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# The Dress Conjectures on Rank in the 3-Dimensional Rigidity Matroid

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Bill Jackson* and Tibor Jordán**

**Abstract**

A. Dress has made two conjectures concerning the rank function of the 3-dimensional rigidity matroid. The first would give a min-max formula for this rank function and hence a good characterization for independence. We show that the first conjecture is false for all graphs with at least 56 vertices. On the other hand we show that the second conjecture and a modified form of the first conjecture are true for certain families of graphs of maximum degree at most five.

## 1 Introduction

A *framework* $(G,p)$ in $d$-space is a graph $G=(V,E)$ and an embedding $p:V\to\mathbb{R}^d$. The *rigidity matrix* of the framework is the matrix $R(G,p)$ of size $|E|\times d|V|$, where, for each edge $v_iv_j\in E$, in the row corresponding to $v_iv_j$, the entries in the $d$ columns corresponding to vertex $i$ ($j$) contain the $d$ coordinates of $(p(v_i)-p(v_j))$ $((p(v_j)-p(v_i))$, respectively), and the remaining entries are zeros. See [12] for more details. The rigidity matrix of $(G,p)$ defines the *rigidity matroid* of $(G,p)$ on the ground set $E$ by independence of rows of the rigidity matrix. A *framework* $(G,p)$ is *generic* if the coordinates of the points $p(v)$, $v\in V$, are algebraically independent over the rationals. Any two generic frameworks $(G,p)$ and $(G,p')$ have the same rigidity matroid. We call this the $d$-dimensional *rigidity matroid* $\mathcal{R}_d(G)=(E,r_d)$ of the graph $G$. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$.

**Lemma 1.1.** *[12, Lema 11.1.3] Let $(G,p)$ be a framework in $\mathbb{R}^d$. Then* rank $R(G,p)\le S(n,d)$*, where $n=|V(G)|$ and*

$$S(n,d)=\begin{cases} nd-\binom{d+1}{2} & \text{if } n\ge d+1\\ \binom{n}{2} & \text{if } n\le d+1.\end{cases}$$

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We say that a graph $G = (V, E)$ is *rigid* in $\mathbb{R}^d$ if $r_d(G) = S(n, d)$. (This definition is motivated by the fact that if $G$ is rigid and $(G, p)$ is a generic framework on $G$, then every smooth deformation of $(G, p)$ which preserves the edge lengths $||p(u) - p(v)||$ for all $uv \in E$, must preserve the distances $||p(w) - p(x)||$ for all $w, x \in V$, see [12].) We say that $G$ is *$M$-independent*, *$M$-dependent* or an *$M$-circuit* in $\mathbb{R}^d$ if $E$ is independent, dependent or a circuit, repectively, in $\mathcal{R}_d(G)$. For $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in $G[X]$, that is, in the subgraph induced by $X$ in $G$. We use $E(X)$ or $i(X)$ when the graph $G$ is clear from the context. A *cover* of $G$ is a collection $\mathcal{X}$ of subsets of $V$, each of size at least two, such that $\cup_{X \in \mathcal{X}} E(X) = E$.

Lemma 1.1 implies the following necessary condition for $G$ to be $M$-independent.

**Lemma 1.2.** *If $G = (V, E)$ is $M$-independent in $\mathbb{R}^d$ then $i(X) \leq S(|X|, d)$ for all $X \subseteq V$.*

It also gives the following upper bound on the rank function.

**Lemma 1.3.** *If $G = (V, E)$ is a graph then*

$$r_d(G) \leq \min_{\mathcal{X}} \sum_{X \in \mathcal{X}} S(|X|, d)$$

*where the minimum is taken over all covers $\mathcal{X}$ of $G$.*

The converse of Lemma 1.2 also holds for $d = 1, 2$. The case $d = 1$ follows from the fact that the 1-dimensional rigidity matroid of $G$ is the same as the cycle matroid of $G$, see [4, Theorem 2.1.1]. The case $d = 2$ is a result of Laman [7]. Similarly, the inequality given in Lemma 1.3 holds with equality when $d = 1, 2$. The case $d = 2$ is a result of Lovász and Yemini [8]. Neither of these statements hold for $d \geq 3$. Indeed, it remains an open problem to find good characterizations for independence or, more generally, the rank function in the $d$-dimensional rigidity matroid of a graph when $d \geq 3$.

# 2 Preliminary lemmas

We will be concerned with the case $d = 3$. We need the following results for this special case. We state them for general $d$ for the sake of completeness. The first and third lemmas appear in [12]. The second is folklore.

