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A note on parity constrained orientations

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Abstract

This note extends the results of Frank, Jordán, and Szigeti [1] on parity constrained orientations with connectivity requirements. Given a hypergraph H, a non-negative intersecting supermodular set function p, and a preferred in-degree parity for every node, a formula is given on the minimum number of nodes with wrong in-degree parity in an orientation of H covering p. It is shown that the minimum number of nodes with wrong in-degree parity in a strongly connected orientation cannot be characterized by a similar formula.

1 Introduction

In [1], Frank, Jordán, and Szigeti proved that the existence of a parity-constrained k-rooted-connected orientation of a graph can be characterized by a partition-type condition. In this note it is shown that the requirement of k-rooted-connectivity can be replaced by any requirement given by a non-negative intersecting supermodular set function. We also extend the characterization to hypergraphs, and show a min-max formula on the minimal number of nodes violating the parity condition. The proof is based on the ideas in [1]. In the last chapter we show that it is not possible to give a similar characterization for parity-constrained strongly connected orientations.

For a hypergraph $H = (V, \mathcal{E})$ and a set $X \subseteq V$, $i_H(X)$ denotes the number of hyperedges of H spanned by X. For a partition $\mathcal{F}$, $e_H(\mathcal{F})$ denotes the number of cross-hyperedges of H; in other words,

$$e_H(\mathcal{F}) = |\mathcal{E}| - \sum_{X \in \mathcal{F}} i_H(X). \qquad (1)$$

A *directed hypergraph* consists of *hyperarcs*: hyperedges that have one node designated as *head node*. An *orientation* of a hypergraph H is a directed hypergraph obtained by selecting a node from each hyperedge of H as head node. For a directed

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hypergraph $D = (V, \mathcal{A})$ and a set $X \subseteq V$, $\varrho_D(X)$ denotes the number of hyperarcs in D which have their head node inside X and have at least one node outside of X.

A set function $p : 2^V \to \mathbb{Z}$ is called *intersecting supermodular* if

$$p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y)$$

holds whenever $X \cap Y \neq \emptyset$. A set function $p : 2^V \to \mathbb{Z}$ is *monotone decreasing* if $p(X) \ge p(Y)$ whenever $\emptyset \neq X \subseteq Y$. We always assume that $p(\emptyset) = 0$. Clearly, if $p(V) = 0$ and p is monotone decreasing, then p is non-negative. For intersecting supermodular functions the converse is also true:

Claim 1.1. *If p is intersecting supermodular, non-negative, and $p(V) = 0$, then p is monotone decreasing.*

Proof. Let $\emptyset \neq X \subsetneq Y \subseteq V$, and let $Z := (V - Y) \cup X$. By the intersecting supermodularity and non-negativity of p, $p(Y) \le p(Y) + p(Z) \le p(Y \cap Z) + p(Y \cup Z) = p(X) + p(V) = p(X)$. □

An orientation $D = (V, \mathcal{A})$ of a hypergraph $H = (V, \mathcal{E})$ *covers* a set function p if $\varrho_D(X) \ge p(X)$ for every $X \subseteq V$. If the in-degrees of the orientation are specified, then the following is true (see e.g. [2]):

Lemma 1.2. *Let $H = (V, \mathcal{E})$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a non-negative set function, and $m : V \to \mathbb{Z}_+$ an in-degree specification such that $m(V) = |\mathcal{E}|$. Then H has an orientation covering p such that the in-degree of each node $v \in V$ is $m(v)$ if and only if*

$$m(X) \ge i_H(X) + p(X) \quad \textit{for every } X \subseteq V.$$

For non-negative intersecting supermodular set functions, the following can be proved using basic properties of polymatroids:

Theorem 1.3. *Let $H = (V, \mathcal{E})$ be a hypergraph, and $p : 2^V \to \mathbb{Z}_+$ an intersecting supermodular and non-negative set function. Then H has an orientation covering p if and only if*

$$e_H(\mathcal{F}) \ge \sum_{X \in \mathcal{F}} p(X) \quad \textit{for every partition } \mathcal{F}. \tag{2}$$

