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A characterisation of weakly four-connected graphs

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Abstract

A graph $G = (V, E)$ is called weakly four-connected if G is 4-edge-connected and $G - x$ is 2-edge-connected for all $x \in V$. We give sufficient conditions for the existence of 'splittable' vertices of degree four in weakly four-connected graphs. By using these results we prove that every minimally weakly four-connected graph on at least four vertices contains at least three 'splittable' vertices of degree four, which gives rise to an inductive construction of weakly four-connected graphs. Our results can also be applied in the problem of finding 2-connected orientations of graphs.

1 Introduction

A graph $G = (V, E)$ is called (k, l)-*connected*, for a pair of positive integers k, l, if $|V| \geq k + 1$ and for any pair of subsets $S \subseteq V$, $L \subseteq E$ with $l|S| + |L| < kl$ the graph $G - S - L$ is connected. This notion of mixed connectivity was introduced by Kaneko and Ota [7], see also [3]. By taking $l = 1$ ($k = 1$) we obtain k-connectivity (l-edge-connectivity, respectively) as a special case.

It turns out that several extremal results on minimally k-connected graphs extend to minimally (k, l)-connected graphs in a natural way [7]. One may ask whether the well-known inductive constructions for k-connected graphs (for $k \leq 3$, see e.g. [1, 13]) and l-edge-connected graphs ([8, 9], see also [4]) can be generalized to (k, l)-connectivity, at least for some pairs of k and l. A closely related question is whether one can prove 'splitting off' results for (k, l)-connected graphs. In this paper we shall prove a splitting off theorem for $(2, 2)$-connected graphs and use it to show that every $(2, 2)$-connected graph can be obtained from a graph on 3 vertices by adding edges and 'hooking up' pairs of edges.

Our motivation to study inductive constructions for mixed connectivity comes from the problem of characterizing graphs which have k-connected orientations. It was conjectured by Frank [4, Conjecture 7.8] that a graph has a k-connected orientation if and only if it is $(k, 2)$-connected. This conjecture is still open, even for $k = 2$. In

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fact, it is not known whether sufficiently highly connected graphs have k-connected orientations (as conjectured by Thomassen [12]).

By using the inductive construction of $2l$-edge-connected graphs it is not difficult to prove (see [8, Problem 6.54(b)]) that $2l$-edge-connected graphs have l-edge-connected orientations, which follows from a deep result of Nash-Williams [11]. Thus an inductive construction for $(2, 2)$-connected (or equivalently, weakly four-connected) graphs might be useful in the 2-connected orientation problem. We could indeed apply the results of this paper to show that $(2, 2)$-connected Eulerian graphs have 2-connected orientations, and, based on this fact, to show that sufficiently highly connected graphs have 2-connected orientations. See the forthcoming papers [2, 6] for more details.

In the rest of this section we introduce some basic notation and definitions. Let $G = (V, E)$ be a graph (which may contain multiple edges and loops). For two disjoint subsets $X, Y \subset V$ let $d_G(X, Y)$ denote the number of edges connecting X and Y. Let $d_G(X) := d_G(X, V - X)$ denote the *degree* of X. We use $d_G(v)$ to denote the degree of a vertex $v \in V$. We may omit the subscript if the graph is clear from the context. We say that G is *k-edge-connected* if $d(X) \geq k$ for all $\emptyset \neq X \subset X$. (We shall use $\subset$ to denote proper set containment and $\subseteq$ to mean $\subset$ or $=$.) The subgraph induced by a set $X \subseteq V$ in G is denoted by $G[X]$. For some vertex $v \in V$ the set of *neighbours* of v, that is, the set of vertices adjacent to v, is denoted by $N(v)$.

For simplicity we shall use the term *weakly four-connected* instead of saying $(2, 2)$-connected. Thus, by definition, a graph G is weakly four-connected if and only if $|V(G)| \geq 3$, G is 4-edge-connected, and $G - x$ is 2-edge-connected for all $x \in V(G)$. The operation *splitting off* deletes two incident edges su, sv from G and adds a new (copy of) edge uv. We say that the splitting is made at vertex s, and when the common vertex s is clear from the context, we denote the resulting graph by G_{uv}. If $d(s)$ is even, we may consider a *complete splitting at s*, which is a sequence of $d(s)/2$ splittings at s. The operation *hooking up* deletes two specified edges xy, tu, and adds a new vertex s and four edges sx, sy, st, su to the graph. It is the inverse of a complete splitting at a vertex of degree four.