**Lemma 2.1.** *[12, Lemma 11.1.9] Suppose $G = G_1 \cup G_2$.*
*(a) If $|V(G_1) \cap V(G_2)| \geq d$ and $G_1, G_2$ are rigid in $\mathbb{R}^d$ then $G$ is rigid in $\mathbb{R}^d$.*
*(b) If $|V(G_1) \cap V(G_2)| \leq 1$ and $G_1, G_2$ are $M$-independent in $\mathbb{R}^d$ then $G$ is $M$-independent in $\mathbb{R}^d$.*

**Lemma 2.2.** *[6, Lemma 2.5] Let $G = (V, E)$ be a graph.*
*(a) If $G$ is rigid in $\mathbb{R}^d$ then $G$ is either $d$-connected or complete.*
*(b) If $G$ is an $M$-circuit in $\mathbb{R}^d$ then $G$ is 2-connected and $(d + 1)$-edge-connected.*

**Lemma 2.3.** *[12, Lemma 11.1.1] Let $G = (V, E)$ be a graph and $v \in V$ with $d(v) \leq d$. Then $G$ is $M$-independent in $\mathbb{R}^d$ if and only if $G - v$ is $M$-independent in $\mathbb{R}^d$.*

Lemmas 2.3 and 1.2 immediately imply the following elementary result.

**Lemma 2.4.** *Let $G$ be a graph on at most $d + 2$ vertices. If $G \neq K_{d+2}$ then $G$ is $M$-independent in $\mathbb{R}^d$. If $G = K_{d+2}$ then $G$ is an $M$-circuit in $\mathbb{R}^d$.*

Let $v$ be a vertex in a graph $G$. Suppose $w, x \in N(v)$ and $wx \notin E(G)$. We denote the graph $(G - v) + wx$ by $G_v^{wx}$ and say that $G_v^{wx}$ has been obtained by a *splitting of $G$ at $v$ along $wx$*.

**Lemma 2.5.** *[12, Theorem 11.1.7] Let $v$ be a vertex of degree $d + 1$ in a graph $G$. Suppose $w, x \in N(v)$ and $wx \notin E(G)$. If $G_v^{wx}$ is $M$-independent in $\mathbb{R}^d$ then $G$ is $M$-independent in $\mathbb{R}^d$. Furthermore, if $G$ is $M$-independent in $\mathbb{R}^d$, then $G_v^{yz}$ is $M$-independent in $\mathbb{R}^d$ for some pair $y, z \in N(v)$.*

Henceforth we take $d = 3$. To simplify terminology, we will supress explicit reference to this particular value of $d$ and say, for example, that $G$ is rigid to mean $G$ is rigid in $\mathbb{R}^3$.

# 3 The Dress Conjectures

Let $G = (V, E)$ be a graph. A cover $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ of $G$ is *$t$-thin* if $|X_i \cap X_j| \leq t$ for all $1 \leq i \leq m$. For $X_i \in \mathcal{X}$ let $f(X_i) = 1$ if $|X_i| = 2$ and $f(X_i) = 3|X_i| - 6$ if $|X_i| \geq 3$. (Thus $f(X_i) = S(|X_i|, 3)$.) Let $H(\mathcal{X})$ be the set of all pairs of vertices $uv$ such that $X_i \cap X_j = \{u, v\}$ for some $1 \leq i < j \leq m$. For each $uv \in H(\mathcal{X})$ let $d(uv)$ be the number of sets $X_i$ in $\mathcal{X}$ such that $\{u, v\} \subseteq X_i$ and put

$$val(\mathcal{X}) = \sum_{X \in \mathcal{X}} f(X) - \sum_{uv \in H(\mathcal{X})} (d(uv) - 1).$$

In 1983, Dress, Drieding amd Haegi conjectured that 2-thin covers could be used to determine the rank function of $\mathcal{R}(G)$.

**Conjecture 3.1.** *[3, equation (39)] and [11, Conjecture 3] Let $G = (V, E)$ be a graph and $E' \subseteq E$. Then*

$$r(E') = \min\{val(\mathcal{X})\}, \qquad (1)$$

*where the minimum is taken over all 2-thin covers $\mathcal{X}$ of $G[E']$.*

The conjecture is stated in [3, 11] in an equivalent form in terms of the *degrees of freedom* of $G$, defined to be $S(n, d) - r_d(G)$. It is stated in the above form as an open problem by Crapo, Dress and Tay in [1]. Several equivalent forms of the conjecture are given by Tay in [9].

The following example shows that Conjecture 3.1 is false for all connected graphs on at least 56 vertices. It also provides a counterexample to weaker conjectures of Crapo and Tay [2] that the function given on the right hand side of (1) is a matroid rank function on $E$, and of Tay [9, Conjecture 2.1] that the function on the right hand side of (1) is an upper bound on $r(E')$.