2 Main result

Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$ a fixed set, and $p : 2^V \to \mathbb{Z}$ a set function such that $p(V) = 0$. An orientation of H is called *(p, T)-feasible* if it covers p and the in-degree of $v \in V$ is odd if and only if $v \in T$. A set $X \subseteq V$ is called *even* if $|X \cap T| + i_H(X) + p(X)$ is even; X is called *odd* if $|X \cap T| + i_H(X) + p(X)$ is odd (the notion of odd and even sets will be used with respect to different H, T, and p values, but the actual meaning will always be clear from the context). Clearly,

$\varrho_D(X) \geq p(X) + 1$ must hold for an odd set X in a (p, T)-feasible orientation of H. We define the following set function:

$$p^T(X) := \begin{cases} p(X) & \text{if } X \text{ is even,} \\ p(X) + 1 & \text{if } X \text{ is odd.} \end{cases} \tag{3}$$

Note that p^T depends on H too. The definition implies that

$$p^T(X) \equiv |X \cap T| + i(X) \mod 2 \tag{4}$$

for every $X \subseteq V$. Given a partition $\mathcal{F}$, the value

$$\mu_T(\mathcal{F}) := \sum_{Z \in \mathcal{F}} p^T(Z) - e_H(\mathcal{F})$$

is called the *deficiency* of $\mathcal{F}$, which depends also on H and p.

Claim 2.1. *For given H, T, and p, the deficiency of every partition has the same parity.*

Proof. According to (1), the deficiency of a partition $\mathcal{F}$ has the same parity as $|\mathcal{E}| + \sum_{Z \in \mathcal{F}} i_H(Z) + \sum_{Z \in \mathcal{F}} p^T(Z)$, which by (4) has the same parity as $|\mathcal{E}| + |T|$. □

It is easy to see that if an orientation D of H is (p, U)-feasible for some $U \subseteq V$, then $|T \Delta U| \geq \max\{\mu_T(\mathcal{F}) : \mathcal{F} \text{ is a partition}\}$. The main result of this note is that if p is non-negative, intersecting supermodular, and there exists an orientation covering p, then equality can be attained.

Theorem 2.2. *Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$ a fixed set, and $p : 2^V \to \mathbb{Z}_+$ an intersecting supermodular and non-negative set function for which $p(V) = 0$. Suppose that H has an orientation covering p, i.e. (2) holds. Then there exists a set $U \subseteq V$ such that*

$$|T \Delta U| = \max\{\mu_T(\mathcal{F}) : \mathcal{F} \textit{ is a partition}\} \tag{5}$$

and H has a (p, U)-feasible orientation.

Proof. Indirectly, let us consider a counterexample where $|V| + |\mathcal{E}|$ is minimal. A partition $\mathcal{F}$ is called *non-trivial* if $\mathcal{F} \neq \{V\}$. Let $\mu := \max\{\mu_T(\mathcal{F}) : \mathcal{F} \text{ is a non-trivial partition}\}$. If μ is negative and odd, then the deficiency of the trivial partition is 1. Let us delete an arbitrary hyperedge from H. By induction, the remaining hypergraph has a (p, T)-feasible orientation. By adding the deleted hyperedge oriented arbitrarily, we get an orientation satisfying (5).

If μ is negative and even, then we delete an arbitrary hyperedge e from H, and let $T' := T \Delta \{v\}$ for some $v \in e$. By induction, this hypergraph has a (p, T')-feasible orientation. By adding the hyperarc e with head v, we get a (p, T)-feasible orientation.