2 Preliminaries

Let $G = (V + s, E)$ be a graph with a designated vertex s. It will be convenient to work with the following weaker version of weak four-connectivity. We say that G is *weakly four-connected in V* if the following two conditions are satisfied:

$$d(X) \geq 4 \text{ for all } \emptyset \neq X \subset V, \tag{1}$$

$$d_{G-x}(X) \geq 2 \text{ for all } x \in V \text{ and } \emptyset \neq X \subset V - x. \tag{2}$$

Note that if $d(s) = 0$ then (1) and (2) hold if and only if $G[V]$ is weakly four-connected. In the rest of this section let $G = (V + s, E)$ be weakly four-connected in V. We say that splitting off a pair su, sv is *admissible* if G_{uv} is also weakly four-connected in V. A complete splitting is *admissible* if the graph on vertex set V obtained by the sequence of splittings is weakly four-connected. We call s *admissible* if there is an admissible complete splitting at s.

We shall give a necessary and sufficient condition for the admissibility of s when $d(s) = 4$. By using this characterisation we shall give sufficient conditions for the existence of an admissible vertex of degree four in a weakly four-connected graph. This will imply, among others, that minimally weakly four-connected graphs have at least three admissible vertices of degree four. A corollary of this fact is an inductive construction for weakly four-connected graphs.

The graph H on Figure 1 is weakly four-connected, and hence it is weakly four-connected in $V(H) - s$ for all $s \in V(H)$. There is no admissible complete splitting at vertex v. However, vertices a, b, and c are admissible.

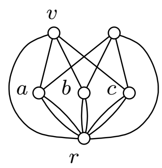

Figure 1: A weakly four-connected graph with three admissible vertices of degree four.

The next proposition follows easily from (1) and (2) and the definition of admissibility.

Proposition 2.1. *If $d(s) = 2$ then s is admissible.*

Thus a vertex s of degree four is admissible if and only if there is an admissible splitting at s.

We now characterise when a given split is non-admissible and prove structural properties of the 'blocking configurations'. We call a set $\emptyset \neq X \subset V$ *edge dangerous* if $d(s, X) \geq 2$ and $d(X) \leq 5$. A pair (X, r) with $r \in V$ and $\emptyset \neq X \subset V - r$ is a *vertex dangerous pair* if $d(s, X) \geq 2$ and $d_{G-r}(X) \leq 3$. A pair (Y, r) with $r \in V$ and $\emptyset \neq Y \subset V - r$ is a *vertex critical pair* if $d(s, r) \geq 1$, $d(s, Y) = 1$, and $d_{G-r}(Y) = 2$.

Lemma 2.2. *A pair su, sv is non-admissible if and only if one of the following holds:*
(a) there is an edge dangerous set X with $u, v \in X$,
(b) there is a vertex dangerous pair (X, r) with $u, v \in X$,
(c) there is a vertex critical pair (Y, r) with $u = r$ and $v \in Y$ (or with $v = r$ and $u \in Y$).

Proof: If (a) holds then G_{uv} violates (1). If (b) or (c) holds then G_{uv} violates (2) by choosing $x = r$. Conversely, suppose that G_{uv} is not weakly four-connected in V. Then (1) or (2) does not hold. It is easy to check that if (1) fails for some $\emptyset \neq X \subset V$ then X satisfies (a). Similarly, if (2) fails for some $x \in V$ and $\emptyset \neq X \subset V - x$ then one of (b) or (c) must hold (depending on whether $x \in \{u, v\}$ or not). •

Lemma 2.3. *Suppose that $d(s) = 4$. Then*
(a) if X is edge dangerous then $d(s, X) = 2$, $G[X]$ is 2-edge-connected, and $V - X$ is also edge dangerous,

(b) if (X, r) is a vertex dangerous pair then $d(s, X) = 2$, $G[X]$ is connected, and either $d(s, r) = 0$ and $(V - X - r, r)$ is also a vertex dangerous pair, or $d(s, r) = 1$ and $(V - X - r, r)$ is a vertex critical pair,
(c) if (Y, r) is a vertex critical pair then $G[Y]$ is connected, and either $d(s, r) = 1$ and $(V - Y - r, r)$ is a vertex dangerous pair, or $d(s, r) = 2$ and $(V - Y - r, r)$ is a vertex critical pair.

Proof: We only prove (a). The proof of (b) and (c) is similar. Let X be an edge dangerous set. Since $d(s, X) \geq 2$ and $d(s) = 4$, we have $d(V - X) = d(X) - d(s, X) + d(s, V - X) \leq d(X) \leq 5$. Thus $V - X$ is also edge dangerous and, by (1), $d(s, X) = 2$ must hold. Now suppose $A \cup B$ is a bipartition of X with $d(A, B) \leq 1$. Then (1) implies that $d(A, V + s - X) \geq 3$ and $d(B, V + s - X) \geq 3$, which gives $d(X) \geq 6$, a contradiction. This proves that $G[X]$ is 2-edge-connected. •

2.1 Maximal edge dangerous sets

In this subsection we summarise (and for completeness, we prove) some properties of edge dangerous and 'critical' sets. These properties have been described earlier (see e.g. [5]) in the context of splitting off edges preserving k-edge-connectivity. We shall need the following well-known equalities. Let $H = (W, F)$ be a graph and $X, Y \subseteq W$. Then

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y), \tag{3}$$

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, V - (X \cup Y)). \tag{4}$$

We say that $G = (V + s, E)$ is *k-edge-connected in V* if $d(X) \geq k$ for all $\emptyset \neq X \subset V$. Let $G = (V + s, E)$ be k-edge-connected in V. We call a set $\emptyset \neq X \subset V$ *k-dangerous* in G if $d(s, X) \geq 2$ and $d(X) \leq k + 1$. We call it *k-critical* if $d(X) = k$ holds.

