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Abstract

In this note, we study a constrained independent set problem for matroids and certain generalizations. The basic problem can be regarded as an ordered version of the matroid parity problem. By a reduction of this problem to matroid intersection, we prove a min-max formula. Studying the weighted case and a delta-matroid generalization, we prove that some of them are not more complex than matroid intersection, but others are as hard as matroid parity. We show how earlier results of Hefner and Kleinschmidt on so called MS-matchings fit in our framework. We also point out another connection to electric networks.

Keywords: matroid, delta-matroid, supermodular function, matroid parity

1 Introduction

In this note, we shall study the following constrained independent set problem for matroids. Let $\mathbf{M} = (V, \mathcal{I})$ be a matroid with $|V|$ even and Π be the partition of V into ordered pairs. An *ideal independent set* is an independent set $I \in \mathcal{I}$ that satisfies the constraint:

$$\text{if } (u, v) \in \Pi \text{ and } u \in I, \text{ then } v \in I \ . \tag{1}$$

Our basic problem, the *ordered matroid parity problem* is to find a maximum cardinality ideal independent set. We shall show that the ordered matroid parity problem can be reduced to matroid intersection, that is to the problem of maximizing the size of a common independent set of two matroids.

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The ordered matroid parity problem looks similar to the matroid parity problem, i.e., the problem of finding a maximum size independent set I of $\mathbf{M}$ so that

$$\text{if } (u, v) \in \Pi, \text{ then } u \in I \text{ if and only if } v \in I\ . \tag{2}$$

In contrast to our ordered version, the matroid parity problem includes NP-hard problems. It is even known to be intractable with an ordinary oracle model of matroids, although it is solvable in polynomial time for linearly represented matroids [9].

If $\mathbf{M}$ is the transversal matroid of some graph $G = (U, V; E)$, then the ordered matroid parity problem is equivalent to the problem of finding a matching M of G that has the property that for any $(u, v) \in \Pi$, the vertex v is covered by M whenever u is covered by M. This is a special case of the so-called MS-matching problem introduced by Hefner and Kleinschmidt [8].

The MS-matching problem involves a graph G and a digraph D on the common vertex set W. The task that was originated from a practical manpower scheduling problem was to find a maximum cardinality matching M of G so that no arc of D leaves the set of vertices that are covered by M. It was shown in [8] that an NP-complete version of the satisfiability problem can be reduced to a restricted MS-matching problem in which each weak component of D has size at most three. They also proved that if all weak components of dependence graph D have size at most two, then even the edge-weighted MS-matching problem can be solved in polynomial time. Furthermore, for the above bipartite special case, Hefner [7] has found a min-max formula. We shall derive this min-max formula from a more general result on ordered parity problem.

It is interesting to observe that the ordered matroid parity problem is also present in the theory of electric networks. In [10], Recski considered the problem of unique solvability of networks that consist of voltage and current sources, resisters, capacitors, inductors, and 2-ports. There is a known necessary and sufficient condition for the unique solvability of such a network in the "general case", and Recski proved that if no 2-port is in a so-called critical situation, then the condition for the general case applies for the above networks. However, a 2-port can belong to five different types of critical situations. As Recski remarked, two of these critical situations are related to the matroid partition problem, one other situation to the matroid parity problem, and the remaining two situations are related to our basic problem. Recski mentioned that Bland had reduced this latter problem to linear matroid parity, which is solvable in polynomial time. As a result of our present work, it follows that a reduction of the problem to the easier matroid intersection problem is also possible.

Our work is organized as follows. In Section 2, we prove a min-maxformula on the ordered matroid parity problem. We also solve the weighted ordered matroid parity problem in a special case. Then we show that the min-max formula extends that of Hefner in Section 3. Section 4 is devoted to a generalized matroid problem that we can also reduce to matroid intersection. Section 5 contains a generalization of the ordered matroid parity problem for delta-matroids. At last, in Section 6, we indicate some other generalizations that are as hard as matroid parity.

2 Ordered Parity Problem

Let us fix a matroid $\mathbf{M}$ and a set of ordered pairs Π for the ordered matroid parity problem. Let S be the set of all the second elements and R be the set of all first elements in the pairs in Π. For any $v \in S$, let $\overline{v}$ denote its mate, i.e., $(\overline{v}, v)$ is a pair in Π. For any subset X of S, we denote $\{\overline{v} : v \in X\}$ by $\overline{X}$.

