Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2004-08. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

On partition constrained splitting off

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May 2004

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Abstract

A short proof is presented for a slight generalization of the partition constrained splitting off theorem of [1].

1 Introduction

Let $G := (V + s, E)$ be a k-edge-connected graph in V with $d(s)$ even. A pair of edges rs, st is called **admissible** if splitting off these edges (replacing rs and st by rt) preserves k-edge-connectivity in V. Let $\mathcal{P} = \{P_1, ..., P_r\}$ be a partition of $\delta(s)$. $e \in P_j$ will also be denoted by $c(e) = j$. An admissible pair $\{e, f\}$ is called **allowed** if $c(e) \neq c(f)$. By a **complete** splitting off we mean that we split off $\frac{d(s)}{2}$ disjoint pairs of edges incident to s. For $X, Y \subset V + s$, $\delta(X)$ denotes the set of edges leaving X, $d(X) = |\delta(X)|$ and $d(X, Y)$ denotes the number of edges between X and Y.

A partition $\{A_1, A_2, A_3, A_4\}$ of V is called a $\boldsymbol{C_4}$**-obstacle** of G if k is odd and

$$d(A_i) = k \qquad \forall 1 \leq i \leq 4, \tag{1}$$
$$d(A_i, A_{i+2}) = 0 \qquad \forall 1 \leq i \leq 2, \tag{2}$$
$$|P_l| = d(s)/2 \qquad \exists 1 \leq l \leq r, \tag{3}$$
$$\delta(A_j \cup A_{j+2}) \cap \delta(s) = P_l \qquad \exists 1 \leq j \leq 2. \tag{4}$$

A partition $\{A_1, A_2, ..., A_6\}$ of V is called a $\boldsymbol{C_6}$**-obstacle** of G if k is odd and

$$d(A_i) = k \qquad \forall 1 \leq i \leq 6, \tag{5}$$
$$d(A_i, A_{i+1}) = (k-1)/2 \qquad \forall 1 \leq i \leq 6, (A_7 = A_1) \tag{6}$$
$$d(s, A_i) = 1 \qquad \forall 1 \leq i \leq 6, \tag{7}$$
$$\delta(A_j \cup A_{j+3}) \cap \delta(s) = P_{l_j} \qquad \forall 1 \leq j \leq 3, \exists 1 \leq l_j \leq r. \tag{8}$$

The following result is a slight generalization of the main theorem on splitting off in [1]. The motivation of this form is that it allows us to contract tight sets and hence it enables us to simplify the proof.

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Theorem 1.1. *Let $G = (V + s, E)$ be a k-edge-connected graph in V with $k \geq 2$ and $d(s)$ is even, let $\mathcal{P} = \{P_1, ..., P_r\}$ be a partition of $\delta(s)$. Then there exists a complete allowed splitting off at s if and only if*

$$|P_i| \leq d(s)/2 \quad \forall 1 \leq i \leq r, \tag{9}$$

$$G \text{ contains no } C_4 \text{ or } C_6\text{-obstacle.} \tag{10}$$

The aim of this note is to present a proof of Theorem 1.1 that is shorter than the proof in [1]. We mention that not all the simplifications are due to the "tight set contraction".

2 Definitions and Preliminary results

In this note $G := (V + s, E)$ is always a k-edge-connected graph in V, that is (11) is satisfied. The fact, that for $X, Y \subset V$, (12) and (13) are satisfied, will be used frequently.

$$d(X) \geq k \ \forall \emptyset \neq X \subset V, \tag{11}$$

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y), \tag{12}$$

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, V + s - (X \cup Y)). \tag{13}$$

Let $X \subset V$. X is called **tight** (resp. **dangerous**) if $d(X) = k$ (resp. $d(X) \leq k + 1$). We say that X is a **singleton** if $|X| = 1$. G/X (resp. $G[X]$) denotes the graph obtained from G by contracting X into one vertex (resp. by deleting the vertices not in X). For $e = rs$ and $f = st$, $G_{e,f} = G_{r,t} = G - rs - rt + rt$.

The followimg two claims are from [2].

