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The Gallai-Edmonds Decomposition for the k-Piece Packing Problem

Marek Janata, Martin Loebl and Jácint Szabó

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Abstract

Generalizing Kaneko's long path packing problem, Hartvigsen, Hell and Szabó [2] consider a new type of undirected graph packing problem, called the *k-piece packing problem.* A k-piece is a simple, connected graph with highest degree exactly k, so when $k = 1$ we get the classical matching problem. They give a polynomial algorithm, a Tutte-type characterization and a Berge-type minimax formula, but they leave open the question of a Gallai-Edmonds type structure theorem. This paper fills this gap by describing such a decomposition. We also prove that the vertex sets coverable by k-piece packings have a matroidal structure in a certain way.

Keywords: graph packing, Gallai-Edmonds decomposition, matroid

1 Introduction

Given a set $\mathcal{F}$ of graphs, an $\mathcal{F}$-packing of G is a subgraph G' of G such that each connected component of G' is isomorphic to a member of $\mathcal{F}$. An $\mathcal{F}$-packing G' is called *maximal* if there is no $\mathcal{F}$-packing G'' with $V(G') \subsetneq V(G'')$. An $\mathcal{F}$-packing is *maximum* if it covers a maximum number of vertices of G, and it is *perfect* if it covers every vertex of G. The graph packing problem is to decide if G has a perfect $\mathcal{F}$-packing, or in general, to determine the size of the maximum $\mathcal{F}$-packings. (The size of a graph is the number of its vertices.) Several polynomial $\mathcal{F}$-packing problems

*Dept. of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranske n. 25, 118 00 Praha 1, Czech Republic. e-mail: `janata@kam.mff.cuni.cz`

**Dept. of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranske n. 25, 118 00 Praha 1, Czech Republic. e-mail: `loebl@kam.mff.cuni.cz`

***Dept. of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). Research is supported by OTKA grants T 037547, N 034040 and by the Egerváry Research Group of the Hungarian Academy of Sciences. e-mail: `jacint@cs.elte.hu`

are known when $K_2 \in \mathcal{F}$ (for references, see [2]). In all these cases the maximal $\mathcal{F}$-packings are maximum too, the vertex sets coverable by $\mathcal{F}$-packings form a matroid, and the analogue of the Gallai-Edmonds structure theorem holds.

The first polynomial $\mathcal{F}$-packing problem with $K_2 \notin \mathcal{F}$ was considered by Kaneko [3], who presented a Tutte-type characterization of those graphs that have a perfect packing by *long paths*, i.e. paths of length at least 2. A shorter proof of Kaneko's theorem and a min-max formula was subsequently found by Kano, Katona and Király [4]. The long path packing problem was generalized by Hartvigsen, Hell and Szabó [2] by introducing the *k-piece packing problem*, calling a simple, connected graph a *k-piece* if it has highest degree exactly k. Observe that a 1-piece is just K_2, thus the 1-piece packing problem is the classical matching problem. Moreover, for $k = 2$, it is immediate that the 2-piece packing problem is equivalent to the long path packing problem. The main result of [2] is a polynomial algorithm for finding a *maximum* k-piece packing. From this algorithm a characterization of those graphs that have a perfect k-piece packing, and a min-max result for the size of a maximum k-piece packing are derived.

Neither the Gallai-Edmonds decomposition, nor the matroidal property is considered in [2]. This paper fills this gap by giving a canonical Gallai-Edmonds type decomposition for the k-piece packing problem, and by showing that the vertex sets coverable by maximal k-piece packings have a certain matroidal structure, see Section 2. This matroidal result holds also for maximum packings. It turns out, that in the k-piece packing problem maximal and maximum packings do not coincide, and the maximal packings are of more interest than the maximum ones.

In this paper all graphs are simple. The vertex set of G is denoted by $V(G)$, the edge set by $E(G)$, the number of connected components of G by $c(G)$, and the highest degree of G by $\Delta(G)$. For $X \subseteq V(G)$ the subgraph induced by X is denoted by $G[X]$, and the degree of $v \in V(G)$ by $\deg_G(v)$. If $U \subseteq V(G)$ then $\Gamma(U)$ denotes the set of vertices in $V(G) - U$ which are adjacent to a vertex in U.

2 The theorems

In this section we introduce an important set of graphs related to the k-piece packing problem, called 'galaxies'. After that we state the main theorems of the paper. These theorems are generalizations of classical results of matching theory. The proofs are contained in Section 4 (Thm. 2.4) and Section 5 (Thm. 2.5). In this section k is a fixed positive integer.

In [2] it was revealed that *galaxies* play a central role in the k-piece packing problem (it will turn out in Section 4 that galaxies are the 'critical' graphs).

Definition 2.1. [2] For an integer $k \geq 1$ the simple, connected graph H is a *k-galaxy* if it satisfies the followings:

- denoting by I the set of vertices of degree at least k, each component of $H[I]$ is a hypomatchable graph,

- for all $v \in I$ there are exactly $k-1$ edges between v and $H-I$, each being a cut edge in H.

Observe that the definition implies that a k-galaxy has no vertex of degree k and that each component of $H[I]$ is a hypomatchable graph on at least 3 vertices. When k is clear from the context, we shall call a k-galaxy simply a *galaxy.* Galaxies generalize hypomatchable graphs when $k=1$, and 'suns' [3] when $k=2$. **Fig. 1** contains examples of galaxies. The vertices of I are drawn as big dots and the edges of $H[I]$ as thick lines. We cite the following important property of galaxies proved in [2]: *a k-galaxy has no perfect k-piece packing.* Later we concern other interesting properties of galaxies related to k-piece packings.

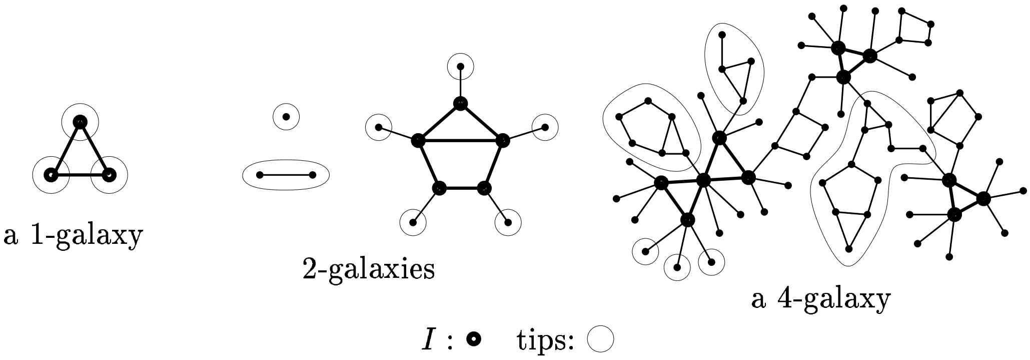


Fig. 1. Galaxies

Now we introduce special subgraphs of galaxies, called *tips.* The importance of this notion will be revealed later (see e.g. Theorem 3.7). Some tips are circled by thin line in **Fig. 1**.

Definition 2.2. [2] For a k-galaxy H the connected components of $H-I$ are called *tips.* Moreover, when $k=1$ we call each vertex a *tip.* The union of vertex sets of tips is denoted by $\mathcal{T}_H \subseteq V(H)$.

When $k \geq 2$, a k-galaxy may consist of only a single tip (a graph with highest degree at most $k-1$), but must always contain at least one tip. When $k=1$, we defined each vertex of a hypomatchable graph to be a tip.

The generalization of the classical Gallai-Edmonds structure theorem can be stated for the k-piece packing problem as follows. The Gallai-Edmonds theorem starts with the set of vertices which can be missed by a maximal matching. As we shall see, here we have to use a different formulation.