Let $\mathcal{J}$ denote the family of all the ideal independent sets. The weighted ordered matroid parity problem is to find for a given weight function $w : V \to \mathbb{R}_+$ an ideal independent set I that is of maximum weight, i.e., we look for $\max\{w(I) : I \in \mathcal{J}\}$ where $w(I) := \sum_{v \in I} w(v)$. A weight function $w : V \to \mathbb{R}_+$ is called *consistent* if $w(u) \leq w(v)$ holds for any pair of $u \in R$ and $v \in S$. Clearly, the ordered matroid parity problem is a weighted ordered matroid parity problem for the consistent weight function $w = \mathbf{1}$.

Lemma 2.1. *If, in the framework of the ordered matroid parity problem, weight function w is consistent, then there is an optimal solution I of the weighted ordered matroid parity problem that contains a base of S.*

Proof. Suppose $I \in \mathcal{J}$ is an optimal solution that satisfies $|I \cap S| < r(S)$, and let J be an arbitrary base of S containing $I \cap S$. For any $v \in J \backslash I$, if $I \cup \{v\} \in \mathcal{I}$, then $I \cup \{v\} \in \mathcal{J}$, which contradicts the optimality of I. Therefore, $I \cup \{v\}$ is not independent, and there exists an element $u \in I \backslash S$ such that $I' = I \cup \{v\} \backslash \{u\} \in \mathcal{I}$. Since w is consistent, we have $w(I') \geq w(I)$. Then I' is another optimal solution with $|I' \cap S| = |I \cap S| + 1$. Thus, we may assert there is an optimal solution I° that satisfies $|I^\circ \cap S| = r(S)$. □

Let $\mathbf{M} \cdot S$ denote the restriction of $\mathbf{M}$ to S, and $\mathbf{M}/S$ denote the contraction of $\mathbf{M}$ by S. Lemma 2.1 implies that the ordered parity problem for $\mathbf{M}$ can be reduced to an ordered parity problem for the direct sum $\mathbf{M}^\circ = \mathbf{M} \cdot S \oplus \mathbf{M}/S$. We now reduce this problem to the matroid intersection problem.

Let $J \subseteq S$ be an independent set in $\mathbf{M} \cdot S$ such that $\overline{J}$ is also independent in $\mathbf{M}/S$. Let K be an arbitrary base in $\mathbf{M} \cdot S$ containing J. Then $I = \overline{J} \cup K$ satisfies the ordered parity condition. Conversely, an optimal solution I for $\mathbf{M}^\circ$ must be in this form. Note that $|I| = r(S) + |J|$ holds independently of the choice of K. Therefore, an optimal solution I of the ordered parity problem can be obtained by finding a maximum cardinality J, which is the matroid intersection problem. The following min-max theorem follows from the matroid intersection theorem of Edmonds [4].

Theorem 2.2. *For the ordered parity problem, we have*

$$\max\{|I| : I \in \mathcal{J}\} = \min\{r(X) + r(V \backslash \overline{X}) : X \subseteq S\}. \tag{3}$$

Proof. The matroid intersection theorem implies that the maximum cardinality of J is given by

$$\begin{aligned} |J| &= \min\{r(X) + r_S(R \backslash \overline{X}) : X \subseteq S\} \\ &= \min\{r(X) + r(V \backslash \overline{X}) : X \subseteq S\} - r(S), \end{aligned}$$

where r_S is the rank function of $\mathbf{M}/S$. Since $|I| = |J| + r(S)$, we obtain (3). □

The weighted ordered matroid parity problem for particular weight $w = \chi^R$ is exactly the NP-hard matroid parity problem. But if weight w is consistent, then by Lemma 2.1, the weighted ordered matroid parity problem for $\mathbf{M}$ can be reduced to the weighted ordered matroid parity problem for the above $\mathbf{M}^\circ$. In what follows, we reduce this latter problem to weighted matroid intersection.