Claim 2.1. *(a) $\{su, sv\}$ is admissible if and only if there is no dangerous set containing u and v. (b) For any edge su, there exist at most two dangerous sets M_1 and M_2 so that $u \in M_1 \cap M_2$ and $\{v : \{su, sv\}$ is not admissible $\} \subseteq M_1 \cup M_2$.* □

Claim 2.2. *For a tight set T, $\{su, sv\}$ is allowed in G if and only if it is allowed in G/T.* □

Claim 2.3. *$d(X) - k \geq 2d(s, X) - d(s)$ $\forall X \subset V$ where equality holds if and only if $d(V - X) = k$.*

Proof. By (11), $d(X) - k = d(V - X) - k + d(s, X) - (d(s) - d(s, X)) \geq 2d(s, X) - d(s)$. □

Claim 2.4. *If $k \geq 3$ and $d(X) \leq k + 2$ then $G[X]$ is connected.*

Proof. For a set $\emptyset \neq Y \subset X$, by (12) and (11), $(k + 2) + 2d(Y, X - Y) \geq d(X) + 2d(Y, X - Y) = d(Y) + d(X - Y) \geq k + k \geq k + 3$, and the claim follows. □

Claim 2.5. *If k is odd, X_1, X_2, X_3 are disjoint tight sets, $d(\cup_{i=1}^3 X_i) = k + 2$ and $d(X_1, X_3) = 0$, then $d(X_1, X_2) = d(X_2, X_3) = \frac{k-1}{2}$.*

Proof. By (12) and (11), $2k = d(X_2) + d(X_i) = d(X_2 \cup X_i) + 2d(X_2, X_i) \geq k + 2d(X_2, X_i)$, thus, by parity, $2d(X_2, X_i) \leq k - 1$ $i \in \{1, 3\}$. $3k = \sum_{i=1}^3 d(X_i) = d(\cup_{i=1}^3 X_i) + \sum_{i \neq j} 2d(X_i, X_j) \leq (k+2) + 2(k-1) + 0 = 3k$, and the claim follows. □

Claim 2.6. *If $\mathcal{A}$ is a C_4-obstacle, then $d(s, A_i) \geq 1$ $\forall A_i \in \mathcal{A}$.*

Proof. Suppose wlog. $d(s, A_1) = 0$. Then, by (2), $d(A_1, A_2) + d(A_1, A_4) = k$, so, since k is odd, wlog. $d(A_1, A_2) \geq \frac{k+1}{2}$. Then, by (11), (12) and (1), $k \leq d(A_1 \cup A_2) = d(A_1) + d(A_2) - 2d(A_1, A_2) \leq k + k - (k+1) = k - 1$, contradiction. □

Claim 2.7. *If $\{A_1, ..., A_6\}$ is a C_6-obstacle, then for every allowed pair $\{sx, sy\}$, $G_{x,y}$ contains a C_4-obstacle.*

Proof. Wlog. $x \in A_1$. By (12), (5), (6), $d(A_i \cup A_{i+1}) = d(A_i) + d(A_{i+1}) - 2d(A_i, A_{i+1}) = k + k - (k-1) = k + 1$. Then, since $\{sx, sy\}$ is admissible, $y \notin A_2 \cup A_6$ by Claim 2.1(a). $\{sx, sy\}$ is allowed so, by (8), $y \notin A_4$. Thus wlog. $y \in A_3$. Then $\{A_1 \cup A_2 \cup A_3, A_4, A_5, A_6\}$ is a C_4-obstacle in $G_{x,y}$. □

The following lemma is a new observation.