A *biplane* $B$ is a collection of subsets of a finite set $V$ such that each pair of subsets intersect in exactly two elements and each pair of elements of $V$ belong to exactly two subsets, see [5]. It can be seen that each subset has the same size, say $k$, that each element belongs to exactly $k$ subsets, and that $|B| = |V| = \binom{k}{2} + 1 =: n$. Thus $B$ is equivalent to a covering of $K_n$ with $n$ subgraphs isomorphic to $K_k$ such that every edge belongs to exactly two subgraphs and every pair of subgraphs intersect in an edge. Let $F = (V, E)$ be a graph on $n$ vertices and without isolated vertices. Let $\mathcal{X}$ be the 2-thin cover of $F$ obtained by taking the above covering of $K_n$. For $k \geq 3$, we have:

$$\sum_{X \in \mathcal{X}} f(X) = n(3k-6) = \left(\binom{k}{2} + 1\right)(3k-6)$$

and

$$\sum_{uv \in H(\mathcal{X})} (d(uv) - 1) = |E(K_n)| = \binom{n}{2} = \frac{1}{2}\left(\binom{k}{2} + 1\right)\binom{k}{2}.$$

Biplanes are known to exist for $k = 3, 4, 5, 6, 9, 11, 13$. Taking $k = 11$ we have $n = 56$ and

$$val(\mathcal{X}) = \sum_{X \in \mathcal{X}} f(X) - \sum_{uv \in H(\mathcal{X})} (d(uv) - 1) < 0.$$

On the other hand $r(E) \geq 0$ since $r$ is the rank function of a matroid. Thus every such graph $F = (V, E)$ is a counterexample to Conjecture 3.1. It follows that every graph $G = (V, E)$ on at least 56 vertices and without isolated vertices is a counterexample to Conjecture 3.1, since we may choose $E' \subseteq E$ such that $F = G[E']$ is a subgraph of $G$ on exactly 56 vertices and without isolated vertices.

At a conference on rigidity held in Montreal in 1987, Dress conjectured that equality is obtained in (1) for the special 2-thin cover defined as follows. For $u, v \in V$, the edge $uv$ is an *implied edge* of $G$ if $uv \notin E$ and $r(E + uv) = r(E)$. The closure $\hat{G}$ of $G$ is the graph obtained by adding all the implied edges to $G$. A *rigid cluster* of $G$ is a set of vertices which induce a maximal complete subgraph of $\hat{G}$. Using Lemma 2.1(a), we can see that any two rigid clusters of $G$ intersect in at most two vertices, see Lemma 4.6. Thus the set of rigid clusters of $G$ is a 2-thin cover of $G$.

**Conjecture 3.2.** *(see [4, Conjecture 5.6.1],[1], and [9, Conjecture 2.3]) Let $G = (V, E)$ be a graph and $\mathcal{X}$ be the set of rigid clusters of $G$. Then*

$$r(E) = val(\mathcal{X}). \tag{2}$$

Note that, while Conjecture 3.1 would have provided a good characterisation for the rank function of $\mathcal{R}(G)$, the same does not seem to be true for Conjecture 3.2.

It is conceivable that Conjecture 3.2 is true because of the special intersection properties of rigid clusters. If so, then it may be possible to resurrect Conjecture 3.1 by only considering 2-thin covers whose intersection properties reflect those of rigid clusters. We will also show in Section 4 that Conjecture 3.2 is true for graphs of low degree. We close this section by showing that a modified version of Conjecture 3.1 is also true for graphs of low degree. We denote the maximum and minimum degrees of a graph $G$ by $\Delta(G)$ and $\delta(G)$, respectively. We use the following result from [6].

**Theorem 3.3.** *[6] Let $G = (V, E)$ be a connected graph with $\Delta(G) \leq 5$ and $\delta(G) \leq 4$. Then*

$$r(E) = \min_{\mathcal{X}} \sum_{X \in \mathcal{X}} f(X)$$

*where the minimum is taken over all 1-thin covers $\mathcal{X}$ of $G$.*

We say that a cover $\mathcal{X}$ of a graph $G = (V, E)$ is *independent* if the graph $(V, H(\mathcal{X}))$ is $M$-independent. The following lemma shows that independent covers of $G$ can be used to give an upper bound on $r(G)$. (Note that the biplane example given above shows that (2-thin) covers which are not independent do not, in general, give an upper bound on $r(G)$.)