Lemma 2.4. *Let $G = (V + s, E)$ be k-edge-connected in V for some even integer k, and let $d(s)$ be even. Suppose that there exists a k-dangerous set X' with $u \in X'$ for some $u \in N(s)$. Then the maximal k-dangerous set X with $u \in X$ is unique.*

Proof: Suppose that M_1, M_2 are intersecting maximal k-dangerous sets contaning u. By (4) we have $5+5 \geq d(M_1)+d(M_2) \geq d(M_1-M_2)+d(M_1-M_2)+2d(s, M_1 \cap M_2) \geq 4 + 4 + 2$. Thus $d(M_1 - M_2) = 4$ and $d(s, M_1 \cap M_2) = 1$. Since $d(s)$ is even, if $V = M_1 \cup M_2$ then this implies that $d(s, M_i) \geq 3$ for some $i \in \{1, 2\}$. But then $d(V - M_i) \leq d(M_i) - 2 \leq 3$ would follow. So $M_1 \cup M_2 \neq V$. Then we can use (3) to obtain $5 + 5 \geq d(M_1) + d(M_2) \geq d(M_1 \cap M_2) + d(M_1 \cup M_2) \geq 4 + 6$, which implies that $d(M_1 \cap M_2) = 4$ and $d(M_1) = 5$. Since $d(M_1 - M_2) = 4$ and $d(M_1) = d(M_1 \cap M_2) + d(M_1 - M_2) - 2d(M_1 \cap M_2, M_1 - M_2)$, this is impossible. •

The proof of the next lemma is similar, by using (4).

Lemma 2.5. *Let $G = (V + s, E)$ be k-edge-connected in V. Suppose that for all $u \in N(s)$ there exists a k-critical set X'_u with $u \in X'$. Then the minimal k-critical set X_u with $u \in X_u$ is unique, for all $u \in N(s)$. If, in addition, $G[V]$ is $(k-1)$-edge-connected, then $X_u \cap X_v = \emptyset$ for all pairs $u, v \in N(s)$, $u \neq v$.*

3 Sufficient conditions for admissibility

Let $G = (V + s, E)$ be weakly four-connected in V and let $d(s) = 4$.

Lemma 3.1. *Suppose that there exists an edge dangerous set in G. Then s is admissible.*

Proof: Let A' be an edge dangerous set and suppose that $u \in A' \cap N(s)$. By Lemma 2.4 there is a unique maximal edge dangerous set A containing u. It follows from Lemma 2.3(a) that $B = V - A$ is also edge dangerous and that we have $d(s, A) = d(s, B) = 2$. Let $v \in B \cap N(s)$. We shall prove that the pair su, sv is admissible. For a contradiction suppose that su, sv is a non-admissible pair. By the maximality of A and by the choice of v there is no edge dangerous set containing both u and v. Thus Lemma 2.2(b) or (c) must hold.

First suppose that there is a vertex dangerous pair (X, r) with $u, v \in X$. By interchanging the role of u and v, if necessary, we can assume that $r \in A$. By Lemma 2.3(b) it follows that $d(s, X) = 2$ and $(V - X - r, r)$ is either vertex dangerous or vertex critical. Since $d(s, X) = 2$, we must have $(N(s) \cap B) - X \neq \emptyset$. Thus $B - X \neq \emptyset$. Lemma 2.3(a) implies that $G[B]$ is 2-edge-connected. Since $B \cap X \neq \emptyset$, we obtain $d_{G-r}(X) \geq 4$, contradicting the fact that (X, r) is vertex dangerous or vertex critical.

Next suppose that there is a vertex critical pair (X, u) with $v \in X$. (The case when there is a vertex critical pair (X, v) with $u \in X$ is similar.) By definition we have $d(s, X) = 1$. Since $d(s, B) = 2$, this implies $(N(s) \cap B) - X \neq \emptyset$, and so $B - X \neq \emptyset$. Since $G[B]$ is 2-edge-connected by Lemma 2.3(a), $v \in X \cap B$ and $u \notin B$, this implies that $d_{G-u}(X) \geq 3$. This contradicts the fact that (X, u) is a vertex critical pair.