Lemma 2.8. *If G contains no C_4-obstacle and (9) is satisfied then each edge su belongs to an allowed pair.*

Proof. Let $S := \{sv \in E : \{su, sv\}$ is admissible$\}$. Suppose su belongs to no allowed pair. Then every $sv \in S$ and su belong to the same P_j. Then, by (9), $\frac{d(s)}{2} \geq |P_j| \geq |S| + 1$, so $|S| \leq \frac{d(s)}{2} - 1$ and if equality holds then $\frac{d(s)}{2} = |P_j|$. It also follows, by Claim 2.1(b), that there are at most two dangerous sets M_1 and M_2 so that $u \in M_1 \cup M_2$ and $\{v_i : sv_i \in \delta(s) - S\} \subseteq M_1 \cup M_2$. In fact there are exactly two, because, by Claim 2.3, $d(M_1 \cup M_2) - k \geq 2d(s, M_1 \cup M_2) - d(s) = 2(d(s) - |S|) - d(s) \geq d(s) - 2(\frac{d(s)}{2} - 1) = 2$, and if equality holds then $d(V - M_1 \cup M_2) = k$ and $|S| = \frac{d(s)}{2} - 1$. The following claim provides a contradiction.

Claim 2.9. $\{A_1 = M_1 \cap M_2, A_2 = M_1 - M_2, A_3 = V - M_1 \cup M_2, A_4 = M_2 - M_1\}$ *is a C_4-obstacle.*

Proof. Note that $A_i \neq \emptyset$ $1 \leq i \leq 4$ and $\cup_{i=1}^4 A_i = V$. By (12), (11) and $d(M_1 \cup M_2) \geq k + 2$, $2(k+1) \geq d(M_1) + d(M_2) = d(A_1) + d(M_1 \cup M_2) + 2d(M_1, M_2) \geq k + (k+2)$, so $d(A_1) = k, d(M_1 \cup M_2) = k + 2$ and hence $d(A_3) = k$ and $\frac{d(s)}{2} = |P_j|$ so (3) is satisfied, and $d(A_2, A_4) = d(M_1, M_2) = 0$. By (13) and (11), $2(k+1) \geq d(M_1) + d(M_2) = d(A_2) + d(A_4) + 2d(A_1, A_3 + s) \geq k + k + 2d(A_1, A_3) + 2d(A_1, s) \geq 2k + 0 + 2$, so $d(A_2) = d(A_4) = k, d(A_1, A_3) = 0$ and $d(s, A_1) = 1$. It also follows that $\delta(A_1 \cup A_3) \cap \delta(s) = P_j$, so (1), (2) and (4) are satisfied. □

Lemma 2.8 shows that there exists an allowed splitting off. The main difficulty of the proof of Theorem 1.1 is to show that there exists an allowed splitting off that creates no C_4- or C_6-obstacle.

3 The proof

Proof. (of the necessity) Suppose there exists a graph that has a complete allowed splitting off $\{\{e_i, f_i\} : 1 \le i \le \frac{d_G(s)}{2}\}$ and violates (9) or (10). Choose such a graph G with $d_G(s)$ minimum. For every $1 \le i \le \frac{d_G(s)}{2}, 1 \le j \le r, |P_j \cap \{e_i, f_i\}| \le 1$ so (9) is satisfied, whence G contains a C_4 or a C_6-obstacle. By Claim 2.6 and (6), $d_G(s) \ne 0$. Then, either by (3) and (4) or by Claim 2.7, G_{e_1,f_1} is a smaller example, contradiction. □

Proof. (of the sufficiency) Induction on $|V|$. By Claim 2.2, we may assume that

$$\text{every tight set is a singleton.} \tag{14}$$

Wlog. $|P_1|$ is maximum. By Lemma 2.8, there is an allowed pair $\{e = sx, f = sy\}$ with $sx \in P_1$.

Lemma 3.1. *Suppose that $G' := G_{e,f}$ contains a C_6-obstacle $\mathcal{A} = \{A_1, ..., A_6\}$. Then there exists an edge $f' = sy'$ such that $\{e, f'\}$ is allowed and $G'' := G_{e,f'}$ satisfies (9) and (10).*

Proof. Since $sx \in P_1$, G'' satisfies (9). Wlog. $x \in A_1$. Since $xy \in E(G')$, either $y \in A_1$ (Case a) or wlog. $y \in A_2$ (Case b). By (5) and (14), $A_j = a_j \ \forall\, 2 \le j \le 6$. By (8), $c(sa_3) \ne c(sa_5)$ so either $c(sa_3) \ne 1$ (let $y' := a_3$) or $c(sa_5) \ne 1$ (let $y' := a_5$).