**Lemma 3.4.** *Let $G = (V, E)$ be a graph, and $\mathcal{X}$ be an independent cover of $G$. Then $r(E) \leq val(\mathcal{X})$.*

**Proof:** Let $H = H(\mathcal{X})$, $E^* = E \cup H$ and $G^* = (V, E^*)$. For each $X_i \in \mathcal{X}$ let $S_i = E_{G^*}(X) \cap H$. Since $(V, H)$ is $M$-independent, $(X_i, S_i)$ is $M$-independent and hence $S_i$ can be extended to a basis $B_i$ for the rigidity matroid of $G^*[X_i]$. Let $S = \cup_{X_i \in \mathcal{X}} B_i$. Then $S$ spans $E^*$ since, if $e \in E^*$ then $e \in E_{G^*}(X_i)$ for some $X_i \in \mathcal{X}$ and hence $e$ is spanned by $B_i \subseteq S$. Thus $r(E^*) \leq |S|$. On the other hand, $|B_i| \leq f(X_i)$ for all $X_i \in \mathcal{X}$ by Lemma 1.2. Since $S$ covers each $uv \in S - H$ exactly once and covers each $uv \in H$ exactly $d(uv)$ times, we have

$$|S| = \sum_{X_i \in \mathcal{X}} |B_i| - \sum_{uv \in H} (d(uv) - 1) \leq val(\mathcal{X}).$$

The lemma now follows since $r(E) \leq r(E^*)$. •

**Theorem 3.5.** *Let $G = (V, E)$ be a connected graph with $\Delta(G) \leq 5$ and $\delta(G) \leq 4$. Then $r(E) = \min_{\mathcal{X}} val(\mathcal{X})$ where the minimum is taken over all independent 2-thin covers $\mathcal{X}$ of $G$.*

**Proof:** By Lemma 3.4, it suffices to show that there exists an independent 2-thin cover $\mathcal{X}$ of $G$ such that $val(\mathcal{X}) = r(E)$. Let $\mathcal{X}$ be a 1-thin cover of $G$ for which equality occurs in Theorem 3.3. Then $H(\mathcal{X}) = \emptyset$ so $\mathcal{X}$ is independent and $val(\mathcal{X}) = r(E)$. •

The following construction due to Tay [10] shows that Theorem 3.5 becomes false if we remove the restriction on the maximum degree of $G$. If $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ are graphs such that $V_1 \cap V_2 = \{u, v\}$ and $E_1 \cap E_2 = \{uv\}$, then we say that $G = (G_1 \cup G_2) - uv$ is a 2-*sum* of $G_1, G_2$. We denote this by $G = G_1 \oplus_2 G_2$.

**Lemma 3.6.** *[10, Theorem 4.1] Suppose $G_1, G_2$ are graphs and $G = G_1 \oplus_2 G_2$. Then $G$ is an $M$-circuit if and only if $G_1$ and $G_2$ are $M$-circuits.*

Let $G_0 = (V_0, E_0)$ be a complete graph on five vertices with $V_0 = \{v_i : 1 \le i \le 5\}$. For $1 \le i < j \le 5$ let $G_{i,j} = (V_{i,j}, E_{i,j})$ be a complete graph on five vertices with $V_{i,j} \cap V_0 = \{v_i, v_j\}$ and $E_{i,j} \cap E_0 = \{v_i v_j\}$ for $1 \le i < j \le 5$. Let

$$G = (G_0 \cup (\cup_{1 \le i < j \le 5} G_{i,j})) - E_0.$$

Then $G$ is an iterated 2-sum of $K_5$'s and hence is an $M$-circuit by Lemma 3.6 and the fact that $K_5$ is an $M$-circuit by Lemma 2.4. Thus $r(G) = |E(G)| - 1 = 89$. On the other hand, $\min_{\mathcal{X}} val(\mathcal{X})$ over all independent 2-thin covers $\mathcal{X}$ of $G$ is 90. Note however that the set of implied edges of $G$ is $E_0$, and hence the rigid clusters of $G$ are $V_0$ and the sets $V_{i,j}$ for $1 \le i < j \le 5$. Hence, if $\mathcal{X}$ is the set of rigid clusters of $G$, then we have $H(\mathcal{X}) = E_0$ and $val(\mathcal{X}) = 89$. Thus Conjecture 3.2 holds for $G$.

The above example has maximum degree 12. It is conceivable that Theorem 3.5 can be extended to all graphs of maximum degree at most 11. On the other hand, Theorem 3.3 cannot be extended to graphs of maximum degree six. This can be seen by considering the $M$-circuit $G = K_5 \oplus_2 K_5$. We have $r(G) = |E(G)| - 1 = 17$ but $\min_{\mathcal{X}} val(\mathcal{X})$ over all 1-thin covers $\mathcal{X}$ of $G$ is 18.

# 4 Rigid clusters in graphs of maximum degree at most five

Let $G = (V, E)$ be a graph. We say that $G$ is *Laman* if $G$ is simple and $i(X) \le 3|X| - 6$ for all $X \subseteq V$ with $|X| \ge 3$. Let $v \in V$ with $d(v) = 4$. Splitting $v$ along two neighbours $u, w$ in a Laman graph $G$ is *admissible* if the resulting graph $G_v^{uw}$ is also Laman.

We shall need the following results from [6].