Define matroid $\mathbf{M}_R := \mathbf{M}/S \oplus \mathbf{F}_S$ where $\mathbf{F}_S$ is the free matroid on S, i.e. each element of S is a coloop in $\mathbf{M}_R$. Let $\mathbf{M}_S$ be the matroid on V so that elements v and $\overline{v}$ are parallel and $\mathbf{M}_S \cdot S = \mathbf{M} \cdot S$. Define weight function w' on V by $w'(s) := w(s)$ and $w'(\overline{s}) := w(s) + w(\overline{s})$ for all $s \in S$.

Let K be a base of $\mathbf{M} \cdot S$ and $J \subseteq K$. Clearly, if $I = \overline{J} \cup K$ is an independent set of $\mathbf{M}^\circ = \mathbf{M} \cdot S \oplus \mathbf{M}/S$, then $I' := I \setminus J$ is a common independent set of $\mathbf{M}_R$ and $\mathbf{M}_S$ with $w(I) = w'(I')$. On the other hand, if $I' = \overline{J} \cup L$ is a common independent set of $\mathbf{M}_R$ and $\mathbf{M}_S$ for $J, L \subseteq S$, then J and L are disjoint and $I := I' \cup J$ is an independent set of $\mathbf{M}^\circ$ satisfying (1) and $w'(I') = w(I)$. That is, if I' is a maximum w'-weight common independent set of $\mathbf{M}_R$ and $\mathbf{M}_S$, then I is a maximum w-weigth independent set of $\mathbf{M}^\circ$ with (1). Note that the above argument did not use the consistency of w.

As the maximum weight matroid intersection problem can be solved in polynomial time, we have the following theorem.

Theorem 2.3. *The weighted ordered matroid parity problem for consistent weight can be solved in polynomial time.* □

3 MS-matchings

In this section, we derive the min-max theorem of Hefner on bipartite MS-matchings from Theorem 2.2.

Let $G = (U, V; E)$ be a bipartite graph with the vertex set $W = U \cup V$ and the edge set E. Suppose the vertex set V is of even cardinality and partitioned into ordered pairs Π, and just like in Section 2, R and S denote the set of first and second elements of pairs in Π, respectively. For a subset $M \subseteq E$, we denote by ∂M the set of vertices covered by M. A matching M in G is called an MS-matching if $\overline{v} \in \partial M$ implies $v \in \partial M$ for every ordered pair $(\overline{v}, v) \in \Pi$. The problem of finding a maximum cardinality MS-matching in G is nothing but an ordered parity problem for the transversal matroid on V.

An *MS-cover* is a vector $y \in \mathbf{Z}^W$ that satisfies

$$y(u) + y(v) \geq 1, \qquad \forall (u, v) \in E, v \in R, \tag{4}$$
$$y(u) + y(v) - y(\overline{v}) \geq 1, \qquad \forall (u, v) \in E, v \in S. \tag{5}$$

The *value of an MS-cover* y is defined by

$$\mathrm{val}(y) = \sum_{u \in U} y(u) + \sum_{v \in S} y(v).$$

Then it is easy to see that $|M| \leq \mathrm{val}(y)$ for any pair of an MS-matching M and an MS-cover y. Hefner [7] showed that the equality holds for an optimal pair of M and y.

Theorem 3.1 (Hefner[7]). *The maximum cardinality of an MS-matching is equal to the minimum value of an MS-cover.* □

We prove this by applying Theorem 2.2 to the transversal matroid. For $Y \subseteq V$, let $\Gamma(Y)$ denote the set of vertices in U adjacent to Y. The rank function τ of the transversal matroid is given by

$$\tau(X) = \min\{\Gamma(Y) - |Y| \mid Y \subseteq X\} + |X|. \tag{6}$$

Let M be a maximum MS-matching. Theorem 2.2 asserts that there exists a subset $X \subseteq S$ such that $|M| = \tau(X) + \tau(V\backslash\overline{X})$. Since $|M| \leq \mathrm{val}(y)$ holds for any MS-cover y, the follwoing lemma completes the proof of Theorem 3.1.

Lemma 3.2. *For any $X \subseteq S$, there exists an MS-cover y such that* $\mathrm{val}(y) = \tau(X) + \tau(V\backslash\overline{X})$.