Claim 3.2. *If $x, y' \in X \ne V$ and $d_{G'}(X) \le k+2$ then $d_{G'}(X) = k+2$ and $X \cup A_1$ is the union of three consecutive sets in $\mathcal{A}$.*

Proof. Let $X^* := X \cup A_1$. By (6), $d_{G'-s}(X^*) \ge k-1$ where equality holds if and only if X^* is the union of $2 < l < 6$ consecutive sets in $\mathcal{A}$. By Claim 2.3, $d_{G'}(s, X) \le 4$, by (7), $d_{G'}(s, A_1) = 1$ and $d_{G'}(s, V) = 6$ so $X^* \ne V$. By (5), $d_{G'}(A_1) = k$, by $x \in X \cap A_1$ and (11) for G', $d_{G'}(X \cap A_1) \ge k$, so by (12), $(k+2) + k \ge d_{G'}(X) + d_{G'}(A_1) \ge d_{G'}(X \cap A_1) + d_{G'}(X^*) \ge k + d_{G'}(X^*)$, so $k + 2 \ge d_{G'}(X^*)$ and if equality holds then $d_{G'}(X) = k+2$. Then, by Claim 2.4, $G'[X^*]$ is connected. Since $d_{G'-s}(y', A_1) = 0$, $X' := X - (y' \cup A_1) \ne \emptyset$. Then $k+2 \ge d_{G'}(X^*) = d_{G'}(s, X^*) + d_{G'-s}(X^*) \ge d_{G'}(s, y') + d_{G'}(s, X') + d_{G'}(s, A_1) + d_{G'-s}(X^*) \ge 1+1+1+(k-1)$, thus $d_{G'}(X^*) = k+2$ and hence $d_{G'}(X) = k+2$, $d_{G'}(s, X') = 1$ and $d_{G'-s}(X^*) = k-1$, thus $X \cup A_1$ is the union of three consecutive sets in $\mathcal{A}$. □

Claim 3.3. $\{e, f'\}$ *is admissible (and hence allowed).*

Proof. If not then, by Claim 2.1(a), there exists a set X with $x, y' \in X \ne V$ and $k+1 \ge d_G(X)$. Since $d_G(X) \ge d_{G'}(X)$, Claim 3.2 implies that $d_{G'}(X) = k+2$, contradiction. □

Case a: Wlog. $y' = a_5$. Suppose that G'' contains a C_4 (Case (i)) or a C_6-obstacle (Case (ii)) $\mathcal{A}'$. Wlog. $x \in A'_1$. Suppose $y' \notin A'_1$. Then, by (5), $k+2 = d_{G'}(A_1) + 2 = d_G(A_1)$. By (1) or (5) and (14), $|A'_i| = 1 \ \forall A'_i \in \mathcal{A}'$ so $A'_1 = A_1$ and hence $k = d_{G''}(A'_1) = d_{G''}(A_1) = d_G(A_1)$, contradiction. Thus $y' \in A'_1$ and $d_G(A'_1) = k+2$. Since $d_{G'}(A'_1) \le d_G(A'_1)$, $A'_1 \cup A_1$ is the union of three consecutive sets in $\mathcal{A}$ by Claim 3.2.

(i): Then $3 = |V - (A_1 \cup A'_1)| \leq |V - A'_1| = 3$ by (14) so $A_1 \subset A'_1$ thus wlog. $A'_j = a_j$ $2 \leq j \leq 4$. By (8) for $\mathcal{A}$, there is a $w \in A_1$ with $c(sw) = c(sa_4)$ but $w \in A'_1$ and $a_4 \in A'_4$, contradiction by (4) for $\mathcal{A}'$.

(ii): Then $a_6 \in A'_1$. Wlog. $A'_2 = a_4$ and $A'_3 = a_3$. Then, by (8) for $\mathcal{A}$ and $\mathcal{A}'$, $c(sa_3) = c(sa_6) \neq c(sa_3)$, contradiction.