In the following we assume that μ is non-negative. Let $\mathcal{F}^* = \{V_1, \ldots, V_t\}$ be a non-trivial partition of maximal cardinality for which $\mu_T(\mathcal{F}^*) = \mu$ holds. Let $H_* = (V_*, \mathcal{E}_*)$

denote the hypergraph obtained by contracting each partition member V_i to a node v_i, let p_* denote the contracted set function, and let $T_* \subseteq V_*$ consist of the nodes v_i for which $|V_i \cap T| + i_H(V_i)$ is odd (thus $p_*^{T_*}(v_i) = p^T(V_i)$ for every i). It follows from the choice of $\mathcal{F}^*$ that $\sum_{v\in X} p_*^{T_*}(v) - p_*^{T_*}(X) - i_{H_*}(X)$ is non-negative and even for every $X \subseteq V_*$.

First we transform T into a set U such that $|T\Delta U| = \mu$, $\sum_{i=1}^t p_*^{U_*}(v_i) = |\mathcal{E}_*|$, and $\sum_{v\in X} p_*^{U_*}(v) - p_*^{U_*}(X) - i_{H_*}(X) \geq 0$ for every $X \subseteq V_*$. If $\mu = 0$ then $U := T$ is appropriate, so suppose that $\mu > 0$. An even set $X \subseteq V_*$ is called *critical* if $\sum_{v\in X} p_*^{T_*}(v) - p_*^{T_*}(X) - i_{H_*}(X) = 0$; thus every even singleton is critical. By the intersecting supermodularity of p, the intersection and union of intersecting critical sets are critical. If every node of V_* is covered by a critical even set, then there is a partition of V_* consisting of critical even sets, which induces a partition on V that violates (2). Thus there is an odd singleton $v_i \in V_*$ that is not covered by a critical even set. Let $T' := T\Delta\{v\}$ for an arbitrary $v \in V_i$. Then $\sum_{i=1}^t p_*^{T'_*}(v_i) = |\mathcal{E}_*| + \mu - 1$, and $\sum_{v\in X} p_*^{T'_*}(v) - p_*^{T'_*}(X) - i_{H_*}(X) \geq 0$ holds for every $X \subseteq V_*$. If we repeat the above procedure μ times, we obtain the required U.

Claim 2.3. *There exists a (p_*, U_*)-feasible orientation D_* of H_* for which the in-degree of v_i is $p_*^{U_*}(v_i)$ $(i = 1, \ldots, t)$.*

Proof. We know that $\sum_{i=1}^t p_*^{U_*}(v_i) = |\mathcal{E}_*|$. Lemma 1.2 implies that a good orientation exists if and only if $\sum_{v\in X} p_*^{U_*}(v) \geq p_*^{U_*}(X) + i_{H_*}(X)$ for every $X \subseteq V_*$. This is satisfied due to the way U was constructed. □

Let D_* be a fixed (p_*, U_*)-feasible orientation of H_*, and let D_0 denote the directed hypergraph on V corresponding to the hyperarcs of D_*. From now on, the parity of sets is determined with respect to U or U_*. The next step is to show that it is possible to obtain a directed hypergraph D'_* by deleting exactly one hyperarc entering each odd singleton $\{v_i\}$, such that $\varrho_{D'_*}(X) \geq p_*(X)$ still holds for every $X \subseteq V_*$. Let $\{v_i\}$ be an odd singleton. If a hyperarc a with head v_i cannot be deleted, then there exists an even set $X_a \subseteq V_*$ such that a enters X_a and $\varrho_{D_*}(X_a) = p_*(X_a)$. We call such a set *tight* – notice that every tight set is even. Since D_* is a feasible orientation and p_* is intersecting supermodular, the intersection and union of intersecting tight sets are also tight sets. Thus if no hyperarc with head v_i can be deleted, then there exists a tight set $X \subseteq V_*$ such that every hyperarc of D_* with head v_i enters X. But this is impossible, since $\varrho_{D_*}(X) = p_*(X) \leq p(V_i) < p^U(V_i) = \varrho_{D_*}(v_i)$ by the monotone decreasing property of p and the fact that V_i is an odd set. Therefore we can delete a hyperarc a with head v_i, and change U_* by adding/deleting v_i, so that $\{v_i\}$ becomes an even set.