Proof. Let Y be the unique minimal minimizer that determines $\tau(X)$ in the right-hand side of (6). Similarly, let Z be a minimizer that determines $\tau(V\backslash\overline{X})$. Then we claim that $Y \subseteq Z$. Note that $|\Gamma(X)| + |\Gamma(Y)| \geq |\Gamma(Y \cap Z)| + |\Gamma(Y \cup Z)|$ and $|Y| + |Z| = |Y \cap Z| + |Y \cup Z|$ hold. Since $|\Gamma(Z)| - |Z| \leq |\Gamma(Y \cup Z)| - |Y \cup Z|$, we have $|\Gamma(Y \cap Z)| - |Y \cap Z| \leq |\Gamma(Y)| - |Y|$, which implies $Y \subseteq Z$ by the minimality of Y.

We now construct an MS-cover y. For each $u \in U$, we assign

$$y(u) = \begin{cases} 2 & \text{if} \quad u \in \Gamma(Y), \\ 1 & \text{if} \quad u \in \Gamma(Z)\backslash\Gamma(Y), \\ 0 & \text{if} \quad u \in U\backslash\Gamma(Z). \end{cases}$$

For each $v \in R$, we also assign $y(v)$ by

$$y(v) = \begin{cases} 1 & \text{if} \quad v \in R\backslash Z, \\ 0 & \text{if} \quad v \in R \cap Z. \end{cases}$$

For each $v \in S$, we assign $y(v) = z(v) + y(\overline{v})$, where $z(v)$ is defined by

$$z(v) = \begin{cases} 1 & \text{if} \quad v \in S\backslash Z, \\ 0 & \text{if} \quad v \in Z\backslash Y, \\ -1 & \text{if} \quad v \in Y. \end{cases}$$

Then the resulting $y \in \mathbf{Z}^W$ is an MS-cover and its value is given by

$$\begin{aligned} \mathrm{val}(y) &= |\Gamma(Y)| + |\Gamma(Z)| + |R\backslash Z| + |S\backslash Z| - |Y| \\ &= |\Gamma(Y)| + |\Gamma(Z)| + |V\backslash\overline{X}| + |X| - |Z| - |Y| \\ &= \tau(X) + \tau(V\backslash\overline{X}). \end{aligned}$$

Thus we obtain an MS-cover y with $\mathrm{val}(y) = \tau(X) + \tau(V\backslash\overline{X})$. □

4 Supermodular Functions

In this section, we study another generalization of our basic ordered matroid parity problem. Let $\mathbf{M}$ be a matroid on ground set V and let (S, R) be a 2-partition of V. Further, let $\mathcal{D}$ be a *ring family* on V. That is, $\mathcal{D}$ is a subset of 2^V with $\emptyset \in \mathcal{D}$ so that the intersection and the union of any two members of $\mathcal{D}$ belongs to $\mathcal{D}$.

A function $p : \mathcal{D} \to \mathbb{R}$ is *supermodular* if $p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y)$ for any members X, Y of $\mathcal{D}$. Given an integral supermodular function p on $\mathcal{D}$ with $p(\emptyset) = 0$, we call a subset J of V *balanced* if

$$|J \cap X \cap S| - |J \cap X \cap R| \geq p(X) \text{ for any member } X \text{ of } \mathcal{D}. \tag{7}$$

The *balanced base problem* is the decision problem whether there exists a basis of $\mathbf{M}$ that is balanced. In the *maximum balanced independent set problem* we have to find balanced independent set of M of maximum size. Let us denote the family of balanced independent sets by $\mathcal{J}$.

Observe that the ordered matroid parity problem is a special case of the maximum balanced independent set problem. Namely, if $p = \mathbf{0}$ and $\mathcal{D} := \{Z \cup \overline{Z} : Z \subseteq S\}$, then a maximum balanced independent set is exactly an independent set with property (1). If a weight function $w : V \to \mathbb{R}_+$ is given, then we can talk about the maximum weight balanced independent set problem.

Similarly to Lemma 2.1, we have the following observation.