Case b: Then, by (5) and (14), $A_1 = a_1$ so $|V| = 6$. Note that, by (7), $d_G(s, a_1) = d_G(s, a_2) = 2$ and $d_G(s, a_h) = 1$ $(3 \leq h \leq 6)$. $d_{G''}(s, a_2) = 2$ so, by (7), G'' contains no C_6-obstacle. Suppose that G'' contains a C_4-obstacle $\mathcal{A}'$. Wlog. $x, y' \in A'_1$ and $d_G(A'_1) = k + 2$. Since $d_{G'}(A'_1) \leq d_G(A'_1)$, $d_{G'}(A'_1) = k + 2$ and $A'_1 \cup A_1$ is the union of three consecutive sets in $\mathcal{A}$ by Claim 3.2. Then $d_{G'}(A'_1) = d_G(A'_1)$ so $y' = a_5$. Thus $A'_1 = \{a_5, a_6, a_1\}$. Wlog. $A'_j = a_j$ $2 \leq j \leq 4$ by (6) for $\mathcal{A}$ and (2) for $\mathcal{A}'$. By (8) for $\mathcal{A}$, $c(sa_1) = c(sa_4)$, contradiction by (4) for $\mathcal{A}'$. □

Lemma 3.4. *Suppose that $G' := G_{e,f}$ contains a C_4-obstacle $\mathcal{A} := \{A_1, A_2, A_3, A_4\}$. Then there exists an allowed pair $e' = sx', f' = sy'$ such that $G'' := G_{e',f'}$ satisfies (9) and (10).*

Proof. Wlog. $x \in A_1$. Since $xy \in E(G')$, either $y \in A_1$ (Case a) or wlog. $y \in A_2$ (Case b). By (1), (14), $A_j = a_j$ $\forall\, 2 \leq j \leq 4$, in Case a $d_G(A_1) = k + 2$ and in Case b $A_1 = a_1$ so $|V| = 4$.

Case a: Let $g := sa_3$. If $c(g) \neq c(e)$ then let $e' := e, f' := g$, otherwise let $e' := g, f' := f$. Since $c(e') = c(e)$, G'' satisfies (9).

Claim 3.5. *If $x', y' \in X \neq V$ and $d_G(X) \leq k + 2$ then $d_G(X) = k + 2$. Moreover if $|V - X| \geq 2$, then (a) $X \cup A_1 = V - a_i$ $\exists i \in \{2, 4\}$, (b) $d_G(X \cap A_1) = k$.*

Proof. By Claim 2.4, $G[X]$ is connected, so, by (2), $\exists i \in \{2, 4\} : V - a_i \subseteq X \cup A_1$. First suppose that $X \cup A_1 = V$. Then, by Claims 2.3 and 2.6, (3), (4), $1 + \frac{d_G(s)}{2} \geq \frac{d_G(X) - k + d_G(s)}{2} \geq d_G(s, X) = d_{G'}(s, a_2 \cup a_4) + d_{G'}(s, a_3) + d_G(s, X \cap A_1) \geq (\frac{d_G(s)}{2} - 1) + 1 + 1 \geq 1 + \frac{d_G(s)}{2}$, so $d_G(X) = k + 2$ and $d_G(V - X) = k$ thus, by (14), $|V - X| = 1$. Now suppose that $|V - X| \geq 2$. Then it follows that $X \cup A_1 \neq V$ and (a) is satisfied. Then, by (3), (4) and Claim 2.6, $d_G(s, X \cup A_1) = d_{G'}(s, a_3 \cup A_1) + 2 + d_G(s, a_4) \geq (\frac{d_G(s)}{2} - 1) + 2 + 1$. Thus, by Claim 2.3, $d_G(X \cup A_1) \geq k + 2d_G(s, X \cup A_1) - d_G(s) \geq k + 4$. Then, by (12) and (11), (b) is satisfied and $d_G(X) = k + 2$. □