Definition 2.3. For a graph G let U_G be the set of vertices which can be missed by a maximal k-piece packing of G.

Theorem 2.4. *For a graph G let $D = \{v\colon |U_{G-v}| < |U_G|\}$, $A = \Gamma(D)$ and $C = V(G) - D - A$. Now*

1. *the components of $G[D]$ are k-galaxies,*

2. *the bipartite graph B_A^D has k-surplus (for definitions see pages 10 and 11),*

3. *$G[C]$ has a perfect k-piece packing,*

4. *each maximal k-piece packing P of G has the following structure:*

   (a) *exactly $k|A|$ components of $G[D]$ are entered by P, and these components are completely covered by P,*

   (b) *if H is a component of $G[D]$ not entered by P then there is a tip T of H such that $P[H]$ is a perfect k-piece packing of $H - T$,*

   (c) *$P[C]$ is a perfect matching of $G[C]$, and finally*

5. *for each maximal k-piece packing P, the graph $G - P$ has exactly $c(G[D]) - k|A|$ components.*

We could also choose $D = \{v : U_{G-v} \subsetneq U_G\}$ by Theorem 4.16.

It is a well known fact in matching theory that the vertex sets which can be covered by matchings form a matroid. In the k-piece packing problem this matroidal property holds only in the following weaker form.

Theorem 2.5. *There exists a partition π on $V(G)$ and a matroid $\mathcal{M}$ on π, such that the maximal vertex sets coverable by k-piece packings are exactly the vertex sets of the form $\bigcup\{X : X \in \pi'\}$ for a base π' of $\mathcal{M}$.*

3 Preliminaries

In this section we summarize in a compact way the results and notions of [2] which are needed in our considerations, together with an outline of the k-piece packing algorithm of [2]. In some places we use different formulations than the original one, for sake of simplicity. Most of the differences come from the fact that in contrast to [2], here we do not consider *k-matchings*. E.g. some statements and terms are contained only implicitly in [2] and treated in detail in [6] (which argues in formally different but, in fact, identical way.)

For proving the properties of the galaxies we need to introduce two other class of graphs which are 'near' to galaxies, see **Fig. 2**. It is convenient to do this only for $k \geq 2$.

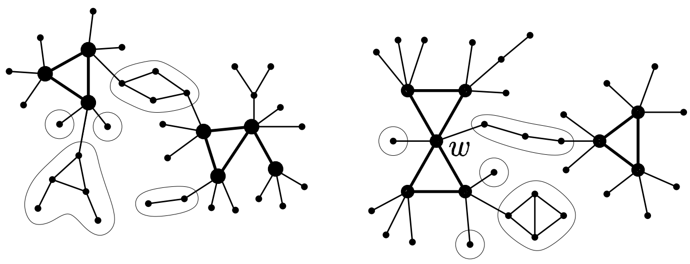
Definition 3.1. For an integer $k \geq 2$ the simple, connected graph G is an *almost k-galaxy of type 1* if it satisfies the followings:

- denoting by I the set of vertices of degree at least k, one of the components of $G[I]$ is perfectly matchable and the others are hypomatchable,

- for all $v \in I$ there are exactly $k - 1$ edges between v and $G - I$, each being a cut edge in G.

Definition 3.2. For an integer $k \geq 2$ the simple, connected graph G is an *almost k-galaxy of type 2* if it satisfies the followings:

- denoting by I the set of vertices of degree at least k, each component of $H[I]$ is a hypomatchable graph,
- there is a distinguished vertex $w \in I$, such that for all $v \in I$ all edges between v and $G - I$ are cut edges in G, and the number of these edges is $k - 1$ for $v \neq w$ and $k - 2$ for w.



almost k-galaxy of type 1 almost k-galaxy of type 2

Fig. 2. Almost galaxies, $k = 4$

Fig. 2 shows almost k-galaxies for $k = 4$. Just like in the case of galaxies, we need to introduce the notion of the tip for the almost galaxies. Some tips are circled by thin lines in **Fig. 2**.

Definition 3.3. For an almost galaxy G the connected components of $G - I$ are called *tips*.

Many properties of the galaxies are explained by the following lemma, which is implicit in [2].

Lemma 3.4. *The almost k-galaxies have perfect k-piece packings.*

Proof. First, let G be an almost k-galaxy of type 2. We proceed by induction on the number of vertices. Let K be the component of $G[I]$ containing the specified vertex w. K is hypomatchable so w has two neighbors w' and w'' in K with the property that $K - \{w', w, w''\}$ has a perfect matching M. For all edges $uv \in M$ let P_{uv} be the k-piece induced by u, v and the vertex sets of tips adjacent to $\{u, v\}$. Moreover, let P_w be the k-piece induced by w', w, w'' and the vertex sets of tips adjacent to $\{w', w, w''\}$, after deleting the edge $w'w''$ (if any). Deleting these k-pieces from G all the connected components of the remaining graph are almost k-galaxies of type 2, so we are done by induction.

Now let G be an almost k-galaxy of type 1. Let K be the perfectly matchable component of $G[I]$. For all edges uv of a perfect matching of K let P_{uv} be the k-piece induced by u, v and the vertex sets of tips adjacent to $\{u, v\}$. Deleting these k-pieces from G all the connected components of the remaining graph are almost k-galaxies of type 2, so we are done by the first part of the proof. □

It is easy to see, that for $k \geq 2$, if we delete a tip from a galaxy, all the components of the remaining graph are almost galaxies of type 2. Hence we proved the following statement, which is well-known for $k = 1$.

Lemma 3.5. [2] *If T is a tip of the k-galaxy H, then $H - T$ has a perfect k-piece packing.*

A further relation is considered in the following lemma (for proof, see [2]).

Lemma 3.6. [2] *If P is a k-piece packing of the k-galaxy H, then P do not intersect all tips of H.*

A hypomatchable graph H has the defining property, that for any vertex $v \in V(H)$ the graph $H - v$ has a perfect matching M_v. This implies that the maximum (and also the maximal) matchings of H are exactly the matchings M_v. The analogous property for galaxies can be stated by means of the tips in the next theorem. This important characterization for the maximal k-piece packings of a k-galaxy is implied by Lemmas 3.5 and 3.6.

Theorem 3.7. [2] *The maximal k-piece packings of a k-galaxy H are exactly the perfect k-piece packings of $H - T$ for a tip T.*

Another generalization of the above defining property of the hypomatchable graphs is the next lemma, which is implicit in [2]. In Theorem 4.19 we will see that the property stated in Lemma 3.8 is characteristic for the galaxies.

Lemma 3.8. *If v is a vertex of the k-galaxy H, then there is a k-piece packing P of H not covering v, such that $H - P$ is connected with highest degree at most $k - 1$.*

Proof. The statement is trivial for $k = 1$ so we assume that $k \geq 2$. If v is contained in a tip T then let P be a perfect k-piece packing of $H - T$ guaranteed by Lemma 3.5.

If $v \in I$ then let H' be the connected subgraph of H induced by v and the vertex sets of the $k - 1$ tips adjacent to v. Now $\Delta(H') = k - 1$. By the tree-like structure of H it is easy to check that some components of $H - H'$ are almost k-galaxies of type 1, and the other components are almost k-galaxies of type 2. Hence $H - H'$ has a perfect k-piece packing P by Lemma 3.4. □

In the investigations of the k-piece packings we frequently use the notion of a *solar-system*, see **Fig. 3**.

Definition 3.9. A connected graph is a *k-solar-system* if it has a mid-vertex y with degree k, such that $G - y$ has k connected components, each being a k-galaxy.