Thus su, sv is an admissible pair, as claimed. Since $d(s) = 4$, the lemma now follows from Proposition 2.1. •

Lemma 3.2. *Suppose that $d(s, u) = 2$ for some $u \in N(s)$. Then s is admissible.*

Proof: By Lemma 3.1 we may suppose that there is no edge dangerous set in G. If su, su is an admissible pair then we are done by Proposition 2.1. So we may also suppose that the pair su, su is non-admissible. Now it follows from Lemma 2.2 that there is a vertex dangerous pair (X, r) with $u \in X$. (Clearly, Lemma 2.2(c) cannot hold when $v = u$.) By Lemma 2.3(b) we have $d(s, X) = 2$. Thus we have $u \neq w, y$ for the other two neighbours w, y of s. (But $w = y$ is possible.)

By Lemma 2.3(b) we have $d(s, V - r - X) \geq 1$. We may assume that $w \in N(s) - r - X$. We shall prove that su, sw is an admissible pair. Suppose, for a contradiction, that this is not the case. Then we can use Lemmas 2.2, 2.3, and the fact that $d(s, u) = 2$ to deduce that there is a vertex critical pair (X', u) with $w \in X'$. Let $X'' = V - u - X'$. By Lemma 2.3(c) the pair (X'', u) is also vertex critical and we must have $y \in X''$. By relabelling w and y, if necessary, we may assume that $r \notin X'$.

Since $d_{G-u}(X') = 2$ and $d(X') \geq 4$ by (1), we have $d(X', u) \geq 2$. If $X \cap X' = \emptyset$ then this implies $d_{G-r}(X) \geq 4$, contradicting the fact that (X, r) is a vertex dangerous pair. If $X \cap X' \neq \emptyset$ then, since $G[X']$ is connected by Lemma 2.3(c), we have $d(X \cap X', X' - X) \geq 1$. If equality holds here then $d_{G-u}(X' - X) = 2$ and

$d(u, X' - X) \geq 2$ also follow, since $d(X' - X) \geq 4$ by (1). Hence we can deduce $d_{G-r}(X) \geq 4$, contradicting the fact that (X, r) is vertex dangerous. Thus su, sw is an admissible pair, as claimed. This completes the proof of the lemma by Proposition 2.1. •

4 Obstacles

Let $G = (V + s, E)$ be weakly four-connected in V and let $d(s) = 4$. Now we define a general configuration in G which precludes the existence of an admissible (complete) splitting at s. By Lemma 3.2 we may assume that s has four distinct neighbours. Let $N(s) = \{t, v, w, y\}$.

Suppose that for $t \in N(s)$ there exist three pairwise disjoint sets $A, B, C \subset V - t$ with

(i) $v \in A, w \in B, y \in C$, and

(ii) $d_{G-t}(A) = d_{G-t}(B) = d_{G-t}(C) = 2$.

In this case vertex t and the sets A, B, C form a configuration that we call a *t-star obstacle* at s in G and denote by (t, A, B, C). An *obstacle at s* is a t-star obstacle at s for some $t \in N(s)$. Note that if (t, A, B, C) is a t-star obstacle then $(A, t), (B, t), (C, t)$ are vertex critical pairs. By Lemma 2.2 this implies that if there is a t-star obstacle at s then there is no admissible pair of edges containing the edge st, and hence there is no admissible complete splitting at s. For example, in the graph of Figure 1, $(r, \{a\}, \{b\}, \{c\})$ is an r-star obstacle at v.

It follows from (2) that $G - x$ is 2-edge-connected in $V - x$ for any $x \in N(s)$. Furthermore, $G[V - x]$ is connected, since $d(s) = 4$ and $x \in N(s)$. Thus the next lemma follows from Lemma 2.5, applied to $G - t$ and $k = 2$.

Lemma 4.1. *Let $t \in N(s)$ and suppose that every edge su ($u \neq t$) enters a set $X \subset V - t$ with $d_{G-t}(X) = 2$. Then for every edge su ($u \neq t$) there exists a unique minimal set $X_u \subset V - t$ with $d_{G-t}(X_u) = 2$. Furthermore, for any two distinct neighbours $u, v \in N(s) - t$ we have $X_u \cap X_v = \emptyset$.*

Lemma 4.1 has the following corollaries. A t-star obstacle (t, A, B, C) is *minimal* if the sets $A, B, C \subset V - t$ are all inclusionwise minimal 2-critical sets in $G - t$.

Lemma 4.2. *(a) If there exist vertex critical pairs (A', t), (B', t) and (C', t) with $v \in A', w \in B', y \in C'$, then there is a t-star obstacle at s.*
(b) If there is a t-star obstacle at s then the minimal t-star obstacle is unique.

Lemma 4.3. *Let (t, A, B, C) be a minimal t-star obstacle at s. Then*
(a) $G[A], G[B], G[C]$ are 2-edge-connected,
(b) there is no r-star obstacle at s for $r \in N(s) - t$,
(c) the subgraphs $G[A+s], G[B+s], G[C+s]$ are connected and pairwise vertex-disjoint apart from vertex s.