**Lemma 4.1.** *[6] Let $G = (V, E)$ be a Laman graph, $V_6$ be the set of all vertices of $G$ of degree at least six and suppose that $G[V_6]$ is a (possibly empty) complete graph. Let $v$ be a vertex of degree four in $G$. Then $G$ has an admissible split at $v$.*

**Theorem 4.2.** *[6] Let $G = (V, E)$ be a connected graph with $\Delta(G) \le 5$ and $\delta(G) \le 4$. Then $G$ is $M$-independent if and only if $G$ is Laman.*

Theorem 4.2 does not seem to be strong enough to determine the rigid clusters in an $M$-independent graph $G$ with $\Delta(G) \le 5$ and $\delta(G) \le 4$. In order to determine the rigid clusters of $G$ we need to determine the closure of $G$ and hence we need to determine the implied edges of $G$. Thus we need to be able to determine when $G + uv$ is $M$-dependent for each pair $u, v \in V$. The problem is that we may not be able to apply Theorem 4.2 to $G + uv$ because it may no longer satisfy the hypotheses that $\Delta \le 5$, or that $\delta \le 4$. The second problem can be easily avoided by requiring $G$ to have at least three vertices of degree four. To overcome the first problem we need to obtain a version of Theorem 4.2 which allows at most two vertices of degree six.

**Theorem 4.3.** *Let $G = (V, E)$ be a 3-edge-connected graph with $\Delta(G) \le 6$. Let $V_i = \{v \in V : d(v) = i\}$. Suppose that $|V_6| \le 2$, $G[V_6]$ is a (possibly empty) complete graph, and $|V_3| + |V_4| \ge \max\{1, |V_6|\}$. Then $G$ is $M$-independent if and only if $G$ is Laman.*

**Proof:** Necessity follows from Lemma 1.2.

To prove sufficiency, we proceed by induction on $|V|$. Let $G$ be a Laman graph satisfying the hypotheses of the theorem. Since $G$ is 3-edge-connected and $K_4$ is $M$-independent we may assume that $|V| \geq 5$. Let $v$ be a vertex of minimum degree in $G$. If $d(v) = 3$ let $G' = G - v$. If $d(v) = 4$ then, by Lemma 4.1, there is an admissible split $G_v$ of $G$ at $v$ and we let $G' = G_v$. In both cases $G'$ is Laman. If $G'$ is $M$-independent then $G$ is $M$-independent by Lemmas 2.3 and 2.5. Thus we may assume that $G'$ is not $M$-independent.

Let $C$ be an $M$-circuit in $G'$. Then $C$ is 4-edge connected by Lemma 2.2(b). Thus $C$ is contained in a maximal 4-edge-connected subgraph $G_1 = (V_1, E_1)$ of $G'$. Since $G$ is 3-edge-connected we have $d_G(V_1, V - V_1) \geq 3$. The facts that $G_1$ is 4-edge-connected and $\Delta(G) \leq 6$ now imply that $G_1$ satisifies the hypotheses of the theorem. Moreover $G_1$ is Laman since $G'$ is Laman and $G_1$ is a subgraph of $G'$. By induction, $G_1$ is $M$-independent. This contradicts the fact that $G_1$ contains the $M$-circuit $C$. •

Using Theorem 4.3 we may deduce:

**Corollary 4.4.** *Let $G = (V, E)$ be a graph with $\Delta(G) \leq 6$. Let $V_i = \{v \in V : d(v) = i\}$. Suppose that $|V_6| \leq 2$, $G[V_6]$ is a (possibly empty) complete graph, and $|V_4| \geq \max\{1, |V_6|\}$. Then $G$ is an $M$-circuit if and only if $G$ is 4-edge-connected, $|E| = 3|V| - 5$, and $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $3 \leq |X| \leq |V| - 1$.*

**Proof:** Suppose $G$ is an $M$-circuit. Then $G$ is 4-edge-connected by Lemma 2.2(b). Hence $G - e$ is $M$-independent and 3-edge-connected for all $e \in E$. By Theorem 4.3, $G - e$ is Laman. Thus $i_{G-e}(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $3 \leq |X| \leq |V|$. Hence $i_G(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $3 \leq |X| \leq |V| - 1$. Furthermore, since $G$ is $M$-dependent and $G$ satisfies the hypotheses of Theorem 4.3, we must also have $i_{G-e}(V) = 3|V| - 6$ and thus $|E| = 3|V| - 5$.