By repeating the above operation for every odd singleton $\{v_i\}$ (always considering the updated parameters when deciding the parity of sets), we get a directed hypergraph D'_*. Let D'_0 denote the directed hypergraph on V corresponding to the hyperarcs of D'_*, let H' denote the hypergraph obtained from H by deleting the hyperedges corresponding to hyperarcs in $D_0 - D'_0$, and let U' be the new parity requirement set, i.e. $U' := U\Delta\{v : \varrho_{D_0 - D'_0}(v) = 1\}$. It is easy to see that $\varrho_{D'_0}(X) \geq p(X)$ holds if X is the union of some members of $\mathcal{F}^*$, and $\varrho_{D'_0}(V_i) = p(V_i) = p^{U'}(V_i)$ for every i.

Furthermore, if D'_0 can be extended to a (p, U')-feasible orientation of H', then D_0 can be extended similarly to a (p, U)-feasible orientation of H.

In the following we construct an orientation of $H[V_i]$ for every i, which together with D'_0 will give a (p, U')-feasible orientation of H'. Let $p_i : 2^{V_i} \to \mathbb{Z}$ be defined as

$$p_i(X) := p(X) - \varrho_{D'_0}(X) \quad (X \subseteq V_i).$$

Then p_i is intersecting supermodular, monotone decreasing, and $p_i(V_i) = 0$ since $\varrho_{D'_0}(V_i) = p(V_i)$. We define $U_i \subseteq V_i$ by

$$U_i := (U' \cap V_i)\Delta\{v \in V_i : \ \varrho_{D'_0}(v) \equiv 1 \mod 2\}.$$

Let $p_i^{U_i} : 2^{V_i} \to \mathbb{Z}$ be the set function defined similarly to (3) but with respect to $H[V_i]$, p_i, and U_i.

Claim 2.4. *The following holds for each V_i and for every partition $\mathcal{F}$ of V_i:*

$$e_{H[V_i]}(\mathcal{F}) \geq \sum_{Z\in\mathcal{F}} p_i^{U_i}(Z). \tag{6}$$

Proof. Suppose that there is a partition $\mathcal{F}$ for which the inequality does not hold. Then $e_{H[V_i]}(\mathcal{F}) \leq \sum_{Z\in\mathcal{F}} p_i^{U_i}(Z) - 2$ by Claim 2.1. We define the following partition of V: $\mathcal{F}^i := \mathcal{F} \cup \mathcal{F}^* - \{V_i\}$. We consider the original deficiency of $\mathcal{F}^i$: $\mu_T(\mathcal{F}^i) = \mu_T(\mathcal{F}^*) - p^T(V_i) + \sum_{Z\in\mathcal{F}} p^T(Z) - e_{H[V_i]}(\mathcal{F}) \geq \mu_T(\mathcal{F}^*) + \sum_{Z\in\mathcal{F}} p_i^{U_i}(Z) - e_{H[V_i]}(\mathcal{F}) - 2 \geq \mu_T(\mathcal{F}^*) = \mu$, since $\sum_{Z\in\mathcal{F}} p_i^{U_i}(Z) \leq \sum_{Z\in\mathcal{F}} p^T(Z) - \varrho_{D'_0}(V_i) + 1 \leq \sum_{Z\in\mathcal{F}} p^T(Z) - p^T(V_i) + 2$. Thus $\mathcal{F}^i$ would be a partition of deficiency μ with more elements than $\mathcal{F}^*$, in contradiction with the way $\mathcal{F}^*$ was chosen. □

By induction, Theorem 2.2 is true for $H[V_i]$, p_i, and U_i. Thus Claim 2.4 implies that there is an orientation D_i of $H[V_i]$ such that $\varrho_{D_i}(X) \geq p_i(X)$ for every $X \subseteq V_i$, and $\varrho_{D_i}(v)$ is odd if and only if $v \in U_i$.