Proof: The proof of (a) is similar to the proof of Lemma 2.3. (c) follows from (a) and the definition of a t-star obstacle. Consider (b). For a contradiction suppose, without loss of generality, that there is a y-star obstacle (y, X, Y, Z) at s in G with $t \in X$. We can also assume that $v \in A$. Now $v \in A - X$. Since $d_{G-t}(A) = 2$ and $d(A) \geq 4$, we have $d(t, A) \geq 2$. If $X \cap A = \emptyset$ then, since $d(t, A) \geq 2$, we have $d_{G-y}(X) \geq 3$, a contradiction. Otherwise, since $v \in A - X$ and so $A - X \neq \emptyset$, it follows from (a) that $d(A \cap X, A - X) \geq 2$, which gives $d_{G-y}(X) \geq 3$, a contradiction. •

5 The characterisation of admissible vertices

In this section we show that if $d(s) = 4$ and s is non-admissible then there is a t-star obstacle for some $t \in N(s)$.

Lemma 5.1. *Let $G = (V + s, E)$ be weakly four-connected in V, such that $d(s) = 4$ and s has four distinct neighbours, denoted by $N(s) = \{t, v, w, y\}$. If there exists a vertex dangerous pair (X, r) with $r \notin N(s)$ and $t, v \in X$, then there is no vertex dangerous pair (Y, r') with $r' \notin N(s)$ and $v, w \in Y$.*

Proof: For a contradiction suppose that there exists a vertex dangerous pair (Y, r') with $r' \notin N(s)$ and $v, w \in Y$. Lemma 2.4 implies that the maximal set $M \subset V - r$ with $d_{G-r}(M) \leq 3$ and $v \in M$ is unique, and has $d(s, M) = 2$. Thus $r \neq r'$.

Let $X' = V - r - X$ and $Y' = V - r' - Y$. It follows from Lemma 2.3 that (X', r) is a vertex dangerous pair with $w, y \in X'$ and (Y', r') is a vertex dangerous pair with $t, y \in Y'$. By relabelling the neighbours of s, if necessary, we can assume that $r' \in X'$ and $r \in Y'$. Consider X and Y.

By Lemma 2.3(b) $G[X]$ is connected, and hence $d(X \cap Y, X - Y) \geq 1$. If $d(X \cap Y, X - Y) \geq 2$ then $d_{G-r'}(Y) \geq d(s, Y) + d(X \cap Y, X - Y) \geq 4$, contradicting the fact that (Y, r') is a vertex dangerous pair. If $d(X \cap Y, X - Y) = 1$ then $d_{G-r}(X \cap Y) \leq 3$, and hence (1) implies $d(r, X \cap Y) \geq 1$. Thus $d_{G-r'}(Y) \geq d(s, Y) + d(X \cap Y, X - Y) + d(r, Y) \geq 4$, a contradiction. •

Theorem 5.2. *Let $G = (V + s, E)$ be weakly four-connected in V such that $|V| \geq 3$ and $d(s) = 4$. Then s is non-admissible if and only if there is a t-star obstacle at s for some $t \in N(s)$.*

Proof: We have already verified the 'if' direction when we defined obstacles. To prove the 'only if' part suppose that s is non-admissible and, for a contradiction, suppose also that there is no u-obstacle at s for all $u \in N(s)$. It follows from Proposition 2.1 that there is no admissible pair of edges incident to s. Thus, by using Lemma 3.1 and Lemma 3.2, we may assume that there is no edge dangerous set in G and $d(s, u) = 1$ for all $u \in N(s)$. Let $N(s) = \{t, v, w, y\}$ denote the four distinct neighbours of s.

First suppose that there is a vertex critical pair (A, t) for $t \in N(s)$. By symmetry we may assume that $v \in A$.

Claim 5.3. *There exists a vertex critical pair (X, t) with $w \in X$.*

Proof: For a contradiction suppose that there is no vertex critical pair (X, t) with $w \in X$. Since there is no edge dangerous set in G, and the pair st, sw is non-admissible, it follows from Lemma 2.2 that either (i) there is a vertex dangerous pair (Y, r) with $t, w \in Y$, or (ii) there is a vertex critical pair (Y, w) with $t \in Y$.

Case 1. There is a vertex dangerous pair (Y, r) with $t, w \in Y$.

Let $B := V - t - A$. Clearly, $y \in B - Y$. By Lemmas 2.3(b),(c) and 3.2, (B, t) is a vertex dangerous pair and $G[B]$ is connected. We have two subcases to consider.

Subcase 1: $r \notin N(s)$

First suppose that $r \in A$. If $d(Y \cap B, B - Y) \geq 2$, then $d_{G-r}(Y) \geq 4$, a contradiction. If $d(Y \cap B, B - Y) = 1$ then $d_{G-t}(B - Y) \leq d_{G-t}(B) = 3$ and hence $d(t, B - Y) \geq 1$ by (2). Thus $d_{G-r}(Y) \geq d(s, Y) + d(Y \cap B, B - Y) + d(t, B - Y) \geq 4$, a contradiction.