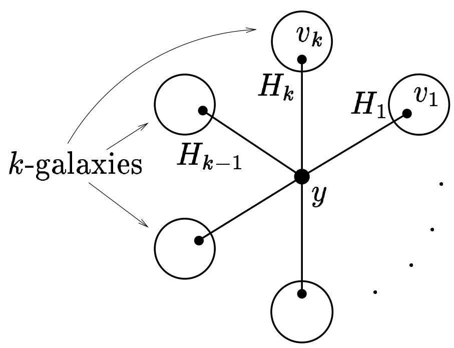


Fig. 3. A k-solar system

Lemma 3.10. *A k-solar-system has a perfect k-piece packing.*

Proof. Let G be a k-solar-system with mid-vertex y. Denote the neighbors of y by v_i $(1 \leq i \leq k)$ and suppose that v_i is contained in the k-galaxy component H_i of $G - y$. Lemma 3.8 implies that for $1 \leq i \leq k$ there exists a k-piece packing P_i of H_i not covering v_i, such that $H_i - P_i$ is connected and $\Delta(H_i - P_i) \leq k - 1$. Observe, that $G - P_1 - \ldots - P_k$ is a k-piece, so we are done. □

Now we describe the essence of the k-piece packing algorithm of [2]. The only difference between this and the original algorithm is that here we do not consider *k-matchings* for finding a maximum k-piece packing. In the description we have to use some more statements. The comprehension of these statements are not needed for the rest of the paper, so we omit their proofs. The algorithm maintains a special subgraph of the input graph, called *alternating structure*. Let G be a graph and P a k-piece packing of G. Let S be a subgraph of G, and A a set of vertices in S (called the *odd vertices* of S). For a connected component B of S, we denote by A_B the (possibly empty) set of odd vertices of S which belong to B.

Definition 3.11. [2] The pair (S, A) is an *alternating structure* with respect to P, if for each connected component B of S the following properties hold:

1. $B - A_B$ consists of $k|A_B| + 1$ k-galaxies, which are induced subgraphs of G,
2. there is no edge in B between the vertices of A_B, and
3. for all $y \in A_B$ it holds that $\deg_B(y) = k + 1$ and all these $k + 1$ edges are cut edges in B.

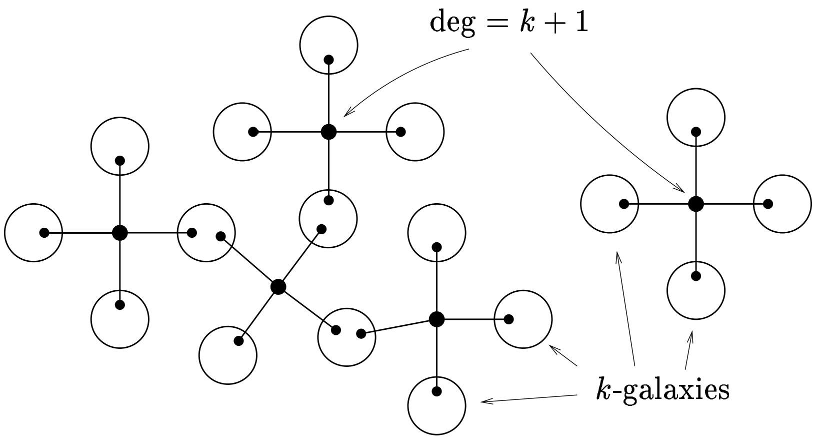


Fig. 4. An alternating structure, $k = 3$

Fig. 4 contains an example of an alternating structure with two connected components (when $k = 3$). In **Figs. 4-8** the big dots are the odd vertices.

We call the connected components of $S - A$ the *galaxies of S* and the connected components of $B - A_B$ the *galaxies of B*. Note, that if B is a component of an alternating structure, then deleting a galaxy from B, the remaining graph has a perfect k-piece packing. This is because it can be decomposed into k-solar-systems, which have perfect k-piece packings. An alternating structure when $k = 1$ is just an alternating forest in Edmonds' algorithm.

The k-piece packing algorithm of [2] maintains a k-piece packing P of G and an alternating structure (S, A) with respect to P. In every step P misses at least one vertex in each component of S, the edges of P do not leave the vertex set of S and $P[V(G-S)]$ is a perfect k-piece packing of $G-S$. If the galaxies of S are not connected components in $G - A$ then either the algorithm increases S or finds a k-piece packing covering more vertices than P.

Now we outline the steps of the algorithm.

Start: Let $P = \emptyset$, and let the connected components of S be the vertices of G as single vertex galaxies (i.e. $V(S) = V(G)$, $E(S) = \emptyset$). Do one of the following steps, until they apply.

1. Suppose that there is an edge $e \in E(G)$ between two distinct galaxies of a component B of S. Let J be the subgraph of G induced by the vertices shown in **Fig. 5**. (In **Figs. 5-8** solid edges are edges of the alternating structure, while broken edges are not.) It is proved in [2] that J is either a k-galaxy or has a perfect k-piece packing. If J is a k-galaxy then the new galaxies of B will be J and the original galaxies outside J. (This operation generalizes *shrinking* in Edmonds' algorithm.) Otherwise take a perfect k-piece packing of J, and take perfect k-piece packings of the solar-systems, to which $B - J$ can be decomposed. Delete B from S.

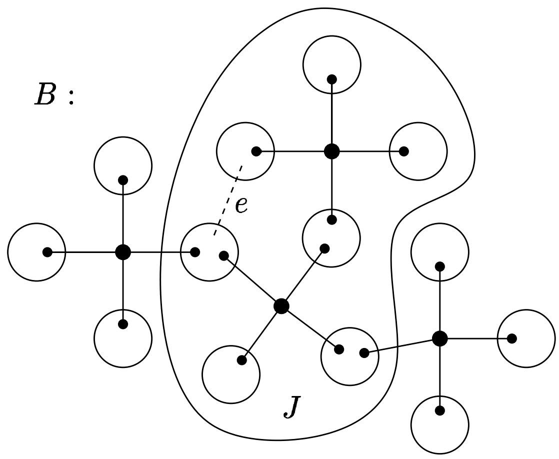


Fig. 5. $k = 3$

2. Suppose that there is an edge $e \in E(G)$ between two galaxies H_1, H_2 of two distinct components B_1, B_2 of S. Let $J = G[V(H_1) \cup V(H_2)]$, as shown in **Fig. 6**. It is implicit in [2] that J is either a k-galaxy or has a perfect k-piece packing. If J is a k-galaxy then we glue B_1, B_2 into a new component of S with galaxies J and the original galaxies of B_1, B_2 outside J. Otherwise take a perfect k-piece packing of J, and take perfect k-piece packings of the solar-systems, to which $B_1 - J$ and $B_2 - J$ can be decomposed. Delete B_1, B_2 from S.

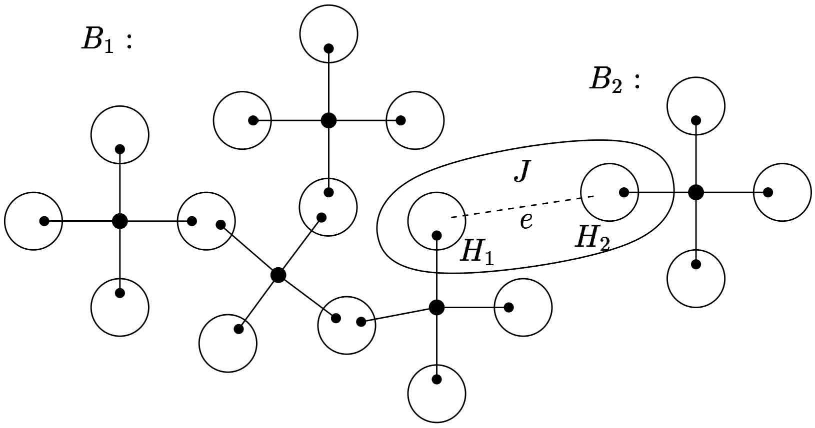


Fig. 6. $k = 3$

3. Suppose that there is an edge $e \in E(G)$ between a galaxy H of a component B and a vertex v outside S. Let the k-piece covering v be R, and let the vertex sets of the components of $R - v$ be $V_1, V_2, \ldots, V_l$. If $l = k$ and $\Delta(G[V_i]) \leq k - 1$ for $1 \leq i \leq k$ as shown in **Fig. 7**, then v will be a new odd vertex of A_B and $G[V_i]\,(1 \leq i \leq k)$ will be new galaxies of B. It is proved in [2], that otherwise $H \cup \{e\} \cup R$ has a perfect k-piece packing. Add perfect k-piece packings of the solar-systems, to which $B - H$ can be decomposed. Delete B from S.

