Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2003-10. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

On constructive characterizations of (k, l)-sparse graphs

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December 2003

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Abstract

In this paper we study constructive characterizations of graphs satisfying tree-connectivity requirements. The main result is the following: if k and l are positive integers and $l \leq \frac{k}{2}$, then a necessary and sufficient condition is proved for a node beeing the last node of a construction in a graph having at most $k|X| - (k + l)$ induced edges in every subset X of nodes.

Keywords: sparse graph, constructive characterization

1 Constructive characterizations

A constructive characterization of a graph property is meant to be a building procedure consisting of some simple operations so that the graphs obtained from some specified initial graph by these operations are precisely those having the property. For example, a graph is connected if and only if it can be obtained from a node by the operation: add a new edge connecting an existing node with either an existing node or a new one. Another well-known result is the so called ear-decomposition of 2-connected graphs.

A graph is said to be *k-edge-connected* if the deletion of at most $k-1$ edges results in a connected graph. From now on, adding an edge means adding a new edge connecting two existing nodes. This new edge can be parallel to existing ones, but it cannot be a loop unless otherwise stated. In 1976 Lovász [10] proved the following result.

Theorem 1.1. *An undirected graph $G = (V, E)$ is $2k$-edge-connected if and only if G can be obtained from a single node by the following two operations:*

(i) *add a new edge (possibly a loop),*

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(ii) *add a new node z, subdivide k existing edges by new nodes, and identify the k subdividing nodes with z.*

Operation (ii) is called *pinching* k edges.

Similar constructive characterizations for $2k+1$-edge-connectivity were given by Mader. A directed counterpart of the previous results is also due to Mader [11]. This kind of characterizations can be very useful. For example, Lovász used his result to derive Nash-Williams' theorem [12] on k-edge-connected orientations of graphs, while Mader used his result to derive Edmonds' theorem [2] on disjoint arborescences.

k-edge-connectivity is the common way to formulate one's intuitive feeling for high 'edge-connection' of an undirected graph but there may be other possibilities, as well.

An undirected graph is called *k-tree-connected* if it contains k edge-disjoint spanning trees. The following constructive characterization of k-tree-connected graphs was given by Frank in [3] by observing that a combination of a theorem of Mader and a theorem of Tutte gives rise to the following. (For a direct proof, see Tay [14].)

Theorem 1.2. *An undirected graph $G = (V, E)$ is k-tree-connected if and only if G can be built from a single node by the following two operations:*

(i) *add a new edge,*

(ii) *add a new node z and k new edges ending at z,*

(iii) *pinch i $(1 \leq i \leq k-1)$ existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.*

Which constructive characterization can be considered to be good. Jüttner [8] gave the following building procedure for graphs having a Hamiltonian cycle. Beginning from K_3 use the following two operations: adding a new edge between two existing nodes and subdividing an edge incident to a node of degree 2 by a new node. It is clear that this procedure builds up a graph G if and only if G has a Hamiltonian cycle.

Why do not we think that this is a good constructive characterization? We did not accept the characterization of k-tree-connected graphs by taking immediately k edge-disjoint trees because it does not take the nodes one by one. Here it is satisfied. The main problem of this characterization here is that it cannot be checked for a graph in polynomial time if it can be obtained this way or not.

Nash-Williams [13] proved the following theorem concerning coverings by trees. For a graph $G = (V, E)$, $\gamma_G(X)$ denotes the number of edges of G with both end-nodes in $X \subseteq V$.

Theorem 1.3 (Nash-Williams). *A graph $G = (V, E)$ is the union of k edge-disjoint forests if and only if $\gamma_G(X) \leq k|X| - k$ for all nonempty $X \subseteq V$.*

In [5] two variants of the notion of k-tree-connectivity were considered. A graph G (with at least 2 nodes) is called *nearly k-tree-connected* if G is not k-tree-connected but adding any new edge to G results in a k-tree-connected graph. Let K_2^{k-1} denote the graph on two nodes with $k-1$ parallel edges. (Based on the work of Henneberg

[6] and Laman [9], Tay and Whiteley [16] gave the proof of the following theorem in the special case of $k = 2$.)

Theorem 1.4. *An undirected graph $G = (V, E)$ is nearly k-tree-connected if and only if G can be built from K_2^{k-1} by applying the following operations:*

(O1') *add a new node z and k new edges ending at z so that no k parallel edges can arise,*

(O2') *choose a subset F of i existing edges $(1 \leq i \leq k-1)$, pinch the elements of F with a new node z, and add $k - i$ new edges connecting z with other nodes so that there are no k parallel edges in the resulting graph.*

Actually, we proved this result in a slightly more general form. We proved the following conjecture in case $l = 1$. Let k, l be two integers such that $k \geq 2$ and $\frac{k}{2} \geq l \geq 0$. A graph $G = (V, E)$ is said to be (k, l)-*sparse* if $\gamma_G(X) \leq k|X| - (k + l)$ for all $X \subseteq V, |X| \geq 2$. (By convention the graph with one single node is (k, l)-sparse.)

Conjecture 1.5. *Let $1 \leq l < \frac{k+2}{3}$. An undirected graph $G = (V, E)$ is (k, l)-sparse if and only if G can be built from a single node by applying the following operations:*

(P1) *add a new node z and at most k new edges ending at z so that no $k - l + 1$ parallel edges can arise.*

(P2) *Choose a subset F of i existing edges $(1 \leq i \leq k-1)$, pinch the elements of F with a new node z, and add $k - i$ new edges connecting z with other nodes so that there are no $k - l + 1$ parallel edges in the resulting graph.*

(If $l = 0$ is allowed, then Theorem 1.2 is also a special case which has been already verified.) By the fundamental Theorem 1.3 of Nash-Williams, a graph is (k, l)-sparse if and only if the edge-set can be covered by k spanning trees after adding l new edges arbitrarily.