Now suppose $r \in B$. If $Y \cap A = \emptyset$ then, since we have $d(t, A) \geq 2$ by (1), we obtain $d_{G-r}(Y) \geq 4$, a contradiction. If $Y \cap A \neq \emptyset$ then, since $G[A]$ is connected by Lemma 2.3(c), either $d(Y \cap A, A - Y) \geq 2$, and hence $d_{G-r}(Y) \geq 4$, or $d(Y \cap A, A - Y) = 1$, and hence $d(t, A - Y) \geq 2$ and $d_{G-r}(Y) \geq 5$ follow. These contradictions complete the proof in the first subcase.

Subcase 2: $r \in N(s)$

Now we have $r \in \{y, v\}$. First suppose $r = y$. Then by Lemma 2.3(c) there is a vertex critical pair (Z, y) with $v \in Z$. Focus on A and Z. If $A \subseteq Z$ then $d(t, A) \geq 2$ implies $d_{G-y}(Z) \geq 3$, a contradiction. We have a similar contradiction when $Z \subset A$. Otherwise, by similar arguments that we used above, we obtain $d(t, A \cap Z) + d(A \cap Z, A - Z) \geq 2$, which gives $d_{G-y}(Z) \geq 3$, a contradiction.

Now suppose $r = v$. Then by Lemma 2.3(c) there is a vertex critical pair (Z', v) with $y \in Z'$. Focus on B and Z'. Since $w \in B - Z'$, we have $B - Z' \neq \emptyset$. If $d(B - Z', B \cap Z') \geq 2$ then $d_{G-v}(Z') \geq 3$ follows, a contradiction. If $d(B - Z', B \cap Z') = 1$, then $d(t, B \cap Z') \geq 1$, and hence $d_{G-v}(Z') \geq d(s, Z') + d(t, B \cap Z') + d(B - Z', B \cap Z') \geq 3$, a contradiction.

Case 2. There is a vertex critical pair (Y, w) with $t \in Y$.

Consider Y and A. If $Y \cap A = \emptyset$ then, since $d(t, A) \geq 2$ and $t \in Y$, we must have $d_{G-w}(Y) \geq d(s, Y) + d(t, A) \geq 3$, a contradiction. If $A \cap Y \neq \emptyset$ then either $d(A - Y, A \cap Y) \geq 2$ or $d(t, A - Y) \geq 2$. In both cases $d_{G-w}(Y) \geq d(s, Y) + d(A - Y, A \cap Y) + d(t, A - Y) \geq 3$, a contradiction. This completes the proof of the claim. •

Claim 5.3 and Lemma 4.2(a) imply that there is a t-star obstacle at s, which contradicts our assumption.

Thus we may assume that there is no vertex critical pair (X, r) with $r \in N(s)$. Hence, by Lemma 2.3(b), there is no vertex dangerous pair (X', r) with $r \in N(s)$ either. Consider the pair st, sv. Since this pair is non-admissible, it follows from Lemma 2.2 that there is a vertex dangerous pair (Y, r) with $r \notin N(s)$ and $t, v \in Y$. Now Lemma 5.1 implies that the pair sv, sw is admissible, a contradiction. This completes the proof of the theorem. •

6 Weakly four-connected graphs

In this section we shall apply the results of the previous sections to find vertices of degree four in (globally) weakly four-connected graphs that can be split off preserving weak four-connectivity. Let $G = (V, E)$ be weakly four-connected. We call a vertex $s \in V$ with $d(s)$ even *admissible* if there is a complete splitting at s for which the graph obtained by the splittings on vertex set $V - s$ is weakly four-connected. Recall that we used the same definition of admissibility in the case when G was assumed to be weakly four-connected only in $V - s$. Observe that if G is weakly four-connected then G is weakly four-connected in $V - s$ for every $s \in V$. Thus we may apply the previous results to vertices of degree four in G. In particular, we can use Theorem 5.2 to deduce that if $d(s) = 4$ then s is admissible if and only if there is no obstacle at s.

A subset $\emptyset \neq X \subset V$ is called an *edge fragment* in G if $d(X) = 4$. It is called a *mixed fragment* if $d_{G-x}(X) = 2$ for some $x \in V - X$. A *fragment* is an edge fragment or a mixed fragment. Note that if (A, t) is a vertex critical pair (in particular, if A belongs to a t-start obstacle (t, A, B, C)) then A is a mixed fragment.

Lemma 6.1. *Let Y be a fragment and suppose that (u, A, B, C) is a minimal u-star obstacle at s for some $u \in N(s)$, $s \in Y$. Then if*
(a) Y is a mixed fragment, or
(b) Y is an edge fragment with $|Y| \geq 2$,
then $A \subset Y$ or $B \subset Y$ or $C \subset Y$ holds.