Let D' be the directed hypergraph obtained as the union of D'_0 and $D_1, \ldots, D_t$. The above property means that $\varrho_{D'}(X) \geq p(X)$ if $X \subseteq V_i$ for some i, and $\varrho_{D'}(v)$ is odd if and only if $v \in U'$. The construction method of D'_0 implies that $\varrho_{D'}(X) \geq p(X)$ also holds if X is the union of some members of $\mathcal{F}^*$.

Suppose that there are sets for which $\varrho_{D'}(X) < p(X)$; let X be such a set, with the property that $X \subseteq V_i$ or $V_i \subseteq X$ or $X \cap V_i = \emptyset$ holds for a maximum number of members of $\mathcal{F}^*$. There must be a member V_i of $\mathcal{F}^*$ for which none of those relations are true, since X is neither a subset of a member of $\mathcal{F}^*$, nor the union of some members of $\mathcal{F}^*$. Since $\varrho_{D'}(V_i) = p(V_i)$, the intersecting supermodularity of p implies that either $\varrho_{D'}(X \cap V_i) < p(X \cap V_i)$ or $\varrho_{D'}(X \cup V_i) < p(X \cup V_i)$. But both cases are impossible due to the way X was chosen.

We obtained that D' is a (p, U')-feasible orientation of H'. This means that if D is the directed hypergraph obtained as the union of D_0 and $D_1, \ldots, D_t$, then D is a (p, U)-feasible orientation of H. This completes the proof of Theorem 2.2 □

3 Remarks

The Berge-Tutte formula on the size of a maximum matching in a graph $G = (V, E)$ easily follows from Theorem 2.2. To see this, we define the graph $G' = (V', E')$ by adding one node v_e to V for every $e \in E$, and by replacing every edge $e = uv$ in E by edges uv_e and vv_e. For $v \in V$, let $p(\{v\}) := d_G(v) - 1$, and let $p(X) := 0$ on every other set. Let T consist of the nodes in V for which $d_G(v) - 1$ is odd. It is easy to see that every orientation of G' covering p determines a matching of G (an edge $e = uv$ is in the matching if the orientation contains the directed edges uv_e and vv_e), and the number of nodes not covered by the matching equals the number of nodes that do not match the parity-specification T. Therefore Theorem 2.2 implies that if $\mathcal{F}$ is a partition of V' for which $\mu_T(\mathcal{F}) := \sum_{Z \in \mathcal{F}} p^T(Z) - e_{G'}(\mathcal{F})$ is maximal, then $2\nu(G) \geq |V| - \mu_T(\mathcal{F})$. The following Claim proves the Berge-Tutte formula.

Claim 3.1. *Let $W := \{v \in V : \{v\} \in \mathcal{F}\}$. then*

$$\mu_T(\mathcal{F}) \leq odd_G(W) - |W|, \tag{7}$$

where $odd_G(W)$ denotes the number of components of $G[V - W]$ having an odd number of nodes. Thus $2\nu(G) \geq |V| + |W| - odd_G(W)$.

Proof. Let $\mathcal{F}'$ consist of the members of $\mathcal{F}$ which are not singletons in W. Then $p(X) = 0$ for every $X \in \mathcal{F}'$. We can assume that there is no edge between two members of $\mathcal{F}'$, otherwise we can replace them by their union. By this assumption, the number of sets $X \in \mathcal{F}'$ for which $p^T(X) = 1$ is at most $\text{odd}_G(W)$. Thus $\mu_T(\mathcal{F}) = \sum_{Z \in \mathcal{F}} p^T(Z) - e_{G'}(\mathcal{F}) = \sum_{v \in W}(d_G(v) - 1) + \sum_{Z \in \mathcal{F}'} p^T(Z) - e_{G'}(\mathcal{F}) \leq \text{odd}_G(W) - |W|$. □

It the next paragraphs we describe a few negative results concerning possible generalizations of Theorem 2.2. First, let us state a corollary which follows easily from Theorem 2.2.