We call a graph *highly k-tree-connected* if the deletion of any existing edge leaves a k-tree-connected graph. Frank and Király [4] gave a constructive characterization (among others) for highly 2-tree-connected graphs. In [5] this was extended for arbitrary $k \geq 2$.

We mention a recent result of Berg and Jordán [1] who proved a conjecture of Connelly. A 2-connected undirected graph $G = (V, E)$ is a *generic circuit* if $|E| = 2|V| - 2$ and $\gamma_G(X) \leq 2|X| - 3$ for all $2 \leq |X| \leq |V| - 1$.

Theorem 1.6. *An undirected graph $G = (V, E)$ is a generic circuit if and only if G can be built up from K_4 by the following operation:*

- *subdivide an edge uv by a new node z and add an edge zw so that $w \neq u, v$.*

These graphs have a role in rigidity theory. We also remark that Whiteley in [17] provided some rigidity property of nearly k-tree-connected graphs.

Jackson and Jordán considers sparse graphs in connection with rigidity properties in [7]. In [15] Tay proved for inductive reasons that a node of degree at most $2k - 1$

either can be "split off", or "reduced" to obtain a smaller nearly k-tree-connected graph. Theorem 1.4 says that there always is a node which can be "split off".

We have the following theorem which follows easily from the definition of (k,l)-sparse graphs.

Theorem 1.7. *Let $1 \le l \le \frac{k}{2}$. If an undirected graph $G=(V,E)$ can be built up from a single node by applying the operations* (P1) *and* (P2)*, then G is (k,l)-sparse.*

Inspired by the previous constructive characterizations we would conjecture that the reverse of the above theorem is also true for all k and l satisfying $\frac{k}{2} \ge l$. But as we will show in Section 4, this is not true if $l \ge \frac{k+2}{3}$. We believe that Conjecture 1.5 will be proved soon.

2 Splittings for (k,l)-sparse graphs

In the definition of (k,l)-sparse graphs why do not we allow bigger l values? The answer is that, if $\frac{k}{2} < l$ and $|E| = 3k-(k+l) = 2k-l$, then there is no graph on 3 nodes satisfying $\gamma_G(X) \le k|X|-(k+l)$ for all $X \subseteq V, |X| \ge 2$. Indeed, if there was one $G=(V,E)$, then $|E| \le 3(k-l)$ since an edge may have multiplicity at most $k-l$. Since $2k-l > 3k-3l$, we get a contradiction.

With the same reasoning the following can be proved.

Lemma 2.1. *There is no graph on $m \ge 3$ nodes with $|E| = km-(k+l)$ satisfying $\gamma_G(X) \le k|X|-(k+l)$ for all $X \subseteq V, |X| \ge 2$ if $\frac{m-1}{m+1}k < l$.*

Proof. Since $|E| \le \frac{m(m-1)}{2}(k-l)$ by the maximal multiplicity of an edge, we have $km-(k+l) = |E| \le \frac{m(m-1)}{2}(k-l)$. But

$$km-(k+l)-\frac{m(m-1)}{2}(k-l) =$$

$$\frac{(m^2-m-2)l-(m^2-3m+2)k}{2} = \frac{(m-2)((m+1)l-(m-1)k)}{2} >$$

$$\frac{1}{2}\left((m+1)\frac{m-1}{m+1}k-(m-1)k\right) = 0,$$

a contradiction. □

That is why we study here only the case of $l \le \frac{k}{2}$.

In graph G *splitting off* a pair zu and zv of edges for distinct u and v means that we delete these two edges and add a new edge uv (maybe parallel to the other existing edges) to G. After applying this operation, uv is called a *split edge*. A splitting off in a (k,l)-sparse graph G is *admissible* if the resulting graph on node set $V-z$ is (k,l)-sparse.

Definition 2.2. Let b_G denote the following function for any $X \subseteq V, |X| \geq 2$

$$b_G(X) := k|X| - (k+l) - \gamma_G(X).$$

By this definition a graph $G = (V, E)$ is (k,l)-sparse if and only if $b_G(X) \geq 0$ for all subsets $X \subseteq V, |X| \geq 2$. If $b_G(X) = 0$ and $X \neq V$, then X is said to be a *G-tight* set. Furthermore G is a union of k edge-disjoint spanning trees after adding arbitrary l edges if and only if G is (k,l)-sparse and $b_G(V) = 0$. We will abbreviate b_G by b.

Observation 2.3. Splitting off zu and zv at node z is not admissible if and only if there exists a tight subset in $V - z$ containing u and v.

We say that splitting off j disjoint pairs of edges $(1 \leq j \leq k-1)$ at node z is *admissible* if it consists of admissible splittings. Obviously the order of the pairs in a *splitting sequence* is irrelevant. The *length* of a splitting sequence $\mathcal{S}$ is the number of its pairs and it is denoted by $|\mathcal{S}|$. $G_{\mathcal{S}}$ denotes the graph obtained after applying the splitting sequence $\mathcal{S}$.