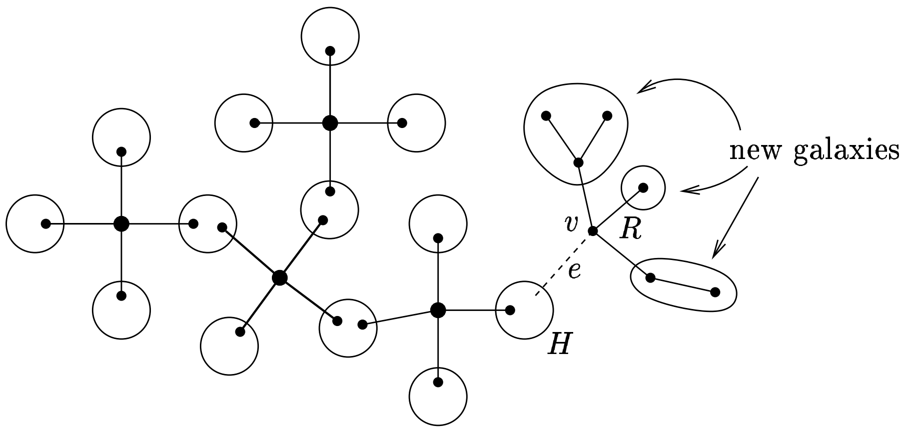


Fig. 7. $k = 3$

When terminating, in $G - A$ each galaxy of S is a connected component, and $G - S$ has a perfect k-piece packing, see **Fig. 8**. Hence, the algorithm implies a Tutte-type existence theorem for the k-piece packing problem.

Definition 3.12. Let k-gal(G) denote the number of those connected components of the graph G that are k-galaxies.

Theorem 3.13. [2] *A graph G has a perfect k-piece packing if and only if*

$$k\text{-gal}(G - A) \leq k|A|$$

for every set of vertices $A \subseteq V(G)$.

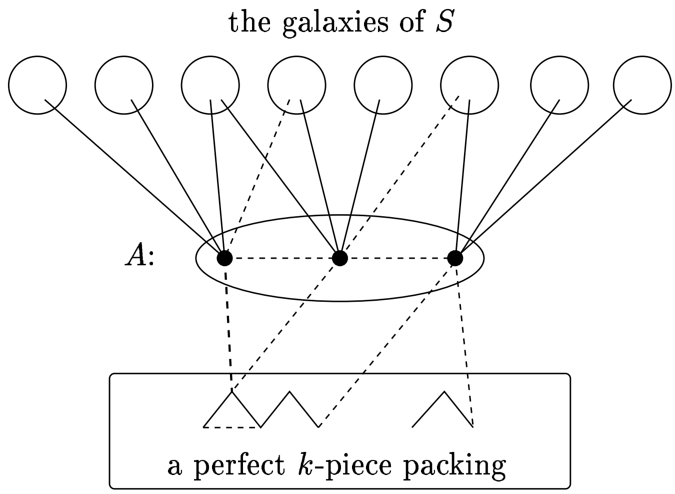


Fig. 8. The output of the k-piece packing algorithm, $k = 2$

Note, that this algorithm is not deterministic in the sense that it may have different runs (depending on the choice of the edge $e \in E(G)$), resulting in different alternating structures when terminating. Still, in the next section we prove that in all runs the followings are uniquely determined: A, $c(S)$ and the galaxies of S.

4 The Gallai-Edmonds decomposition

In this section we prove some results on the outputs of this algorithm, e.g. that the set of odd nodes, A is a 'canonical barrier' for the k-piece packing problem. The investigations of the 'canonical decomposition' result in the Gallai-Edmonds type theorem for the k-piece packing problem, see Theorem 2.4. In this section $k \geq 1$ is a fixed integer.

Definition 4.1. For $A \subseteq V(G)$ let D_A^G (or D_A, for short) denote the set of vertices belonging to the k-galaxy components of $G - A$. Moreover, $C_A^G = V(G) - D_A^G - A$ (shortly C_A).

A is a *barrier* if $c(G[D_A]) \geq k|A|$. The *defect* of a barrier A is $\mathrm{def}(A) = c(G[D_A]) - k|A|$.

Note, that $\mathrm{def}(A) = k\text{-gal}(G - A) - k|A|$. The defect is defined only for barriers so we have $\mathrm{def}(A) \geq 0$ always. A k-piece has no vertex of degree higher than k, thus graphs having a barrier with defect at least 1 have no perfect k-piece packings. The k-piece packing algorithm implies that the reverse also holds, see Theorem 3.13.

For describing the properties of the outputs of the algorithm we need the following definition.

Definition 4.2. The bipartite graph with color classes A and D has *k-surplus* if for all $\emptyset \neq A' \subseteq A$

$$|\Gamma(A')| \geq k|A'| + 1.$$

It has *k-surplus of d* if $|\Gamma(A)| = k|A| + d$.

The usage of this definition is tacitly meant by viewing the bipartite graph from color class A.

For a graph G with vertex sets D, $A \subseteq V(G)$, $D \cap A = \emptyset$, the term '$D - A$ bipartite graph' refers to the graph B_A^D what we get from $G[D \cup A]$ after shrinking each component of $G[D]$ to a vertex and deleting the edges induced by A (and replacing the parallel edge sets by single edges). For a vertex set A let $k \cdot A = \{y^i : y \in A,\ 1 \le i \le k\}$. If B is a bipartite graph with color classes A and D then *k-tupling* A results in the bipartite graph with vertex set $D \cup (k \cdot A)$ and edge set $\{vy^i : 1 \le i \le k,\ vy \in E(B)\}$. *$k$-tupling* A in B_A^D gives the bipartite graph B_{kA}^D.

The following important property (in fact, characterization) of the bipartite graphs with k-surplus is implied by k-tupling A and then applying Hall's theorem.

Lemma 4.3. *If B is a bipartite graph with color classes A and D with k-surplus, then $k \cdot A$ can be matched into $D - v$ for each vertex $v \in D$.*

Definition 4.4. Let $A \subseteq V(G)$ be a barrier. If $B_A^{D_A}$ has k-surplus (of d) then A is said to be a *barrier with k-surplus (of d).* If $G[C_A]$ has a perfect k-piece packing, then A is said to be *perfect.*

Let (S, A) be the alternating structure in any output of the algorithm. Now $D_A = V(S) - A$ and $C_A = V(G - S)$, because $G - S$ has a perfect k-piece packing so it has no k-galaxy component. Moreover, the properties of the alternating structure implies that $B_A^{D_A}$ has k-surplus of $c(S)$, hence A is a perfect barrier with k-surplus of $c(S)$. In the sequel we prove that such a vertex set A is unique, implying that $c(S)$, C_A and the galaxies of S are unique as well. For this we need an important property of the k-galaxies.