Corollary 3.2. *Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$ a fixed set, and $p : 2^V \to \mathbb{Z}_+$ an intersecting supermodular and non-negative set function for which $p(V) = 0$. Suppose that there exists an orientation of H covering p^T (as defined in (3)). Then there exists a (p, T)-feasible orientation of H.*

One may try to extend this corollary to more general set functions. A possibility is to include upper bounds on the in-degrees of the nodes (which may violate intersecting supermodularity). However, Frank, Sebő, and Tardos [3] showed that if p consists of lower and upper bounds on the in-degrees of nodes, then the equivalent of Corollary 3.2 is not necessarily true.

Another problem that is not contained in the intersecting supermodular case is to find a strongly connected orientation of a graph. In this case $p(X)$ equals 1 for every $\emptyset \neq X \subsetneq V$. In the following we describe an example where the equivalent of Corollary 3.2 for strongly connected orientations does not hold.

Let G be the graph on the left side of Figure 1, let T be the set of black nodes. Then G has no (p, T)-feasible orientation (i.e. it has no strongly connected orientation where

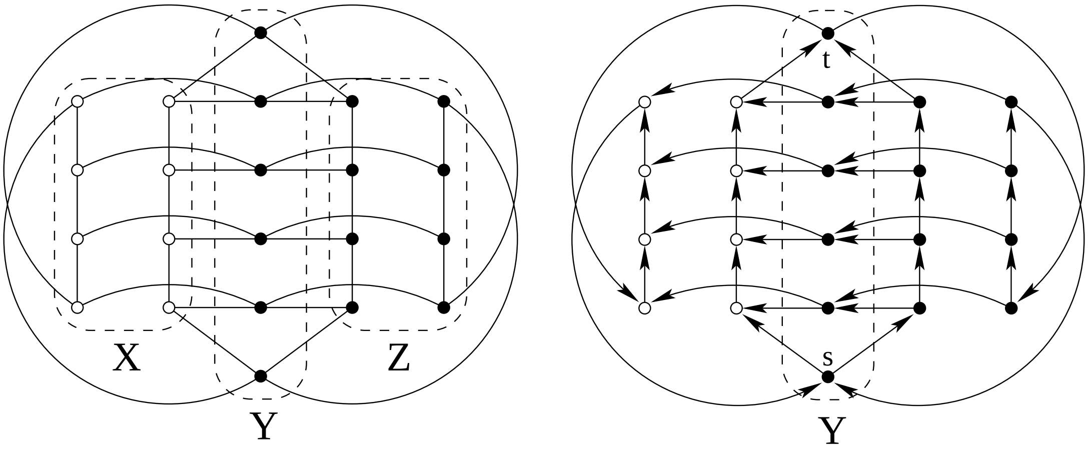

Figure 1

T is the set of nodes with odd in-degree). To see this, observe that in a (p, T)-feasible orientation every node of X must have at least 2 in-edges, every node of Z must have at least 2 out-edges, and every node of Y must either have an in-edge coming from X, or an out-edge going to Z. Thus the graph must have at least $2|X| + 2|Z| + |Y| = 38$ edges, but it has only 36.

On the other hand, G has an orientation covering p^T, as shown on the right side of Figure 1. It is easy to check that the orientation is strongly connected, and the in-degree parity is incorrect only at the nodes of Y. Thus it suffices to show that the in-degree of every set separating Y is at least 2. This can be seen by checking that there are 2 edge-disjoint paths from s to any $v \in Y$, there are 2 edge-disjoint paths from any $v \in Y$ to t, and there are 2 edge-disjoint paths from t to s.

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