Next we suppose that $G$ is 4-edge-connected, $|E| = 3|V| - 5$, and $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $3 \leq |X| \leq |V| - 1$. Then $G$ and $G - e$ satisfy the hypotheses of Theorem 4.3 for all $e \in E$. Thus $G$ is dependent and $G - e$ is independent for all $e \in E$. Hence $G$ is an $M$-circuit. •

**Corollary 4.5.** *Let $G = (V, E)$ be an $M$-circuit with $\Delta(G) \leq 6$. Let $V_i = \{v \in V : d(v) = i\}$. Suppose that $|V_6| \leq 2$, $G[V_6]$ is a (possibly empty) complete graph, and $|V_4| \geq \max\{1, |V_6|\}$. Then $G - e$ is rigid for all $e \in E$.*

**Proof:** By Corollary 4.4, $G$ is 4-edge-connected, $|E| = 3|V| - 5$, and $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $3 \leq |X| \leq |V| - 1$. Hence $G - e$ is 3-edge-connected and $i_{G-e}(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $3 \leq |X| \leq |V|$. Applying Theorem 4.3 to $G - e$ we deduce that $G - e$ is rigid. •

We shall use Corollary 4.5 to determine some structural properties of the rigid clusters in graphs of low degree. We first prove a general result about rigid clusters.

**Lemma 4.6.** *Let $Y_1, Y_2$ be distinct rigid clusters of a graph $G$. Then $|Y_1 \cap Y_2| \leq 2$.*

**Proof:** Suppose $|Y_1 \cap Y_2| \geq 3$. Since $Y_i$ is a rigid cluster of $G$, $\hat{G}[Y_i]$ is a complete, and hence rigid, subgraph of $\hat{G}$ for each $i \in \{1,2\}$. By Lemma 2.1(a), $\hat{G}[Y_1] \cup \hat{G}[Y_2]$ is rigid. It follows that $y_1y_2$ is either an edge or an implied edge of $\hat{G}$ for all $y_1 \in Y_1$ and $y_2 \in Y_2$. Since $\hat{G}$ is closed, we have $y_1y_2 \in E(\hat{G})$ for all $y_1 \in Y_1$ and $y_2 \in Y_2$. This $\hat{G}[Y_1 \cup Y_2]$ is complete. This contradicts the fact that $\hat{G}[Y_i]$ is a maximal complete subgraph of $\hat{G}$. •

**Lemma 4.7.** *Let $G = (V, E)$ be a 3-edge-connected graph with $\Delta(G) \leq 5$ and at least three vertices of degree at most four.*
*(a) Let $Y$ be a rigid cluster of $G$ with with $|Y| \geq 5$. Then $G[Y]$ is rigid.*
*(b) Let $Y_1, Y_2$ be rigid clusters of $G$ with $|Y_1|, |Y_2| \geq 5$. Then $Y_1 \cap Y_2 = \emptyset$.*
*(c) Let $uv$ be an implied edge of $G$. Then $\{u, v\}$ is contained in exactly one rigid cluster of $G$ of size at least five.*

**Proof:** (a) Suppose that $G[Y]$ is not rigid. Then $G[Y] \neq \hat{G}[Y]$, since $\hat{G}[Y]$ is complete. Hence we may choose an implied edge $uv$ of $G$ with $u, v \in Y$. Then $uv \in E(C) \subseteq E + uv$ for some $M$-circuit $C$ of $G + uv$. If $V(C) = V$ then $C$ satisfies the hypotheses of Corollary 4.5 and hence $C - uv$ is rigid. On the other hand, if $V(C) \neq V$ then the 3-edge-connectivity of $G$ implies that $d_G(V(C), V - V(C)) \geq 3$. The fact that $C$ is 4-edge-connected and the hypotheses on the degree in $G$ now imply that $C$ again satisfies the hypotheses of Corollary 4.5 and hence $C - uv$ is rigid. We may apply Lemma 4.6 to $\hat{G}$ to deduce that either $V(C) \subseteq Y$ or $V(C) \cap Y = \{u, v\}$. We shall show that the second alternative cannot hold.

Suppose $V(C) \cap Y = \{u, v\}$. Since $5 \geq d_G(u) \geq d_C(u) - 1 + d_Y(u)$ and $d_C(u) \geq 4$, we have $d_Y(u) \leq 2$. The facts that $|Y| \geq 5$ and $\hat{G}[Y]$ is complete, now imply that $uv'$ is an implied edge of $G$ for some $v' \in Y - v$. Arguing as above we have $uv' \in E(C') \subseteq E + uv'$ for some rigid $M$-circuit $C'$ of $G + uv'$ and either $V(C') \subseteq Y$ or $V(C') \cap Y = \{u, v'\}$. If $V(C') \subseteq Y$ then, since $E(C') - uv' \subseteq E$, we have $d_Y(u) \geq d_{C'}(u) - 1 \geq 3$. This contradicts the fact $d_Y(u) \leq 2$. Hence $V(C') \cap Y = \{u, v'\}$. If there exists a vertex $w \in (V(C) \cap V(C')) - u$ then $w \notin Y$ and, since $C, C'$ are rigid, $wu, wv, wv' \in E(\hat{G})$. But then $\hat{G}[Y + w]$ would be rigid by Lemma 2.1(a), contradicting the fact that $Y$ is a rigid cluster of $G$. Thus $V(C) \cap V(C') = \{u\}$. Hence $5 \geq d_G(u) \geq d_C(u) - 1 + d_{C'}(u) - 1 \geq 6$. This contradiction implies that we must have $V(C) \subseteq Y$.