Lemma 4.5. *If H is a k-galaxy and $\emptyset \ne X \subseteq V(H)$ then k-gal$(H - X) \le k|X| - 1$.*

Proof. Recall the related property for hypomatchable graphs (proof: otherwise $X - x$ would be a barrier of defect at least 1 in $H - x$). It is easier to prove the statement for a broader set of graphs, called *pseudo galaxies.*

Definition. For an integer $k \ge 2$ the simple, connected graph G is a *pseudo k-galaxy* if denoting by I (or I_G) the set of vertices of degree at least k, for all $v \in I$ there are exactly $k - 1$ edges between v and $G - I$, each being a cut edge in G.

Fig. 9 shows a pseudo galaxy. The vertices of I are drawn as big dots and the edges of $G[I]$ as thick lines. Note, that a pseudo galaxy is just like a galaxy with the relaxation that the components of $G[I]$ need not be hypomatchable. What we actually prove is the following.

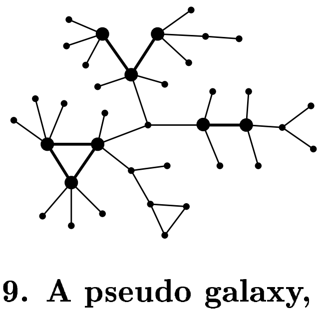


Fig. 9. A pseudo galaxy, $k = 4$

If G is a pseudo k-galaxy and $\emptyset \neq X \subseteq V(G)$ with the property that each vertex of $X \cap I$ is contained in a hypomatchable component of $G[I]$, then $k\text{-gal}(G - X) \leq k|X| - 1$ holds.

Suppose G is a pseudo galaxy of minimum size failing this statement. Let $\emptyset \neq X \subseteq V(G)$ be a minimal set for which $k\text{-gal}(G - X) \geq k|X|$. First, $X \subseteq I$ holds: it is obvious if $|X| = 1$, and if $|X| \geq 2$ and $x \in X \setminus I$ then $X - x$ would be a smaller set for which the statement fails.

Let F be a hypomatchable component of $G[I]$ with $X_F = X \cap V(F) \neq \emptyset$. Denote the k-galaxy components of $G - X_F$ by $H_1, \dots, H_s$. It is easy to see that the other components are pseudo k-galaxies, denoted by $G_1, \dots, G_t$. Let X_h (resp. X_g) be the set of vertices of X in a galaxy (resp. pseudo galaxy) component of $G - X_F$.

Now we bound the number of k-galaxy components of $G - X_F$. Let F' be a non-hypomatchable component of $F - X_F$ and G' be the component of $G - X_F$ containing F'. Now F' is a connected component of $G'[I_{G'}]$, hence G' is not a galaxy. So the related property of the hypomatchable graphs imply that the number of the galaxy components of $G - X_F$ meeting F is at most $|X_F| - 1$. On the other hand, due to the tree-like structure of G, the number of those components of $G - X_F$ which are disjoint from F is $(k-1)|X_F|$, because a vertex in $V(F)$ is incident with exactly $k-1$ cut edges in G and $X_F \subseteq V(F)$. So $s = k\text{-gal}(G - X_F) \leq |X_F| - 1 + (k-1)|X_F| = k|X_F| - 1$.

Let $X \cap V(G_i) = X_i$. In G_i each vertex of $X_i \cap I_{G_i}$ is contained in a hypomatchable component of $G_i[I_{G_i}]$, since $I_{G_i} = I_G \cap V(G_i)$. By the minimality of G we get that $k\text{-gal}(G_i - X_i) \leq k|X_i|$ (independently of the emptiness of X_i). On the other hand, let s' be the number of those k-galaxy components H_i of $G - X_F$, for which $X^i := X \cap V(H_i) \neq \emptyset$. For such a component $k\text{-gal}(H_i - X^i) \leq k|X^i| - 1$ holds by the minimality of G. Hence

$$k\text{-gal}(G - X) \leq k|X_g| + s - s' + (k|X_h| - s') \leq$$

$$\leq k|X_h \cup X_g| + s \leq k|X - X_F| + k|X_F| - 1 = k|X| - 1.$$

□

Theorem 4.6. *If A_1, A_2 are perfect barriers with k-surplus then $A_1 = A_2$.*

Proof. Let $D_i = D_{A_i}$ and $C_i = C_{A_i}$. Denote by c_i the number of components of $G[D_i]$ intersecting A_{3-i}. We prove that $c_1 = c_2 = 0$. Suppose $c_1 \geq c_2$ and that $A_2' := A_2 \cap D_1 \neq \emptyset$. A_2 is a barrier with k-surplus, so A_2' is adjacent to at least $k|A_2'| + 1$ galaxy components of $G[D_2]$. Now $A_1 \cap A_2' = \emptyset$ and in $G - A_1$ the set A_2' is adjacent to at most $k|A_2'| - c_1$ galaxy components by Lemma 4.5. The remaining $c_1 + 1$ components of $G[D_2]$ necessarily intersect A_1, so $c_2 \geq c_1 + 1$, a contradiction. This implies $c_1 = c_2 = 0$.

Suppose $A_1 \setminus A_2 \neq \emptyset$. The components of $G[D_1]$ are connected in $G - A_2$ because $A_2 \cap D_1 = \emptyset$. Let $D_1' \subseteq V(G)$ be the set of vertices in those components of $G[D_1]$ which are adjacent to $A_1 \setminus A_2$. By the k-surplus of A_1 we get that $c(G[D_1']) \geq k|A_1 \setminus A_2| + 1$. The components of $G[D_1']$ are not connected components in $G - A_2$, so $c_2 = 0$ implies that $D'_1 \cup (A_1 \setminus A_2) \subseteq C_2$. Hence $A_1 \setminus A_2$ is a barrier of defect at least 1 in C_2, a contradiction, because $G[C_2]$ has a perfect k-piece packing.

So $A_1 \subseteq A_2$ and by symmetry, $A_1 = A_2$. □

As we noted in page 11, the set of odd vertices A in any output of the algorithm is a perfect barrier with k-surplus, so A is unique by Theorem 4.6. Hence $V(S) - A$ (this is D_A) and $V(G - S)$ (this is C_A) are unique, too. So the following definition is sound:

Definition 4.7. Let (S, A) be the alternating structure in any output of the algorithm. Denote by D the set of vertices in the galaxies of S, and let $C = V(G - S)$. The decomposition $V(G) = D \dot{\cup} A \dot{\cup} C$ is said to be *canonical* to the k-piece packing problem.

From now on, the sets D, A, C denote the canonical decomposition of G for the k-piece packing problem. Theorem 4.6 implies also, that though the alternating structure S is not unique, the number of its components is, because A is a perfect barrier with k-surplus of $c(S)$. In Theorem 4.10 we give a characterization for the canonical barrier by other barriers.

Definition 4.8. The maximum defect of a barrier is denoted by $\mathrm{def}(G)$.

By Theorem 3.13, G has a perfect k-piece packing if and only if $\mathrm{def}(G) = 0$ (note, that in that case $\emptyset$ is a barrier with defect 0).