Proof: For a contradiction suppose that $A-Y$, $B-Y$, $C-Y$ are all non-empty. First consider the case when Y is a mixed fragment. By definition, we have $d_{G-x}(Y) = 2$ for some $x \in V - Y$. Since $s \in Y$ and Y is a mixed fragment (and hence $|N(Y)| \leq 3$), Lemma 4.3(c) implies that $u \in Y$, $|N(Y)| = 3$, and each of the sets A, B, C contains precisely one neighbour of Y. Let $N(Y) = \{x, b, c\}$. We may assume that $x \in A, b \in B, c \in C$. Since G is 4-edge-connected and $d_{G-u}(B) = d_{G-u}(C) = 2$, we must have $d(u, B), d(u, C) \geq 2$. Since $d_{G-x}(Y) = 2$, it follows that $d(Y, B-Y) = d(Y, C-Y) = 1$. Thus $B \cap Y \neq \emptyset \neq C \cap Y$. By Lemma 4.3(a) $G[B]$ and $G[C]$ are 2-edge-connected. This implies $d_{G-x}(Y) \geq d(B \cap Y, B - Y) + d(C \cap Y, C - Y) \geq 4$, a contradiction.

Next consider the case when Y is an edge fragment with $|Y| \geq 2$. Now we have $d(Y) = 4$. Since G is 4-edge-connected, we must have $d(s, Y - s) \geq 2$. Thus we may assume that $A \cap Y \neq \emptyset$. By Lemma 4.3(a) $G[A]$ is 2-edge-connected, and hence $d(A \cap Y, A - Y) \geq 2$. Since $d(Y) = 4$, Lemma 4.3(c) now implies $d(Y, B - Y) = d(Y, C - Y) = 1$ and $u \in Y$. Since $G[B]$ and $G[C]$ are also 2-edge-connected, this gives $B \cap Y = \emptyset = C \cap Y$. Since G is 4-edge-connected and $d_{G-u}(B) = 2$, we must have $d(u, B) \geq 2$. This contradicts the fact that $u \in Y$ and $d(Y, B) = 1$. •

Lemma 6.2. *Suppose that every mixed fragment of G contains a vertex of degree four and let (t, A, B, C) be a t-star obstacle at s for some $s \in V$. Then each of the sets A,B,C contains an admissible vertex of degree four.*

Proof: Let $A_4 = \{v \in A : d(v) = 4\}$. By our assumption $A_4 \neq \emptyset$. For a contradiction suppose that each vertex of A_4 is non-admissible. By Theorem 5.2 there is a minimal $t(s)$-star obstacle $(t(s), X_1^s, X_2^s, X_3^s)$ at s for some $t(s) \in N(s)$ for each vertex $s \in A_4$. Let us choose s and X_i^s in such a way that no other vertex $s' \in A_4$ and set $X_j^{s'}$ satisfies $X_j^{s'} \subset X_i^s$. Since X_i^s is a mixed fragment, our assumption implies that there is a vertex $q \in A_4 \cap X_i^s$. Now we can use Lemma 6.1 to deduce that $X_l^q \subset X_i^s$ for some $1 \leq l \leq 3$. This contradicts the choice of s and X_i^s. •

Theorem 5.2 and Lemma 6.2 imply the following sufficient condition for the existence of an admissible vertex of degree four. (This condition will be used in [2].)

Theorem 6.3. *Let $G = (V, E)$ be weakly four-connected with $|V| \geq 4$ and suppose that V as well as every mixed fragment of G contains a vertex of degree four. Then G has an admissible vertex of degree four.*

Theorem 6.4. *Let $G = (V, E)$ be weakly four-connected with $|V| \geq 4$ and suppose that every fragment of G contains a vertex of degree four. Then every fragment Y of G with $|Y| \geq 2$ contains an admissible vertex of degree four.*

Proof: Let $Y_4 = \{v \in V : d(v) = 4\}$. For a contradiction suppose that each vertex in Y_4 is non-admissible. Let $s \in Y_4$. By Theorem 5.2 there is a t-star obstacle (t, A, B, C) at s for some $t \in N(s)$. We may assume, by using Lemma 6.1, that $A \subset Y$. Thus Y contains a mixed fragment. Let A' be a minimal mixed fragment in Y and let s' be a vertex of degree four in A'. By Lemma 6.1(a), and the choice of A', we can deduce that s' is admissible, a contradiction. •

6.1 Minimally weakly four-connected graphs

Let $G = (V, E)$ be weakly four-connected. We say that G is *minimally weakly four-connected* if $G - e$ is not weakly four-connected for all $e \in E$. By specialising a more general result [7, Lemma 7] to the case of weakly four-connected graphs, we obtain:

Lemma 6.5. *[7, Lemma 7] Let G be a minimally weakly four-connected graph and let A be a fragment of G. Then there is a vertex $v \in A$ with $d(v) = 4$.*

Since every minimally weakly four-connected graph contains a fragment (in fact, every edge enters a fragment), Lemma 6.5 and Theorems 6.3, 6.4 imply:

Theorem 6.6. *Let $G = (V, E)$ be a minimally four-connected graph with $|V| \geq 4$. Then G has an admissible vertex of degree four and every fragment Y of G with $|Y| \geq 2$ contains an admissible vertex of degree four.*

Mader [10, Theorem 20] proved that a minimally (k, l)-connected graph has at least $k + 1$ vertices of degree kl. For minimally weakly four-connected graphs this implies the existence of at least three vertices of degree four. By using this fact and Theorems 5.2, 6.3 we can slightly improve on the first part of Theorem 6.6.

Theorem 6.7. *Let $G = (V, E)$ be a minimally four-connected graph with $|V| \geq 4$. Then G has at least three admissible vertices of degree four.*

7 The inductive construction

Let K_3^2 denote the graph which has three vertices $\{a,b,c\}$ and six edges $\{ab, ab, ac, ac, bc, bc\}$. This graph is (minimally) weakly four-connected. We omit the proof of the following simple lemma.

Lemma 7.1. *Let $G = (V, E)$ be weakly four-connected and let $e, f \in E$ be distinct edges such that if one of them is a loop then they have no common end-vertex. Then the graph obtained by hooking up e and f is weakly 4-connected.*

Theorem 7.2. *A graph $G = (V, E)$ with $|V| \geq 3$ is weakly four-connected if and only if it can be obtained from K_3^2 by the following operations:*
(a) adding a new edge, which connects existing vertices,
(b) hooking up two edges (such that if one of them is a loop then they have no common end-vertex).

Proof: The 'if' direction follows from Lemma 7.1. To prove the 'only if' direction it suffices to show that if G is weakly four-connected and $G \neq K_3^2$ then it is possible to perform the inverse operations of (a) or (b) on G in such a way that the resulting graph is also weakly four-connected. The inverse operations are edge deletion and splitting off at a vertex of degree four. If G is not minimally weakly four-connected then there is an edge $e \in E$ such that $G' = G - e$ is weakly four-connected. If G is minimally weakly four-connected (and $|V| \geq 4$) then there is an admissible vertex s of degree four in G by Theorem 6.6, so it is possible to split off a vertex of degree four preserving weak four-connectivity. This completes the proof of the theorem by noting that for such an admissible vertex s we must have $d(s, u) \leq 2$ for all $u \in N(s)$, and hence if one of the split edges is a loop then the two split edges have no common end-vertex. •

References

[1] D.W. Barnette, H. Grünbaum, On Steinitz's theorem concerning convex 3-polytopes and on some properties of planar graphs, in: The many facets of graph theory, Lecture Notes in Mathematics, Vol. 110, eds. G. Chartrand and S.F. Kapoor, Springer, pp. 27-40.

[2] A. Berg, T. Jordán, Two-connected orientations of Eulerian graphs, EGRES Technical Report 2004-3, `http://www.cs.elte.hu/egres/`, submitted.

[3] Y. Egawa, A. Kaneko, and M. Matsumoto, A mixed version of Menger's theorem, Combinatorica 11, No. 1 (1991), 71-74.

[4] A. Frank, Connectivity and network flows. Handbook of combinatorics, Vol. 1, 2, 111–177, Elsevier, Amsterdam, 1995.

[5] A. Frank, Augmenting graphs to meet edge-connectivity requirements, SIAM J. Discrete Math. 5 (1992), no. 1, 25–53.

[6] T. Jordán, On the existence of k edge-disjoint 2-connected spanning subgraphs, EGRES Technical Report 2004-5, submitted. `http://www.cs.elte.hu/egres/`

[7] A. Kaneko, K. Ota, On minimally (n, λ)-connected graphs, J. Combin. Theory, Series B 80, 156-171 (2000).

[8] L. Lovász, Combinatorial problems and exercises, North-Holland, 1979.

[9] W. Mader, A reduction method for edge-connectivity in graphs, Ann. Discrete Math. 3 (1978) 145-164.

[10] W. Mader, On vertices of degree n in minimally n-connected graphs and digraphs, Combinatorics, Paul Erdős is eighty (Vol. 2), Bolyai Society Math. Studies 2, 1996, pp. 423-449.

[11] C. St. J. A. Nash-Williams, On orientations, connectivity, and odd vertex pairings in finite graphs, Canad. J. Math. 12 (1960), 555-567.

[12] C. Thomassen, Configurations in graphs of large minimum degree, connectivity, or chromatic number, Annals of the New York Academy of Sciences 555 (1989) 402-412.

[13] W.T. Tutte, A theory of 3-connected graphs, Indag. Math. 23, 441-455.