It follows that $uv$ is an implied edge of $G[Y]$. Since this holds for all implied edges $uv$ of $G$ with $u, v \in Y$, we may deduce that the closure of $G[Y]$ is $\hat{G}[Y]$. Thus the closure of $G[Y]$ is complete and hence $G[Y]$ is rigid. This completes the proof of (a).

(b) By (a), $G[Y_i]$ is rigid for $i = 1, 2$. Lemma 2.1(a) implies that $|Y_1 \cap Y_2| \leq 2$.

Suppose $Y_1 \cap Y_2 = \{x\}$. By Lemma 2.2(a), $d(x) \geq d_{Y_1}(x) + d_{Y_2}(x) \geq 3 + 3 = 6$. This contradicts the fact that $\Delta(G) \leq 5$.

Suppose $Y_1 \cap Y_2 = \{u, v\}$. By Lemma 2.2(a), $5 \geq d(y) \geq d_{Y_1}(y) + d_{Y_2}(y) - 1 \geq 3 + 3 - 1 = 5$ for each $y \in \{u, v\}$. Thus equality must hold throughout, $d_{Y_1}(u) = 3 = d_{Y_1}(v)$ and $uv \in E$. Let $G' = G[Y_1]$. Applying Lemma 2.3 to $G'$ we deduce that $G' - u$ is rigid. This contradicts Lemma 2.2(a) since $|Y - u| \geq 4$ and $d_{G'-u}(v) = 2$.

Thus $Y_1 \cap Y_2 = \emptyset$.

(c) Since $uv$ is an implied edge of $G$, $uv$ belongs to an $M$-circuit $C$ in $G + e$. We may deduce, as in the proof of (a), that $C - e$ is rigid. Thus $\{u, v\} \subseteq V(C) \subseteq Y$ for some rigid cluster $Y$ of $G$. Furthermore $|Y| \geq |V(C)| \geq 5$ by Lemma 2.4. Uniqueness follows from (b). •

We next use Lemma 4.7 to show that Conjecture 3.2 holds for this family of graphs.

**Theorem 4.8.** *Let $G = (V, E)$ be a 3-edge-connected graph with $\Delta(G) \leq 5$ and at least three vertices of degree at most four. Let $\mathcal{X}$ be the set of rigid clusters of $G$. Then $val(\mathcal{X}) = r(E)$.*

**Proof:** Let $H = H(\mathcal{X})$ and $F = (V, H \cap E)$. We shall show that $F$ is $M$-independent. Suppose to the contrary that $F$ is $M$-dependent and let $C$ be an $M$-circuit contained in $F$. Since $C$ is a subgraph of $G$, Corollary 4.5 implies that $C$ is rigid. Hence $V(C) \subseteq X$ for some $X \in \mathcal{X}$. Since $|X| \geq |V(C)| \geq 5$, $G[X]$ is rigid by Lemma 4.7(a). Choose $u \in V(C)$. By Lemma 2.2(b), $d_C(u) \geq 4$. Thus we may choose vertices $v_i \in V(C)$ such that $uv_i \in E(C)$ for $1 \leq i \leq 4$. Since $E(C) \subseteq H$, we may choose $X_i \in \mathcal{X} - \{X\}$ such that $u, v_i \in X_i$ for each $1 \leq i \leq 4$. Since $X_i \not\subseteq X$, we we may choose $w_i \in X_i - X$ for $1 \leq i \leq 4$. If $w_i = w_j$ for some $1 \leq i < j \leq 4$ then $w_iu, w_iv_i, w_iv_j \in E(\hat{G})$. This would imply, by Lemma 2.3, that $\hat{G}[X + w_i]$ is rigid and contradict the maximality of $X$. Hence $w_i \neq w_j$ for all $1 \leq i < j \leq 4$. Since $|X| \geq 5$, Lemma 4.7(b) implies that $|X_i| \leq 4$ for $1 \leq i < j \leq 4$. If $w_iu \notin E$ for some $1 \leq i \leq 4$, then $w_iu$ is an implied edge of $G$ and Lemma 4.7(c) implies that $w_i, u \subseteq Y$ for some rigid cluster $Y$ of $G$ with $|Y| \geq 5$. This contradicts Lemma 4.7(b) since $u \in X \cap Y$. Thus $w_iu \in E$ for all $1 \leq i \leq 4$. This gives $d_G(u) \geq d_C(u) + \sum_{i=1}^{4} d_{X_i}(u) \geq 4 + 4 = 8$. This contradicts the hypothesis that $\Delta(G) \leq 5$. Thus $F$ is $M$-independent.