An admissible splitting sequence at node z of length $d_G(z) - k$ (which number is denoted by i) is called a *full splitting* for $d_G(z) \geq k+1$. For the sake of convenience, at a node z with degree at most k the inverse of operation (P1) (that is, the deletion of z and all of its adjacent edges) is also called a *full splitting.* The main result of this chapter is a necessary and sufficient condition of a node admitting a full splitting. We hope that it will lead to a proof of Conjecture 1.5 just like in the special case of $l = 1$.

Note that $b_G(X)$ is an upper bound for the number of split edges induced by $X \subseteq V - z$ provided by an admissible sequence of splittings at some node z.

The next four claims are about (k,l)-sparse graphs. ($d_G(X,Y)$ is defined to be the number of edges between the node-sets X and Y.)

Claim 2.4. *If $X, Y \subseteq V$ and $|X \cap Y| \geq 2$, then*

$$b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d(X,Y).$$

Proof. $b(X) + b(Y) = k|X| - (k+l) - \gamma_G(X) + k|Y| - (k+l) - \gamma_G(Y) = k(|X| + |Y|) - 2(k+l) - (\gamma_G(X \cap Y) + \gamma_G(X \cup Y) - d_G(X,Y)) = k|X \cap Y| - (k+l) - \gamma_G(X \cap Y) + k|X \cup Y| - (k+l) - \gamma_G(X \cup Y) + d_G(X,Y) = b(X \cap Y) + b(X \cup Y) + d(X,Y)$. □

Claim 2.5. *If $X, Y \subseteq V$ and $|X \cap Y| = 1$, then*

$$b(X) + b(Y) = b(X \cup Y) - l + d(X,Y).$$

Proof. $b(X) + b(Y) = k|X| - (k+l) - \gamma_G(X) + k|Y| - (k+l) - \gamma_G(Y) = k(|X| + |Y| - 1) - (k+l) - l - (\gamma_G(X) + \gamma_G(Y)) = k|X \cup Y| - (k+l) - l - (\gamma_G(X \cup Y) - d_G(X,Y)) = b(X \cup Y) - l + d(X,Y)$. □

Claim 2.6. *If $X_1, X_2, X_3 \subseteq V$ and $|X_j \cap X_m| = 1$ for $1 \leq j < m \leq 3$ and $|X_1 \cap X_2 \cap X_3| = 0$, then*

$$b\Big(\bigcup_{j=1}^{3} X_j\Big) \leq \sum_{j=1}^{3} b(X_j) - k + 2l.$$

Proof. $b(\bigcup_{j=1}^{3} X_j) = k|\bigcup_{j=1}^{3} X_j| - (k+l) - \gamma_G(\bigcup_{j=1}^{3} X_j) \leq k(\sum_{j=1}^{3}|X_j| - 3) - (k+l) - \sum_{j=1}^{3}\gamma_G(X_j) = \sum_{j=1}^{3}(k|X_j| - (k+l) - \gamma_G(X_j)) - k + 2l = \sum_{j=1}^{3} b(X_j) - k + 2l$. □

Remark. Especially, all of X_1, X_2, X_3 cannot be tight at the same time for $k \geq 2l+1$. If $k = 2l$ and X_1, X_2, X_3 are tight sets, then $\bigcup_{j=1}^{3} X_j$ is also tight.

Claim 2.7. *Let $z \in V$ and $X \subset V - z$ be a maximal tight set containing the distinct nodes c_1, c_2. Let d be a node in $V - X - z$. If there is a tight set in $V - z$ containing c_1 and d, then there is no tight set in $V - z$ containing c_2 and d.*

Proof. According to Claim 2.4, $P \cap X = \{c_1\}$ since X is maximal. By Claims 2.4 and 2.6 we obtain that there is no tight set containing c_2 and d. □

Let G be a (k,l)-sparse graph. Since $\sum_{v \in V} d_G(v) = 2|E| \leq 2k|V| - 2(k+l) < 2k|V|$, it follows that there is a node z of G with $d_G(z) \leq 2k - 1$.

Claim 2.8. *Let $G = (V,E)$ be a (k,l)-sparse graph. $d_G(u,v) \leq k - l$ for any two nodes u, v.*

Proof. By the definition of (k,l)-sparse graphs, $\gamma_G(\{u,v\}) \leq k|\{u,v\}| - (k+l) = k - l$ for set $\{u,v\}$. □

3 Full splittings in (k,l)-sparse graphs

In this section we derive a necessary and sufficient condition for an arbitrary specified node to admit a full splitting.

Let $k \geq 2$ and $0 \leq l \leq \frac{k}{2}$. Let G be a (k,l)-sparse graph. Consider a node z with degree at most $2k-1$ for which there is no full splitting. If $d_G(z) \leq k$, then the deletion of z and its adjacent edges results in a (k,l)-sparse graph, hence $d_G(z) \geq k+1$.

Assume that a longest admissible splitting sequence $\mathcal{S}$ at z is not full. Since z does not admit a full splitting, $|\mathcal{S}| < i := d_G(z) - k$.

Let $N_D(w)$ denote the set of the neighbours of a node w in graph D.

Claim 3.1. *If $|N_{G_\mathcal{S}}(z)| \geq 2$, then there exists a maximal $G_\mathcal{S}$-tight subset $P_{\max}$ of $V - z$ including $N_{G_\mathcal{S}}(z)$.*

Proof. Let za and zb denote two non-parallel edges. Since (za, zb) is not an admissible splitting off, there is a $G_\mathcal{S}$-tight set $X \subseteq V - z$ containing a and b. According to Claim 2.4, there is a maximal tight set $P \subseteq V - z$ containing a and b.