Lemma 4.9. *If A' is a barrier with maximum defect then it is perfect.*

Proof. If $G[C_{A'}]$ has no perfect k-piece packing, then it would admit a barrier $A'' \subseteq C_{A'}$ with defect at least 1 by Theorem 3.13. But then $A' \cup A''$ would be a barrier with larger defect than $\mathrm{def}(G)$. □

Theorem 4.10. *A has defect* $\mathrm{def}(G)$*, and A is the intersection of the barriers with maximum defect.*

Proof. Let A' be a barrier with defect $d := \mathrm{def}(G)$. Denote the set of components of $G[D_{A'}]$ by K'. Consider the following function in the bipartite graph $B_{A'}^{D_{A'}}$: for $X \subseteq K'$ let $f(X) = |X| - k|\Gamma(X)|$. Now $f(K') = d$, and $f(X) \leq d$ for $X \subseteq K'$, because otherwise $\Gamma(X)$ would be a barrier of larger defect than d. Suppose, that $f(X_1) = f(X_2) = d$ for $X_1, X_2 \subseteq K'$. The function $X \mapsto |\Gamma(X)|$ is submodular, so $2d = f(X_1) + f(X_2) \leq f(X_1 \cap X_2) + f(X_1 \cup X_2) \leq 2d$, implying $f(X_1 \cap X_2) = d$. Hence there exists a *minimum* set $K_0 \subseteq K'$ with $f(K_0) = d$. Let $A_0 = \Gamma(K_0)$ and let D_0 be the set of vertices contained in a component of K_0. The minimum property of K_0 implies that $B_{A_0}^{D_0}$ has k-surplus of d.

Now $k \cdot (A' - A_0)$ can be matched into $K' - K_0$ in $B_{kA'}^{D_{A'}}$ by Hall's theorem: if $|\Gamma(Y)| < k|Y|$ for $Y \subseteq A' - A_0$ then $A' - Y$ would be a barrier of larger defect than d. Moreover, $k|A' - A_0| = |K' - K_0|$ so this matching gives rise to a perfect k-piece packing in $G - C_{A'} - D_0 - A_0$ using Lemma 3.10. Moreover, by Lemma 4.9, $G[C_{A'}]$ has a perfect k-piece packing, so A_0 is a perfect barrier with k-surplus of d. Hence $A_0 = A$ by Theorem 4.6. This implies also that A has defect d, i.e. maximum defect. □

Note, that Theorem 4.6 and Lemma 4.9 implies that A is the only barrier with k-surplus of maximum defect. Another characterization is the following easy corollary of the proof of the previous theorem.

Corollary 4.11. $D = \bigcap\{D_{A'} : A'$ *is a barrier with maximum defect*$\}$.

In the sequel we prove some properties and characterizations for the canonical decomposition. First, we investigate the structure of *maximal* k-piece packings of G. In the subsequent lemmas we use the notational inaccuracy, that 'D' may denote both the vertex set $D \subseteq V(G)$ and the set of components of $G[D]$. It will be clear from the context which reading is meant.

Lemma 4.12. *Each maximal k-piece packing P of G has the following structure:*

1. *exactly $k|A|$ components of $G[D]$ are entered by P, and these components are completely covered by P,*
2. *if H is a component of $G[D]$ not entered by P then there is a tip T of H such that $P[H]$ is a perfect k-piece packing of $H - T$, and*
3. *$P[C]$ is a perfect k-piece packing of $G[C]$.*

Proof. Let P be a maximal k-piece packing of G. We construct a k-piece packing P' with $V(P') \supseteq V(P)$, such that if P fails one of the above properties then $V(P') \supsetneq V(P)$ would hold.

Let $D_e \subseteq V(G)$ be the set of vertices in those components of $G[D]$ which are entered by an edge of P. The bipartite graph B^D_{kA} has a matching covering $k \cdot A$ (due to the k-surplus of A), and another matching covering D_e (due to P). Hence the Mendelsohn-Dulmage theorem gives a matching M in B^D_{kA} covering $k \cdot A$ and D_e. Using Lemma 3.10, M gives rise to a perfect k-piece packing P_1 in the subgraph induced by A and the vertex sets of $k|A|$ components of $G[D]$ including the components of $G[D_e]$.

If a component H of $G[D]$ is not entered by M then it is not entered by P either. By Theorem 3.7 and by the maximality of P there is a tip T of H such that $P[H]$ is a perfect k-piece packing of $H - T$. The union of these k-pieces is denoted by P_2. Finally, let P_3 be a perfect k-piece packing of $G[C]$ (recall, that A is a perfect barrier). With $P' = P_1 \cup P_2 \cup P_3$ we get that $V(P') \supseteq V(P)$.

It is easy to see that $c(G[D_e]) \leq k|A|$. In fact, $c(G[D_e]) = k|A|$ holds here, because otherwise M would enter strictly more components of $G[D]$ than P, resulting in $V(P') \supsetneq V(P)$, a contradiction. Now *1.* and *2.* are straightforward. For *3.* observe, that there is no edge $e \in E(P)$ from A to $A \cup C$ because otherwise $c(G[D_e]) < k|A|$. □

In the matching case (i.e. when $k = 1$) there is a strong relation between maximal matchings and def(G), namely, each maximal matching misses exactly def(G) vertices of G. In general, when $k \geq 2$, this is not the case, because a maximal k-piece packing of a galaxy may miss an arbitrary number of vertices instead of only one (namely, the vertices of a tip). What is salvaged, is that for each maximal k-piece packing P, the graph $G - P$ has exactly def(G) components. This is implied by Theorem 4.10 and Lemma 4.12.

Note, that Lemma 4.12 (and Lemma 4.13 as well) holds also for decompositions $V(G) = D_{A'} \dot\cup A' \dot\cup C_{A'}$, with the property that A' is a perfect barrier for which $k \cdot A'$ can be matched into $D_{A'}$ in $B^{D_{A'}}_{kA'}$. This observation will be needed in the proof of Theorem 4.16.

Lemma 4.13. *If P is a k-piece packing satisfying 1., 2. and 3. of* Lemma 4.12*, then P is maximal.*

Proof. Recall, that $\mathrm{def}(G) = c(G[D]) - k|A|$. Let $D_e \subseteq V(G)$ be the set of vertices in those components of $G[D]$ which are entered by an edge of P. By property *1.*, $c(G[D_e]) = k|A|$ so P misses only the vertices of $\mathrm{def}(G)$ tips in the $\mathrm{def}(G)$ components of $G[D - D_e]$.

Suppose P' is a k-piece packing covering $V(P)$ and one more vertex $v \notin V(P)$. Now v is in a tip in a galaxy H of $G[D - D_e]$. By property *2.* and Lemma 3.6, P' must enter H, beside the components of $G[D_e]$. This is impossible, because a k-piece packing cannot enter more than $k|A|$ components of $G[D]$. □

These results imply a characterization for the union of the vertex sets of tips in D. Recall, that U_G is the set of vertices which can be missed by a maximal k-piece packing of G, and $\mathcal{T}_H \subseteq V(H)$ is the union of vertex sets of tips in a galaxy H.

Definition 4.14. Let $\mathcal{T}_G = \bigcup\{\mathcal{T}_H : H$ is a galaxy of $G[D]\}$.

Lemma 4.15. $\mathcal{T}_G = U_G$.

Proof. Lemma 4.12 implies $U_G \subseteq \mathcal{T}_G$. On the other hand, let $v \in \mathcal{T}_G$ be a vertex contained in a tip T of a galaxy H of $G[D]$. The bipartite graph B^D_A has k-surplus, so $k \cdot A$ can be matched into $D - H$ in B^D_{kA}. As we have seen in the proof of Lemma 4.12, this matching M gives rise to a perfect k-piece packing in the subgraph induced by $A \cup \bigcup\{V(H') : H'$ is covered by $M\}$. Add a perfect k-piece packing of $G[C]$ and perfect k-piece packings of $H' - T_{H'}$ where H' is a component of $G[D]$ not entered by M and $T_{H'}$ is a tip of H', with $T_H = T$. By Lemma 4.13, this k-piece packing is maximal. □

In the matching case, the uniqueness of the output of the (Edmonds') algorithm is implied by Lemma 4.15 itself (which uses only Lemmas 4.12 and 4.13). To see this, let A be the set of odd vertices and let $D = V(S) - A$ for the alternating structure (S, A) in any output. The proof of Lemma 4.15 works also for decompositions $V(G) = D_{A'} \dot\cup A' \dot\cup C_{A'}$, where A' is a perfect barrier with k-surplus. Hence $\mathcal{T}_G$ is unique, which is equal to D itself in the case $k = 1$. Now $A = \Gamma(D)$ implies that A is unique, too. When $k \geq 2$ the set of vertices missed by a maximal k-piece packing is merely a subset of D, so Theorem 4.6 is needed in proving the uniqueness of the decomposition given by the algorithm.