We complete the proof by showing that $val(\mathcal{X}) = r(E)$. Since $F$ is $M$-independent, we can choose a basis $B$ for $\mathcal{R}(G)$ with $H \cap E \subseteq B$. Let $B_i = B \cap E(X_i)$ for each $X_i \in \mathcal{X}$. We have $|B_i| \leq f(X_i)$ by Lemma 1.2.

**Claim 4.9.** *Suppose $X_i \in \mathcal{X}$ and $|X_i| \geq 5$. Then $|B_i| = f(X_i)$.*

**Proof:** Suppose to the contrary that $|B_i| < f(X_i)$. Since $G[X_i]$ is rigid by Lemma 4.7(a), we have $r(E(X_i)) = f(X_i)$. Since $|B_i| < f(X_i)$, there exists $e \in E(X_i)$ such that $e$ is not spanned by $B_i$ in $\mathcal{R}(G)$. Since $B$ spans $E$, we have $e \in E(C) \subseteq B + e$ for some $M$-circuit $C$ of $G$. Since $C$ is rigid by Corollary 4.5, $V(C) \subseteq X_j$ for some $X_j \in \mathcal{F}$. Since $|X_j| \geq |V(C)| \geq 5$, Lemma 4.7(b) implies that $X_j = X_i$. Thus $V(C) \subseteq X_i$. Since $E(C) \subseteq B + e$, this implies that $E(C) \subseteq B_i + e$, and contradicts the fact that $B_i$ does not span $e$. •

**Claim 4.10.** *Suppose $X_j \in \mathcal{X}$ and $|X_j| \leq 4$. Let $I(X_j)$ be the set of implied edges $uv$ of $G$ with $\{u, v\} \subseteq X_j$. Then $B_j = E(X_j)$ and $|B_j| = f(X_j) - |I(X_j)|$.*

**Proof:** Choose $e \in E(X_j)$. Suppose $e \in E(C)$ for some $M$-circuit $C$ of $G$. Since $C$ is rigid by Lemma 4.7(a), $e \in E(Y)$ for some rigid cluster $Y$ of $G$ with $|Y| \geq 5$. Thus $e \in H$. Hence $e \in H \cap E \subseteq B$ and thus $e \in B_j$. On the other hand, if $e \notin E(C)$ for all $M$-circuits $C$ of $G$ then $r(G-e) = r(G) - 1$. Thus $e \in B$ and we again have $e \in B_j$. Hence $B_j = E(X_j)$. Since $|X_j| \leq 4$, $f(X_j) = |E_{\hat{G}}(X_j)| = |E(X_j)| + |I(X_j)|$. Thus $|B_j| = f(X_j) - |I(X_j)|$. •

Let $\mathcal{X}_1 = \{X \in \mathcal{X} \ : \ |X| \geq 5\}$ and $\mathcal{X}_2 = \mathcal{X} - \mathcal{X}_1$. Let $\mathcal{B}$ be the collection of all sets $B_i$ for $X_i \in \mathcal{X}$. Then $\mathcal{B}$ covers each $uv \in B - H$ exactly once and covers each $uv \in B \cap H$ exactly $d(uv)$ times. Thus Claims 4.9 and 4.10 give

$$\begin{aligned} r(E) &= |B| = \sum_{X_i \in \mathcal{X}} |B_i| - \sum_{uv \in B \cap H} (d(uv) - 1) \\ &= \sum_{X_i \in \mathcal{X}_1} f(X_i) + \sum_{X_j \in \mathcal{X}_2} (f(X_j) - |I(X_j|) - \sum_{uv \in B \cap H} (d(uv) - 1). \end{aligned} \tag{3}$$

Let $\mathcal{I}$ be the collection of all sets $I(X_j)$ for $X_j \in \mathcal{X}_2$, and $uv$ be an implied edge of $G$. Then $uv$ belongs to exactly one set $X_i \in \mathcal{X}_1$ by Lemma 4.7(c), and hence $uv$ belongs to exactly $d(uv) - 1$ sets $X_j \in \mathcal{X}_2$. Thus the collection $\mathcal{I}$ covers each edge $uv \in H - B$ exactly $d(uv) - 1$ times, and $\sum_{X_j \in \mathcal{X}_2} |I(X_j)| = \sum_{uv \in H - B} (d(uv) - 1)$. Substituting into (3) we obtain

$$r(E) = \sum_{X_i \in \mathcal{X}} f(X_i) - \sum_{uv \in H} (d(uv) - 1).$$

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