If there is another neighbour c of z which is not in P, then there is a tight set $Y \subseteq V - z$ containing a and c, since (za, zc) is not an admissible splitting off. Since P is maximal, $Y \cap P = \{a\}$. By Claim 2.7 (zb, zc) is an admissible splitting off, a contradiction, that is, P contains all the neighbours of z. □

Claim 3.2. *If $|N_{G_\mathcal{S}}(z)| \geq 2$, then there exists a split edge which is disjoint from the nodes of $P_{\max}$.*

Proof. Since there is no admissible splitting off at z in $G_{\mathcal{S}}$, according to Claim 3.1 there exists $P_{\max} \subseteq V - z$. Let j, h, m denote the number of split edges with exactly, respectively, 2, 1, 0 end-node in $P_{\max}$. $j + h + m = |\mathcal{S}| < i$ since $\mathcal{S}$ is not full.

$$k|P_{\max} + z| - (k+l) \geq \gamma_G(P_{\max} + z) = \gamma_{G_{\mathcal{S}}}(P_{\max}) + j + h + d_{G_{\mathcal{S}}}(z, P_{\max})$$

$$= \gamma_{G_{\mathcal{S}}}(P_{\max}) + j + h + (k + i - 2(j + h + m))$$

$$= \gamma_{G_{\mathcal{S}}}(P_{\max}) + k + (i - (j + h + m)) - m > k|P_{\max}| - (k+l) + k - m$$

$$= k|P_{\max} + z| - (k+l) - m,$$

which implies $m > 0$. □

Claim 3.3. *If $|N_{G_{\mathcal{S}}}(z)| \geq 2$, then $|N_{G_{\mathcal{S}}}(z)| = 2$. There is a neighbour s of z for which $d_{G_{\mathcal{S}}}(z, s) = 1$.*

Proof. First assume that $|N_{G_{\mathcal{S}}}(z)| \geq 3$. Let a_1, a_2, a_3 denote three of these nodes. By Claim 3.2 there is a split edge uv disjoint from $P_{\max}$. Let $J = \{1, 2, 3\}$.

By Claim 2.7, $\mathcal{S} - (zu, zv) \cup (zu, za_j)$ is an admissible splitting sequence for at least two elements j of J. The same is true for $\mathcal{S} - (zu, zv) \cup (zv, za_j)$. Hence we may assume that $\mathcal{S} - (zu, zv) \cup (zu, za_1)$ and $\mathcal{S} - (zu, zv) \cup (zv, za_2)$ are both admissible splitting sequences. We claim that $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, za_1) \cup (zv, za_2)$ is an admissible splitting sequence. If not, then there is a tight set Y in $G_{\mathcal{S}} - z$ containing u, v, a_1, a_2. Then, according to Claim 2.4, $P_{\max} \cup Y$ is a tight set in $G_{\mathcal{S}} - z$ contradicting the maximality of $P_{\max}$. The length of $\mathcal{S}'$ is greater than the length of $\mathcal{S}$, a contradiction.

Now assume that $|N_{G_{\mathcal{S}}}(z)| = 2$. Let s and t be the two neighbours of z and assume that $d_{G_{\mathcal{S}}}(z, s) \geq 2$ and $d_{G_{\mathcal{S}}}(z, t) \geq 2$. By Claim 3.2 there is a split edge uv disjoint from $P_{\max}$. According to Claim 2.7 $\mathcal{S} - (zu, zv) \cup (zu, zt)$ or $\mathcal{S} - (zu, zv) \cup (zu, zs)$ is an admissible splitting sequence. This also holds for zv instead of zu.

Hence at least one of the following splitting sequences is admissible: $\mathcal{S} - (zu, zv) \cup (zu, zt) \cup (zv, zt), \mathcal{S} - (zu, zv) \cup (zu, zt) \cup (zv, zs), \mathcal{S} - (zu, zv) \cup (zu, zs) \cup (zv, zt), \mathcal{S} - (zu, zv) \cup (zu, zs) \cup (zv, zs))$, a contradiction. □

Now we prove that if $d_G(z)$ is at most $k + l$, then a full splitting always exists at z.

Proposition 3.4. *Let G be a (k,l)-sparse graph.If $z \in V$ has degree at most $k + l$, then there exists a full splitting at z.*

Proof. If $d_G(z)$ is at most k, then if we delete z with its adjacent edges, then we obviously get a (k,l)-sparse graph, that is, z admits a full splitting.

We claim that there always exists a full splitting at a node z with degree $k+i$ where $1 \leq i \leq l$. There is no G-tight set $X \subseteq V - z$ which contains all the neighbours of z because, if there was one, then $b_G(X + z) = b_G(X) + k - d_G(z) \leq 0 + k - (k+1) < 0$ which contradicts that G is (k,l)-sparse. Since there are no edges with multiplicity greater than $k - l$, the neighbour-set of z in G has at least two elements, so by Observation 2.3 there is an admissible splitting off at z. Hence the longest admissible splitting sequence at z has length at least 1.

Let $\mathcal{S}$ be a longest admissible splitting sequence at z. If $|\mathcal{S}| \geq i$, then we are done. If $h := |\mathcal{S}| < i$, then $d_{G_\mathcal{S}}(z) \geq d_G(z) - 2(i-1) = k+i-2i+2 = k-i+2 \geq k-l+2$. Hence by Claim 2.8, $|N_{G_\mathcal{S}}(z)| \geq 3$ or $|N_{G_\mathcal{S}}(z)| = 2$ and both neighbours are joined to z by at least two edges. By Claim 3.3 $\mathcal{S}$ is not longest, a contradiction. □

Let $i := d_G(z) - k$ (here $2 \leq i \leq k-1$). Call a node z *small* if $k+l+1 \leq d_G(z) \leq 2k-1$.

