(v)$ for all $v \in V$ if and only if the following hold for every partition $\mathcal{F}$ of V:*

$$\frac{m(V)}{\nu} \geq (|\mathcal{F}| - 1)k + l - e_H(\mathcal{F}) \ , \tag{29}$$

$$\min_i m(V - X_i) \geq (|\mathcal{F}| - 1)k + l - e_H(\mathcal{F}) \ . \tag{30}$$

Corollary 4.13. *Let $H = (V, \mathcal{E})$ be a hypergraph, $\nu \geq 2$ and $k \geq l$ non-negative integers. There is a ν-uniform hypergraph I with γ edges such that $H + I$ is (k,l)-partition-connected if and only if the following two conditions are met:*

1. *$\gamma \geq (|\mathcal{F}| - 1)k + l - e_H(\mathcal{F})$ for every partition $\mathcal{F}$,*
2. *$\nu\gamma \geq |\mathcal{F}_1|k + |\mathcal{F}_2|l - e_H(\mathcal{F}_1 + \mathrm{co}(\mathcal{F}_2))$ whenever $\mathcal{F}_1$ and $\mathcal{F}_2$ are partitions of some $X \subseteq V$ and $\mathcal{F}_1$ is a refinement of $\mathcal{F}_2$.*

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