But how can we characterize the canonical D when $k \geq 2$? It is not true that deleting $v \in D$ the defect of G gets smaller, even $\mathrm{def}(G - v) = \mathrm{def}(G) + k - 2$ is possible. But something similar to the matching case is still true.

Theorem 4.16. $D = \{v : U_{G-v} \subsetneq U_G\} = \{v : |U_{G-v}| < |U_G|\}$.

Proof. The proof argues as in a stability lemma, we investigate the canonical decomposition of the graph $G - v$. Lemma 4.15 is frequently applied.

1. Let $v \in C$. Denote by $D' \dot\cup A' \dot\cup C'$ the canonical decomposition of $G[C - v]$. We get that $D^{G-v}_{A\cup A'} = D \cup D'$ and $C^{G-v}_{A\cup A'} = C'$, so $A \cup A'$ is a perfect barrier of k-surplus in $G - v$. Hence by Theorem 4.6 the canonical decomposition of $G - v$ is $(D \cup D') \dot\cup (A \cup A') \dot\cup C'$, so $U_{G-v} = \mathcal{T}_{G-v} \supseteq \mathcal{T}_G = U_G$.

2. Let $v \in A$. Now $A - v$ is a perfect barrier of k-surplus in $G - v$ so by Theorem 4.6 the canonical decomposition of $G - v$ is $D \dot\cup (A - v) \dot\cup C$. Hence $U_{G-v} = \mathcal{T}_{G-v} = \mathcal{T}_G = U_G$.

3. Finally, let v be contained in a galaxy H of $G[D]$. To prove that the deletion of v decreases $\mathcal{T}_{G-v}$, we need to show this for galaxies.

   Lemma 4.17. *Each component of $H - v$ is either a k-galaxy or has a perfect k-piece packing. Moreover, $\bigcup\{\mathcal{T}_F : F$ is a k-galaxy component of $H - v\} \subsetneq \mathcal{T}_H$.*

   Proof. The statement is trivial for $k = 1$ so assume $k \geq 2$. Due to the tree-like structure of a galaxy, the following considerations are easy to check. If v is contained in a tip then each component of $H - v$ is either a k-galaxy or an almost k-galaxy of type 2. Moreover, the union of the vertex sets of tips in all components is exactly $\mathcal{T}_H - v$. On the other hand, if $v \in I$ then $H - v$ consists of k-galaxies (the number of which is $k - 1$), and almost galaxies of type 1, the number of which is at least 1. Moreover, the union of the vertex sets of tips in all components is $\mathcal{T}_H$ and each almost galaxy of type 1 contains at least one tip of H. Using Lemma 3.4, we get the statement. □

   In $H - v$ let C_H denote the set of vertices in components with perfect k-piece packings, and let D_H denote the set of vertices in k-galaxy components. Now $D^{G-v}_A = (D \setminus V(H)) \cup D_H$ and $C^{G-v}_A = C \cup C_H$. Since $G[C \cup C_H]$ has a perfect k-piece packing and A is a barrier with k-surplus in G, we get that A is a perfect barrier of $G - v$ with the property that $k \cdot A$ can be matched into D^{G-v}_A. So, as mentioned in page 15, the statement of Lemma 4.12 holds for A as well. Denote by $\mathcal{T}'$ the vertices of tips of galaxies in $G[D^{G-v}_A]$. By Lemma 4.12 each maximal packing of $G - v$ misses only vertices in $\mathcal{T}'$, i.e. $U_{G-v} \subseteq \mathcal{T}'$. On the other hand, $\mathcal{T}' \subsetneq \mathcal{T}_G$ by the above property of the galaxies, so $U_{G-v} \subsetneq \mathcal{T}_G = U_G$.

□

Theorem 4.16 easily implies a characterization for the galaxies.

Theorem 4.18. *The followings are equivalent for a graph G.*

1. *G is a k-galaxy.*

2. *$|U_{G-v}| < |U_G|$ for all $v \in V(G)$.*

3. *$U_{G-v} \subsetneq U_G$ for all $v \in V(G)$.*

At this point, the proof of Theorem 2.4 is straightforward using the results of this section.

In the graph packing terminology a graph K is called *critical* to the $\mathcal{F}$-packing problem, if it does not have a perfect $\mathcal{F}$-packing, but $K-v$ has one for all $v \in V(K)$. In the previous polynomial graph-packing problems $K_2 \in \mathcal{F}$ implies that a critical graph is hypomatchable. In the k-piece packing problem the k-galaxies play the role of the critical graphs. This is because each k-galaxy H satisfies the following two properties. Actually, the Gallai-Edmonds theorem for the k-piece packing problem implies that these properties give another characterization for the k-galaxies, see Lemma 4.19.

1. H has no perfect k-piece packing, and
2. for each $v \in V(H)$ there is a k-piece packing P of H not covering v, such that $H - P$ is connected with highest degree at most $k - 1$.

Theorem 4.19. *A graph H satisfies properties* 1. *and* 2. *if and only if H is a k-galaxy.*

Proof. If H is a k-galaxy, then 1. is implied by Theorem 3.7, and 2. by Lemma 3.8.

For the reverse direction, suppose that H satisfies the above properties. By Theorem 2.4 property 1. implies that H is either a k-galaxy or its canonical barrier A is nonempty. Assuming the latter case to be true, choose a vertex $v \in A$ and let P be the k-piece packing of H guaranteed by property 2. Let W be a new set of $k - \deg_P(v)$ isolated vertices. Connect each vertex of W to v resulting in the new graph H'. Now P together with the new edges and vertices is a perfect k-piece packing of H', contradicting to the fact that A is still a barrier of H' of defect $\mathrm{def}(H) \geq 1$. □

When $k = 1$, property 2. is equivalent to the defining property of hypomatchable graphs H: the graph $H - v$ has a perfect matching for each $v \in H$. This implies property 1. as well by parity arguments (parity has no consequence when $k \geq 2$).

By the Gallai-Edmonds theorem for the k-piece packing problem, every simple graph has a sequence of 'canonical' decompositions $V(G) = D_k \dot\cup A_k \dot\cup C_k$, one for each $k \geq 1$. Here $D_1 \dot\cup A_1 \dot\cup C_1$ is the classical Gallai-Edmonds structure. Observe, that $A_k = C_k = \emptyset$ if $k \geq \Delta(G) + 1$, and $D_k = A_k = \emptyset$ if $k = \Delta(G)$. Besides, there does not seem to be any interesting relation among the decompositions for different k-s.

5 The matroidal property and maximum packings

Loebl and Poljak conjectures [5] that if the $\mathcal{F}$-packing problem is polynomial and $K_2 \in \mathcal{F}$, then the vertex sets coverable by $\mathcal{F}$-packings form a matroid. This conjecture is still open. The previous considerations directly imply, that the k-piece packing problem is not matroidal when $k \geq 2$ (see the counterexample in [2]). Still, the k-piece packing problem *has* the matroidal property in a slightly weaker form. Hence it gives another support for the validity of the conjecture of Loebl and Poljak.

Theorem. 2.5. *There exists a partition π on $V(G)$ and a matroid $\mathcal{M}$ on π, such that the maximal vertex sets coverable by k-piece packings are exactly the vertex sets of the form $\bigcup\{X : X \in \pi'\}$ for a base π' of $\mathcal{M}$. Moreover, the co-rank of $\mathcal{M}$ is* $\text{def}(G)$.