Lemma 4.1. *Let $\mathbf{M} = (V, \mathcal{I})$ be a matroid and let (S, R) be a 2-partition of V. Let moreover $\mathcal{D}$ be a ring family on V, w be a consistent weight function on V and p be supermodular on $\mathcal{D}$. If there exists a balanced independent set, then there exists a maximum w-weight balanced independent set I of $\mathbf{M}$ so that $I \cap S$ spans S.*

Proof. Let I be a maximum weight balanced independent set of $\mathbf{M}$ with $|I \cap S|$ maximum. As $\mathcal{J}$ is nonempty and finite, I is well defined. If I does not span element v of S, then there is an element u of R so that $I' := I \cup \{v\} \setminus \{u\}$ is independent in $\mathbf{M}$. Clearly, $w(I') \geq w(I)$, by the consistency of w. As I was balanced, (7) holds for I', contradicting the choice of I. □

A function $b : \mathcal{D} \to \mathbb{R}$ is *submodular* if $-b$ is supermodular. For a pair of sub- and supermodular functions, we have the following discrete separation theorem [5]. See also [6, Theorem 4.12].

Lemma 4.2. *Let V be a finite set, $b : 2^V \to \mathbb{R}$ be a submodular function and $p : \mathcal{D} \to \mathbb{R}$ be a supermodular function on a ring family $\mathcal{D} \subseteq 2^V$ with $b(\emptyset) = p(\emptyset) = 0$. If $p(X) \leq b(X)$ for any $X \in \mathcal{D}$, then there is a vector $y : V \to \mathbb{R}$ such that $p(X) \leq y(X)$ for any $X \in \mathcal{D}$ and $y(X) \leq b(X)$ for any $X \subseteq V$. If p and b are integer-valued, then y can be chosen integral.* □

Lemma 4.3. *Let $\mathbf{M} = (V, \mathcal{I})$ be a matroid and let (S, R) be a 2-partition of V. Moreover, let $\mathcal{D}$ be a ring family on V and p be a supermodular function on $\mathcal{D}$ with $p(\emptyset) = 0$. There exists a balanced independent set if and only if*

$$p(X) \leq r(X \cap S) \tag{8}$$

holds for any member X of $\mathcal{D}$.

Proof. If I is a balanced independent set, then for every member X of $\mathcal{D}$

$$r(X \cap S) \geq |I \cap X \cap S| \geq |I \cap X \cap S| - |I \cap X \cap R| \geq p(X)$$

by (7). This shows the necessity of (8).

Conversely, suppose (8) holds for any $X \in \mathcal{D}$. Let b be the rank function of the direct sum of $\mathbf{M} \cdot S$ and the trivial matroid on R. Namely, $b(X) = r(X \cap S)$ for any $X \subseteq V$. Then by Lemma 4.2 there is an integral vector y with $p(X) \leq y(X)$ for any $X \in \mathcal{D}$ and $y(X) \leq b(X)$ for any $X \subseteq V$. Then $y(v) \leq 1$ for $v \in S$ and $y(v) \leq 0$ for $v \in R$. The subset $J = \{v \mid y(v) = 1\}$ is an independent set that satisfies $|J \cap X| \geq p(X)$ for any $X \in \mathcal{D}$. Since $J \subseteq S$, this means J is balanced. □

Theorem 4.4. *Suppose there exists a balanced independent set. For a fixed $k \geq r(S)$, there exists a balanced independent set of cardinality k if and only if*

$$r(X \cap S) + r(V - (X \cap R)) \geq p(X) + k \tag{9}$$

holds for every $X \in \mathcal{D}$.

Proof. For a balanced independent set I and a member X of $\mathcal{D}$, we have

$$\begin{aligned} |I| &\leq r(V - (X \cap R)) + |I \cap X \cap R| \\ &\leq r(X \cap S) + r(V - (X \cap R)) + |I \cap X \cap R| - |I \cap X \cap S| \\ &\leq r(X \cap S) + r(V - (X \cap R)) - p(X), \end{aligned}$$

which shows the necessity of (9).