Theorem 3.5. *A small node z of G does not admit a full splitting if and only if z has a neighbour t and there is a family $\mathcal{P}_z$ of subsets of $V-z$ with at least two elements such that:*

$$X \cap Y = \{t\} \quad for \quad X, Y \in \mathcal{P}_z, \tag{$*$}$$

$$\sum_{X \in \mathcal{P}_z} b(X) < d_G(z,t) - (k-i) - d_G(z, V - z - \cup\mathcal{P}_z), \tag{$**$}$$

where $\cup\mathcal{P}_z$ denotes $\bigcup_{X\in\mathcal{P}_z} X$.

Proof. Suppose first that t and $\mathcal{P}_z$ satisfy $(*)$, $(**)$ and let $\mathcal{S}$ be an admissible splitting sequence. The number of split edges incident to t with other end-nodes outside of $\cup\mathcal{P}_z$ is at most $d_G(z, V-z-\cup\mathcal{P}_z)$. The number of split edges incident to t with their other end-nodes in $\cup\mathcal{P}_z$ is at most $\sum_{X\in\mathcal{P}_z} b(X)$. In a full splitting we would have at least $d_G(z,t) - (k-i)$ split edges incident to t which implies by $(**)$ that $\mathcal{S}$ is not full.

To see the other direction, let $\mathcal{S}$ be a longest admissible splitting sequence at z for which the following pair is lexicographically maximal: $(|N_{G_\mathcal{S}}(z)|, |P_{\max}|)$ where $P_{\max}$ denotes a maximal tight set in $G_\mathcal{S}$ which includes $N_{G_\mathcal{S}}(z)$ but does not contain z. If there is no such a tight set, then let $P_{\max} := \emptyset$. Since z does not admit a full splitting, $|\mathcal{S}| < i$. From now on $G_\mathcal{S}$-tight is abbreviated by tight.

By Claim 3.3 there are only the following two Cases. An edge not incident to t is called *t-disjoint*.

CASE 1. $|N_{G_\mathcal{S}}(z)| = 2$ and z has a neighbour s for which $d_{G_\mathcal{S}}(z,s) = 1$.

Let $u \in V-t-s$ be an arbitrary node for which there is a t-disjoint split edge uv. There is a tight set $X \subseteq V-z$ containing u and t, otherwise $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, zt)$ is an other longest admissible splitting sequence for which if $v \neq s$, then $|N_{G_{\mathcal{S}'}}(z)| = 3$, if $v = s$ and $d_{G_\mathcal{S}}(z,t) \geq 3$, then $d_{G_{\mathcal{S}'}}(z,t) \geq d_{G_{\mathcal{S}'}}(z,s) \geq 2$, which is a contradiction by Claim 3.3. If $v = s$ and $d_{G_\mathcal{S}}(z,t) = 2$ and $d_{G_\mathcal{S}}(z,s) = 1$, then by Claim 3.2 there is a split edge ab which is disjont from $P_{max} \cup \{u\}$. Since $\mathcal{S}^* := \mathcal{S} - (za, zb) - (zu, zs) \cup (za, zs) \cup (zb, zs) \cup (zu, zt)$ is not admissible, we have a tight set in $G_\mathcal{S}$ containing a, b, t, s, u contradicting the maximal choice of P_{max} by Claim 2.5 (it also contradits that there is no tight set containing t and u). (By the previous cases and Claim 2.8, there is no tight set containing (a or b) and s.)

Let P_u be such a tight set containing minimal number of t-disjoint split edges which is inclusion-wise maximal. Similarly, there is a tight set $X \subseteq V-z$ containing s and t, otherwise $\mathcal{S} \cup (zs, zt)$ is a longer admissible splitting sequence than $\mathcal{S}$. Let P_s be such a tight set containing minimal number of t-disjoint split edges which is inclusion-wise maximal.

Let $\mathcal{P}_z := \{X \subseteq V - z : \exists u \in V$ incident to a t-disjoint split edge such that $X = P_u$ or $X = P_s\}$. For nodes $u \neq v$, P_u can be equal to P_v, but there is only one copy of them in $\mathcal{P}_z$. Now we prove some essential properties of $\mathcal{P}_z$.

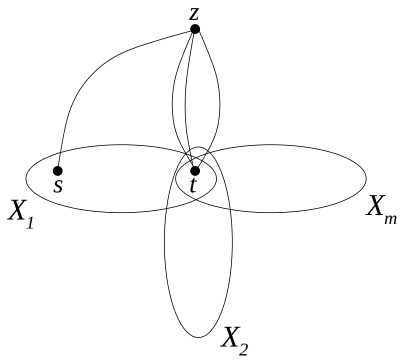

Figure 1: A set-system $\mathcal{P}_z$.

Proposition 3.6. *There is no t-disjoint split edge in any member X of $\mathcal{P}_z$.*

Proof. First let us assume that $X = P_s$. Let us suppose indirectly that there is a t-disjoint split edge ab in P_s. $\mathcal{S}' := \mathcal{S} - (za, zb) \cup (zt, zs)$ is an admissible splitting sequence with three remaining neighbours of z in $G_{\mathcal{S}'}$, which is a contradiction by Claim 3.3.

Now let us assume $X = P_u$ and $u \neq s$. By the definition of P_u we have a t-disjoint split edge uv. Let us suppose indirectly that there is a t-disjoint split edge ab in P_u. We may suppose that $b \neq u$.