Proof. Lemmas 4.12, 4.13 and 4.15 imply that the following considerations hold.

The elements of π are the vertex sets of the tips of the galaxy components in $G[D]$, together with the vertices outside $\mathcal{T}_G$ as single element vertex sets. For $v \notin \mathcal{T}_G$ the element $\{v\} \in \pi$ is a bridge in $\mathcal{M}$. As for the tips, denote the transversal matroid on the components of $G[D]$ in B^D_{kA} by $\mathcal{N}$. Now for each component H of $G[D]$ replace H in $\mathcal{N}$ by a series set consisting of the tips of H. This gives the matroid $\mathcal{M}$. □

So each maximal k-piece packing P is 'compatible' to π, i.e. each vertex set in π is either fully covered or fully missed by P (and the number of the latter sets is def(G)).

The ground set of this matroid is a partition of $V(G)$ to different cardinality sets. Hence, in the k-piece packing problem, a *maximal* packing is not necessarily *maximum*, as it is the case in the previous polynomial packing problems. Still, the vertex sets missed by *maximum* packings admit a similar matroid: take the maximum weight bases of $\mathcal{M}$ with the weight function $w(X) := |X|$ for $X \in \pi$. Though this observation characterizes the size of the maximum k-piece packings, [2] proves a more compact Berge-Tutte type formula. After finding the canonical decomposition of the graph, the k-piece packing algorithm described in [2] has a second phase, which makes transformations on the alternating structure, imitating a method for finding a maximum weight base in the transversal matroid $\mathcal{N}$ of B^D_{kA}. From this method the following Berge-Tutte type theorem can be derived (with a little additional work). Here $k\text{-gal}_i(G)$ denotes the number of k-galaxy components H of the graph G with the property that each tip of H has size at least i.

Theorem 5.1. [2] *If G is a graph of size n, then the size of the maximum k-piece packings of G is*

$$n - \max \sum_{i=1}^{n} \left(k\text{-gal}_i(G - A_i) - k|A_i|\right),$$

taken over all sequences of vertex sets $V(G) \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n$.

We mention that A_1 can be chosen to be the canonical barrier A. The theorem refers to *sequences* of vertex sets because of the related structure of the maximum weight bases of the transversal matroid.

When $k = 1$ we get Berge's theorem on maximum matchings [1]. The case $k = 2$ was proved by Kano, Katona and Király [4].

6 The (l, u)-piece packing problem

As a generalization of the k-piece packing problem, the *(l, u)-piece packing problem* is introduced in [2]. In this section we investigate the matroidal property and the canonical decomposition for this problem, using the reduction to the k-piece packing problem [2]. As one may expect, it turns out that this problem is essentially the same as the k-piece packing problem, and all the above results can be directly applied with straightforward modifications.

Let two integer bounds $u(v) \geq l(v) \geq 0$ be given for each vertex $v \in V(G)$. A connected subgraph P of G is an (l,u)-*piece* if $\deg_P(v) \leq u(v)$ holds for each $v \in V(P)$ and there exists at least one vertex $w \in V(P)$ with $\deg_P(w) \geq l(w)$. Note, that if $l \equiv u \equiv k$ we get the k-piece packing problem.

Galaxies and tips change in the following way:

Definition 6.1. Given the bounds $l, u : V(H) \to \mathbb{N}$, the simple, connected graph H is an (l,u)-*galaxy* if it satisfies that

- denoting by I the set of vertices v with $\deg_G(v) \geq l(v)$, each component of $H[I]$ is a hypomatchable graph,
- $l(v) = u(v) \geq 1$ for $v \in I$, and
- for all $v \in I$ there are exactly $l(v) - 1$ edges between v and $H - I$, each being a cut edge in H.

The *tips* are the connected components of $H - I$ together with the vertices $v \in I$ with $l(v) = u(v) = 1$, as single vertex subgraphs.

The new tips still have the important property of Theorem 3.7, i.e. the maximal (l,u)-piece packings of an (l,u)-galaxy are exactly the perfect (l,u)-piece packings of $H - T$ for a tip T.

The difference in the definition of the galaxies and tips can be explained by the following reduction to the k-piece packing problem, described in [2]. Let $k = 1 + \max\{u(v) : v \in V(G)\}$. For each vertex $v \in V(G)$ let $M_v,\ N_v$ be disjoint sets of new vertices with $|M_v| = u(v) - l(v) + 1$ and $|N_v| = k - u(v) - 1$. Now for $v \in V(G)$ take a complete graph on M_v and connect the vertices of $M_v \cup N_v$ to v. Denote the new graph by G_k. It is easy to see that G_k has a perfect k-piece packing if and only if G has a perfect (l,u)-piece packing, and G is an (l,u)-galaxy if and only if G_k is a k-galaxy. Actually, with the help of this reduction we can see that all the above considerations for the k-piece packings hold in the (l,u)-case as well, with the necessary modifications. For illustrating the essence of this equivalence, we briefly describe how to get the canonical decomposition of G for the (l,u)-piece packing problem.

Let $V(G_k) = D_k \dot\cup A_k \dot\cup C_k$ be the canonical decomposition of G_k for the k-piece packing problem. Due to the k-surplus of A_k, each vertex of A_k has degree at least $k+1$ in G_k. Because the new vertices of G_k (i.e. the vertices in $V(G_k) - V(G)$) have degree at most $u(v) - l(v) + 1 \leq k$, we get that $A_k \subseteq V(G)$. So, the deletion of the new vertices yields in a natural way a partition $V(G) = D \dot\cup A \dot\cup C$ (where $D = D_k \cap V(G)$, $A = A_k$ and $C = C_k \cap V(G)$). This partition has all the properties listed in Theorem 2.4, e.g. the components of $G[D]$ are (l,u)-galaxies, the B_A^D bipartite graph has u-*surplus* (i.e. $|\Gamma(A')| \geq u(A') + 1$ for $\emptyset \neq A' \subseteq A$), and C has a perfect (l,u)-piece packing. The uniqueness of this decomposition can be shown easily: If

$V(G) = D' \dot\cup A' \dot\cup C'$ is another partition with this property, then in G_k the set A' is a perfect barrier with k-surplus, hence by Theorem 4.6 it equals to A.

This Gallai-Edmonds type theorem in the case $l(v) = l < u = u(v)$ for all $v \in V(G)$ becomes quite compact, so we include this. Here an (l, u)-packing is a packing with connected graphs F with $l \leq \Delta(F) \leq u$. Call such a packing an $(l < u)$-packing. The simplicity of this structure theorem comparing to the general (l, u)-case is due to the fact that here an (l, u)-galaxy is just a graph with highest degree at most $l - 1$. So it always consists of only one tip.

Theorem 6.2. *For a graph G let D consists of the vertices which can be missed by a* maximal *$(l < u)$-packing. Let $A = \Gamma(D)$ and $C = V(G) - D - A$. Now*

1. *$\Delta(G[D]) \leq l - 1$,*
2. *the bipartite graph B_A^D has u-surplus, and*
3. *$G[C]$ has a perfect $(l < u)$-packing,*
4. *for each maximal $(l < u)$-packing P, the graph $G - P$ has exactly $c(G[D]) - u|A|$ components.*

We remark that with a bit additional work all the above results can be generalized to the following packing problem. Given two bounds $l, u\colon V(G) \to \mathbb{N}$, $l \leq u$, and a set $\mathcal{F}$ consisting of hypomatchable subgraphs of G, decide if G has a spanning subgraph each component of which is either an (l, u)-piece or an (l, u)-galaxy H such that $H[I]$ is connected and belongs to $\mathcal{F}$. We do not go into details.

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