To show the converse, suppose (9) holds for any $X \in \mathcal{D}$. Let $\mathbf{N}$ be the dual matroid of the $(k - r(S))$-truncation of $\mathbf{M}/S$. The rank function b of the direct sum $\mathbf{M} \cdot S \oplus \mathbf{N}$ is given by

$$b(X) = r(X \cap S) + \min\{|X \cap R|, |X \cap R| + r(V - (X \cap R)) - k\}.$$

It follows from Lemma 4.3 and (9) that $p(X) + |X \cap R| \leq b(X)$ holds for any $X \in \mathcal{D}$. Then by Lemma 4.2 there is an integral vector y that satisfies $p(X) + |X \cap R| \leq y(X)$ for any $X \in \mathcal{D}$ and $y(X) \leq b(X)$ for any $X \subseteq V$. Since b is a rank function, $y(v) \leq 1$ for each $v \in V$. Let J be the positive support of y, i.e., $J = \{v \mid y(v) > 0\}$. Then J is an independent set of $\mathbf{M} \cdot S \oplus \mathbf{N}$. Let K be a base of $R - J$ in $\mathbf{M}/S$. Then $I = (J \cap S) \cup K$ is an independent set of cardinality k. Moreover, we have

$$p(X) \leq |X \cap J| - |X \cap R| = |X \cap J \cap S| - |X \cap (R - J)| \leq |X \cap I \cap S| - |X \cap I \cap R|$$

for any $X \in \mathcal{D}$. Thus I is a balanced independent set of cardinality k. □

The following min-max formula is immediate from Theorem 4.4.

Corollary 4.5. *For the maximum size of a balanced independent set (if such a set exists) we have*

$$\max\{|I| : I \in \mathcal{J}\} = \min\{r(X \cap S) + r(V - (X \cap R)) - p(X) : X \in \mathcal{D}\}.$$

□

In order to find a maximum weight balanced independent set of cardinality k for a consistent weight function w, we have to obtain an integral vector y in the proof of Theorem 4.4 that maximizes $\sum_{v \in S} w(v) - \sum_{u \in R} w(u)$. This can be done in polynomial time by solving a submodular flow problem [5].

5 Delta-Matroids

Let V be a finite set. A *delta-matroid* (or *pseudomatroid*), introduced by Bouchet [1] and Chandrasekaran–Kabadi [3], is a set system $(V, \mathcal{F})$ with $\mathcal{F}$ being a nonempty family of subsets of V that satisfies the following exchange property:

$$\forall F_1, F_2 \in \mathcal{F}, \forall v \in F_1 \triangle F_2, \exists u \in F_1 \triangle F_2 : F_1 \triangle \{u, v\} \in \mathcal{F},$$

where $\triangle$ denotes the symmetric difference. A member of $\mathcal{F}$ is called a feasible set of the delta-matroid. Note that the base and the independent-set families of a matroid satisfy this exchange property. Thus, a delta-matroid is a generalization of a matroid.

The rank function q of $(V, \mathcal{F})$ is defined by

$$q(X, Y) = \max\{|X \cap F| + |Y \backslash F| : F \in \mathcal{F}\}$$

for $X, Y \subseteq V$ with $X \cap Y = \emptyset$.

Let $(V, \mathcal{F})$ be a delta-matroid with $|V|$ even and Π be a partition of V into ordered pairs. For any $F \in \mathcal{F}$, we denote by $\delta_\Pi(F)$ the number of pairs $(u, v) \in \Pi$ with $u \in F$ and $v \notin F$. The ordered parity problem for $(V, \mathcal{F})$ is to find a feasible set $F \in \mathcal{F}$ with minimum $\delta_\Pi(F)$. If $\mathcal{F}$ is a matroid base family, then this is equivalent to the ordered parity problem for the matroid.

Let R denote the set of all the first elements in the pairs in Π, and consider the set family $\mathcal{F} \triangle R = \{F \triangle R : F \in \mathcal{F}\}$. Then $(V, \mathcal{F} \triangle R)$ is a delta-matroid. For any $H \in \mathcal{F} \triangle R$, we denote by $\xi_\Pi(H)$ the number of pairs in Π that contain at least one element in H. Since $\delta_\Pi(F) = |R| - \xi_\Pi(F \triangle R)$, it suffices to find $H \in \mathcal{F} \triangle R$ that maximizes $\xi_\Pi(H)$. We may further restrict H to be a maximal member of $\mathcal{F} \triangle R$. Let $\mathcal{H}$ denote the family of all the maximal feasible sets in $(V, \mathcal{F} \triangle R)$. Then $\mathcal{H}$ forms a matroid base family.