If $v \neq s$, then $v \notin P_u$ (if $v \in P_u$, then $\mathcal{S} - (zu, zv) \cup (zt, zu)$ is an admissible splitting sequence with the same length but with one more remaining neighbour of z). $P_v \cap P_u = \{t\}$ according to Claim 2.4. $\mathcal{S} - (za, zb) - (zu, zv) \cup (zt, zu) \cup (zv, za)$ is an other longest splitting sequence with one more remaining neighbour of z, so it cannot be admissible, that is, there is a set $Y \subseteq V - z$ containing a, u, v, t, which is tight in $G_{\mathcal{S}}$. Y does not contain b, hence the tight set $Y \cap P_u$ contains a smaller number of split edges than P_u, a contradiction. If $v = s$ and $v \notin P_u$, then the proof is the same.

Suppose that $v = s$ and $v \in P_u$. Let us consider a split edge cd which is disjoint from $P_{\max}$ and hence from P_u (such an edge exists according to Claim 3.2). By the previous paragraph tight sets P_c and P_d do not contain t-disjoint split edges. According to Claim 2.4, $P_c \cap P_{\max} = \{t\}$.

According to Claim 2.7, $\mathcal{S}' := \mathcal{S} - (zc, zd) \cup (zc, zs)$ is an admissible splitting sequence. For $\mathcal{S}'' := \mathcal{S}' - (zu, zv) \cup (zt, zu)$, the cardinality of $N_{G_{\mathcal{S}''}}(z) = \{t, s, d\}$ is 3, hence $\mathcal{S}''$ cannot be admissible, that is, there is a tight set $Y \subseteq V - z$ containing c, s, u, t in $G_{\mathcal{S}'}$. $Y \cup P_{\max}$ (in $G_{\mathcal{S}'}$) contradicts the choice of $\mathcal{S}$ by the maximality of $P_{\max}$. □

Now it follows that $(**)$ holds for $\mathcal{P}_z$.

Claim 3.7. *Let X, Y be two distinct members of $\mathcal{P}_z$. $X \cap Y = \{t\}$.*

Proof. Let us suppose $X = P_u$ and $Y = P_v$ for some $u, v \in V$. By Proposition 3.6, $P_u \not\subset P_v$. If $|P_u \cap P_v| \geq 2$, then by Claim 2.4 $d_{G_\mathcal{S}}(P_u, P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any t-disjoint split edge, it contradicts the maximal choice of P_u. □

Hence $(*)$ holds for $\mathcal{P}_z$.

CASE 2. $|N_{G_\mathcal{S}}(z)| = 1$. Let t denote the only neighbour of z in $G_\mathcal{S}$.

Claim 3.8. *There exists a t-disjoint split edge.*

Proof. Let l and m be the number of split edges incident to, respectively, not incident to t. Since $\mathcal{S}$ is not full, $l + m = |\mathcal{S}| < i$. In the original graph G by Claim 2.8:

$$k - 1 \geq d_G(z, t) = d_G(z) - l - 2m = k + i - l - 2m = k + (i - l - m) - m > k - m,$$

which implies that $m > 1$. □

Since $\mathcal{S}$ is not a full splitting: $d_{G_\mathcal{S}}(z) \geq k + i - 2(i - 1) = k - i + 2 \geq 3$. Now we define $\mathcal{P}_z$. Let $u \in V - t$ be an arbitrary node for which there is a t-disjoint split edge uv. There is a tight set $X \subseteq V - z$ containing u and t, otherwise $\mathcal{S}' := \mathcal{S} - (zu, zv) \cup (zu, zt)$ is an other longest admissible splitting sequence for which $|N_{G_{\mathcal{S}'}}(z)| = 2$, which contradicts the choice of $\mathcal{S}$. Let P_u be such a tight set containing minimal number of t-disjoint split edges which is inclusion-wise maximal. Let $\mathcal{P}_z := \{X \subseteq V - z : \exists u \in V$ incident to a t-disjoint split edge such that $X = P_u\}$. (The only difference to Case 1. is that there is no set P_s here.)

Proposition 3.9. *There is no t-disjoint split edge in an arbitrary element of $\mathcal{P}_z$.*

Proof. Assume $X = P_u$. By the definition of P_u we have a t-disjoint split edge uv. Let us suppose indirectly that there is a t-disjoint split edge ab in P_u. We may suppose that $b \neq u$. $v \notin P_u$, otherwise $\mathcal{S} - (zu, zv) \cup (zt, zu)$ is an admissible splitting sequence with the same length but with one more remaining neighbour of z. $P_v \cap P_u = \{t\}$ according to Claim 2.4. $\mathcal{S} - (za, zb) - (zu, zv) \cup (zt, zu) \cup (zv, za)$ is an other longest splitting sequence with one more remaining neighbour of z, so it cannot be admissible, that is, there is a set $Y \subseteq V - z$ containing a, u, v, t, which is tight in $G_\mathcal{S}$. Y does not contain b, hence the tight set $Y \cap P_u$ contains a smaller number of split edges than P_u, a contradiction. □

Now it follows that $(**)$ holds for $\mathcal{P}_z$.

Claim 3.10. *Let X, Y be two distinct members of $\mathcal{P}_z$. $X \cap Y = \{t\}$.*

Proof. Let us suppose $X = P_u$ and $Y = P_v$ for some $u, v \in V$. By Proposition 3.6, $P_u \not\subset P_v$. If $|P_u \cap P_v| \geq 2$, then by Claim 2.4 $d_{G_\mathcal{S}}(P_u, P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any t-disjoint split edge, it contradicts the maximal choice of P_u. □

Hence $(*)$ holds for $\mathcal{P}_z$.

We have showed that if a small node z does not admit a full splitting, then the neighbour t of z and set-system $\mathcal{P}_z$ satisfy both $(*)$ and $(**)$. □ □

We state the following easy consequence of Theorem 3.5. The neighbour t of z in Theorem 3.5 is called the *blocking* node of z.

Corollary 3.11. *Let z be a small node in a (k,l)-sparse graph G. If z does not admit a full splitting, then the blocking node t of z is uniquely determined.*

4 Counterexamples

In this section we give a (k,l)-sparse graph for any $k \geq 2$, $\frac{k+2}{3} \leq l \leq \frac{k}{2}$ which cannot be obtained by the operations of Theorem 1.7. This is surprising because we managed to prove almost all the ingredients of the proof of the constructive characterization of $(k,1)$-sparse graphs also for these graphs. We remark that, for the given graph $G_{(k,l)} = (V_{(k,l)}, E_{(k,l)})$, $|V_{(k,l)}| = 15k - 5l + 10$, which is 60 in the smallest case $(4,2)$ and 85 in case $(6,3)$.

Let us consider $m := 3k - l + 2$ copies of the following graph $G_1 = (V_1, E_1)$ and let the subscripts go from 1 to m. Graph G_1 has $|V_1| = 5$ nodes and $|E_1| = k|V_1| - (k+l) = 4k - l$ edges. Edges $a_1d_1, b_1d_1, c_1d_1, z_1d_1$ have multiplicity $k-l$, b_1z_1, c_1z_1 has l, a_1b_1 has $l-1$, a_1z_1 has 1, and all the other edges multiplicity 0. See Figure 2, the multiplicity of the edges are shown in the figure.

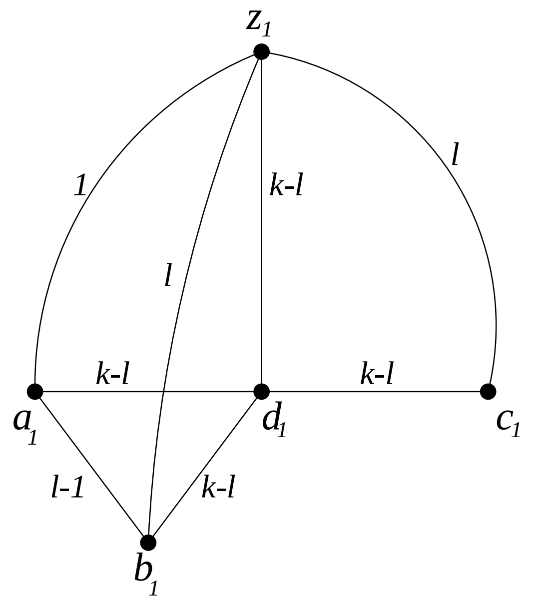

Figure 2: Graph G_1

It is easy to see, that G_1 is (k,l)-sparse since it can be obtained by the operations (i.e. z_1, d_1, c_1, b_1, a_1).

Let $G_{(k,l)} = (V_{(k,l)}, E_{(k,l)})$ where $V_{(k,l)} := \cup_{j=1}^{m} V_j$, $E_{(k,l)} := \cup_{j=1}^{m} E_j \cup E^*$ and $E^* := K_1 \cup K_2 \cup K_3 \cup K_{1,2} \cup K_{3,2} \cup K_{1,3}$, where

$$K_1 = \{a_i a_j : 1 \le i < j \le k+1\}$$

$$K_2 = \{c_1 c_j : 2k-l+3 \le j \le 3k-l+2\} \cup \{c_i c_j : 2k-l+3 \le i < j \le 3k-l+2\}$$

$$K_3 = \{b_1 b_j : k+2 \le j \le 2k-l+2\} \cup \{b_i b_j : k+2 \le i < j \le 2k-l+2\}$$

$$K_{1,2} = \{b_i a_j : 2 \le i \le k+1, k+2 \le j \le 2k-l+2\}$$

$$K_{3,2} = \{b_i c_j : 2k-l+3 \le i \le 3k-l+2, k+2 \le j \le 2k-l+2\}$$

$$K_{1,3} = \{c_i a_j : 2 \le i \le k+1, 2k-l+3 \le j \le 3k-l+2\}$$

See Figure 3. We will use the following two facts about E^*

- $d_{E^*}(v) \le k$ for all $v \in V$,
- $d_{G_{(k,l)}}(V_i, V_j) = 1$ for all $1 \le i < j \le 3k-l+2$.