Theorem 5.1. *For the ordered parity problem on delta-matroid $(V, \mathcal{F})$, we have*

$$\min_{F \in \mathcal{F}} \delta_\Pi(F) = \max\{|X| - q(X, \overline{X}) : X \subseteq S\}. \tag{10}$$

Proof. Let b be the rank function of the matroid $(V, \mathcal{H})$. It follows from the matroid intersection theorem that

$$\max_{H\in\mathcal{H}} \xi_\Pi(H) = \min\{b(X \cup \overline{X}) + |S \backslash X| : X \subseteq S\}.$$

On the other hand, the rank function b is given by $b(W) = q(W \cap S, W \cap R)$ for $W \subseteq V$. Thus we obtain

$$\max_{H\in\mathcal{H}} \xi_\Pi(H) = \min\{q(X, \overline{X}) - |X| : X \subseteq S\} + |S|,$$

which implies (10) by $|S| = |R|$ and $\min\{\delta_\Pi(F) : F \in \mathcal{F}\} = |R| - \max\{\xi_\Pi(H) : H \in \mathcal{H}\}$. □

6 Other Extensions

A possible generalization of the ordered matroid parity problem is the following *closed independent set problem.* Matroid $\mathbf{M} = (V, \mathcal{I})$ and directed graph $D = (V, A)$ is given. We ask for a maximum cardinality independent set I of $\mathbf{M}$ that has outdegree 0 in D (that is, $I \in \mathcal{I}$ so that $u \in I$ implies $v \in I$ for all arcs uv of D). Clearly, the special case of this closed independent set problem in which D consists of oppositely directed arcs on disjoint pairs of vertices contains the NP-hard matroid parity problem.

This shows that we can expect polynomial time solvability only if we put some restrictions on $\mathbf{M}$ or on D. If we try to restrict D, then to avoid the straightforward reduction of matroid parity to the closed independent set problem, we had better consider only acyclic graphs for D. Still, it is not very difficult to see that if a matroid $\mathbf{M} = (V, \mathcal{I})$ and partition Π of V into pairs are given, the problem of finding a basis B of $\mathbf{M}$ that contains a maximum number of pairs from Π can be polynomially reduced to the closed independent set problem such that all components of D are out-stars on three vertices. (Namely, for each pair (u, v) of Π take a new element z_{uv}. Let D consist of arcs $z_{uv}u$ and $z_{uv}v$ for $(u, v) \in \Pi$, and consider the closed independent set problem on $\mathbf{M}' := \mathbf{M} \oplus \mathbf{F}$, where $\mathbf{F}$ is the free matroid on the z_{uv} elements. If I is a maximum size closed independent set of $\mathbf{M}'$, then $I \cap V$ is a basis of $\mathbf{M}$ that contains a maximum number of pairs of Π.)

Similarly, the question, whether there exists an independent set I of $\mathbf{M}$ that is the union of k pairs of Π can also be polynomially reduced to a closed independent set problem with digraph D consisting of disjoint in-stars of size three. (Take for each pair (u, v) of Π a new element z_{uv}. Let D consist of arcs uz_{uv} and vz_{uv} for $(u, v) \in \Pi$, and consider the closed independent set problem on $\mathbf{M}' := \mathbf{M} \oplus \mathbf{U}_k$, where $\mathbf{U}_k$ is the uniform matroid of rank k on the z_{uv} elements. If I is a maximum size closed independent set of $\mathbf{M}'$ of size $3k$, then $I \cap V$ is an independtent set of $\mathbf{M}$ consisting of k pairs of Π. If the maximum size closed independent set has cardinality less than $3k$, then no k pair of Π is independent in $\mathbf{M}$.)

At last, it is also not hard to see that by introducing parallel elements in $\mathbf{M}$, both closed independent set problems above can be polynomially reduced to one in which

D consists of disjoint directed paths on three vertices. Being all these problems NP-hard, if we look for an efficiently solvable class of the closed independent set problem by restricting D, it is reasonable to assume that D consists of disjoint arcs. But this problem is the ordered matroid parity problem that we have studied in Section 2.

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