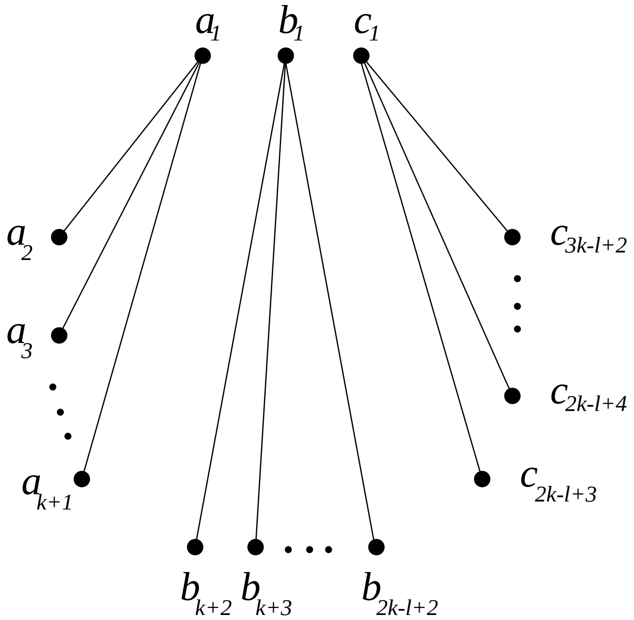

Figure 3: A subgraph of $G_{(k,l)}$

It is clear that $|V_{(k,l)}| = 5m = 5(3k-l+2) = 15k - 5l + 10$ and $|E_{(k,l)}| = m|E_1| + |E^*| = m(4k-l) + \frac{1}{2}m(3k-l+1)$. In $G_{(k,l)}$ we have the following degrees for any $1 \le j \le m$

$$d(a_j) = d(b_j) = d(b_j) = 2k,$$

$$d(d_j) = 4(k-l) \ge 4\frac{k}{2} = 2k,$$

$$d(z_j) = k + l + 1.$$

Hence the only small nodes are z_j-s. Since $\{a_j, d_j\}, \{b_j, d_j\}, \{c_j, d_j\}$ are tight sets, there is no full splitting at z_j, hence graph $G_{(k,l)}$ cannot be obtained by the operations.

It is remained to see that $G_{(k,l)}$ is (k,l)-sparse for the given k and l. We are going to prove that $b(X) \geq 0$ for all $X \subseteq V_{(k,l)}$. It can be shown easily that if $X \subseteq V_{(k,l)}$ includes at least two nodes of V_j for some j, then $b(X) \geq b(X \cup V_j)$. Hence it is enough to prove the condition for subsets X either including V_j or having the cardinality of the intersection with it at most 1 for all j.

Let n denote the number of V_j's that are included entirely in X and r denote the number of V_j's having a one-element intersection with X. $|X| = 5n + r$, hence we must prove

$$|E[X]| \leq k|X| - (k+l) = k(5n+r) - (k+l) = 5kn + kr - k - l. \tag{1}$$

We have

$$|E[X] - E^*| = n|E_1| = n(4k - l).$$

$$|E[X] \cap E^*| \leq \frac{n(n+r-1) + rk}{2},$$

since $d(V_i, V_j) = 1$ and $d(a_i, V - V_i) = d(c_i, V - V_i) = k, d(b_i, V - V_i) = k - l + 1 < k$ for all i, j. Hence

$$|E[X]| = |E[X] - E^*| + |E[X] \cap E^*| \leq n(4k - l) + \frac{n(n+r-1) + kr}{2}. \tag{2}$$

We will prove that the difference of the right hand side of (1) and (2) is at least 0, which will finish the proof that G is (k,l)-sparse. Let us compute, but first multiply by 2,

$$2(5kn + kr - k - l) - 2\left(n(4k - l) + \frac{n(n+r-1) + kr}{2}\right) =$$

$$(10kn + 2kr - 2k - 2l) - (8kn - 2ln + n^2 + nr - n + kr) =$$

$$10kn + 2kr - 2k - 2l - 8kn + 2ln - n^2 - nr + n - kr =$$

$$2kn + kr - 2k - 2l + 2ln - n^2 - nr + n =$$

$$(n+r)(k-n) + n(k + 2l + 1) - 2(k+l). \tag{3}$$

If $2 \leq n \leq k$, then (3) is obviously at least 0. $n + r \leq m = 3k - l + 2$. If $n > k$, then we continue the computation:

$$\geq m(k-n) + n(k + 2l + 1) - 2(k+l) =$$

$$(3k-l+2)(k-n)+n(k+2l+1)-2(k+l)=$$

$$(3k-l+2)k+n(3l-2k-1)-2(k+l)\geq$$

since $3l-2k-1<0$,

$$\geq (3k-l+2)k+(3k-l+2)(3l-2k-1)-2(k+l)=$$

$$(3k-l+2)(3l-k-1)-2(k+l)=$$

$$(3k-l+2)(3l-k-2)+(3k-l+2)-2k-2l=$$

$$(3k-l+2)(3l-k-2)+(k-3l+2)=$$

$$(3k-l+1)(3l-k-2). \quad (4)$$

Since $l \geq \frac{k+2}{3}$, that is, $3l \geq k+2$, (4) is at least 0. If $n=1$ or 0, $E[X] \leq k|X|-(k+l)$ can be shown with a much shorter computation. Hence we proved that G is really (k,l)-sparse.

5 Open problems

The main problem is proving Conjecture 1.5 in the remaining cases. Another important question is finding an appropriate constructive characterization theorem for (k,l)-sparse graphs if $\frac{k+2}{3} \leq l \leq \frac{k}{2}$. One possibility if the following. If we allow $i=k$ in (P2), is the reverse of Theorem 1.7 true?

This operation can be allowed in the cases which are already proved, of course, but it is not necessary.

Are the examples of Section 4 the graphs with the smallest number of nodes? We think they are.

Give a constructive characterization for (k,l)-sparse graphs, if $\frac{k}{2} \leq l \leq k$. We may have to allow operations which glue together bigger graphs and the nodes are not considered one by one.

A graph is said to be $[k,m]$-sparse, if $0 \leq m \leq k$ and $\gamma_G(X) \leq k|X|-m$ for all $X \subseteq V, |X| \geq 2$. These graphs have not a direct connection to covering by trees but may have a similar construction.

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