By Claims 2.1(a) and 3.5, $\{e', f'\}$ is an allowed pair. If G'' satisfies (10) then we are done. If G'' contains a C_6-obstacle then, by Lemma 3.1, we are done. Suppose that G'' contains a C_4-obstacle $\mathcal{A}' := \{A'_1, A'_2, A'_3, A'_4\}$. Wlog. $x' \in \delta(A'_1)$. Wlog. $y' \in A'_1$, otherwise restarting the proof by e' and f' we are in Case b. Then, by (1) and (14), $|A'_j| = 1$ $\forall\, 2 \leq j \leq 4$ and $d_G(A'_1) = k + 2$. By Claim 3.5 applied for $X = A'_1$, wlog. $A'_1 \cup A_1 = V - a_4$ and $d_G(A'_1 \cap A_1) = k$ thus, by (14), $|A'_1 \cap A_1| = 1$, say $A'_1 \cap A_1 = a_1$. Then it follows that $|V| = 6$, say $V = \{a_1, a_2, ..., a_6\}$. Note that $d_G(a_i) = k$ $1 \leq i \leq 6$. By Claim 2.6 for $\mathcal{A}$ and for $\mathcal{A}'$, $1 \leq d_G(sa_i)$ $1 \leq i \leq 6$ so $6 \leq d_G(s)$. The following lemma provides a contradiction.

Lemma 3.6. *$\{a_1, a_2, ..., a_6\}$ forms a C_6-obstacle in G.*

Claim 3.7. *(a) $A_1 = \{a_1, a_5, a_6\}$, $A_i = a_i$ $2 \leq i \leq 4, A'_1 = \{a_1, a_2, a_3\}$, wlog. $A'_2 = a_6, A'_3 = a_5, A'_4 = a_4$, (b) $d_G(a_1, a_2) = d_G(a_2, a_3) = d_G(a_1, a_6) = d_G(a_5, a_6) = \frac{k-1}{2}$, (c) $\{x, y\} = \{a_1, a_5\}$.*

Proof. We know that $A'_1 = \{a_1, a_2, a_3\}$ and $A_1 = \{a_1, a_5, a_6\}$. Then, by (2) for $\mathcal{A}$, $d_G(a_1, a_3) = 0$ so, by Claim 2.5, $d_G(a_1, a_2) = d_G(a_2, a_3) = \frac{k-1}{2}$. Wlog. $A'_2 = a_6$. Suppose that $A'_4 = a_5$. Then, by (2) for $\mathcal{A}'$, $d_G(a_5, a_6) = 0$ so, by Claim 2.5, $d_G(a_1, a_5) = d_G(a_1, a_6) = \frac{k-1}{2}$. Then $k = d_G(a_1) \geq d_G(a_1, a_2) + d_G(a_1, a_5) + d_G(a_1, a_6) + d_G(a_1, s) \geq 3\frac{k-1}{2} + 1$, that is $k \leq 1$, contradiction. Thus $A'_4 = a_4$ and $A'_3 = a_5$, that is (a) is satisfied. Then, by (2) for $\mathcal{A}'$, $d_G(a_1, a_5) = 0$ so, by Claim 2.5, $d_G(a_1, a_6) = d_G(a_5, a_6) = \frac{k-1}{2}$ and (b) is satisfied.

By definition $\{x, y\} \cap \{x', y'\} = a_1$ so $a_1 \in \{x, y\}$. Suppose $a_5 \notin \{x, y\}$. Then $\{x, y\} = \{a_1, a_6\}$. By (12), (5) and (b), $d_G(\{a_1, a_6\}) = d_G(a_1) + d_G(a_6) - 2d_G(a_1, a_6) = k + k - (k-1) = k+1$, hence, by Claim 2.1(a), $\{sx, sy\}$ is not admissible, contradiction, thus (c) is satisfied. □

Proof. By (4) for $\mathcal{A}'$, $c(sa_2) \neq c(sa_4)$ so $\delta_{G'}(A_1 \cup A_3) \cap \delta_{G'}(s) = P'_l$ in (4) for $\mathcal{A}$ for some l with $|P_l| \geq |P'_l| = \frac{d_{G'}(s)}{2} = \frac{d_G(s)}{2} - 1$. By (4) for $\mathcal{A}$, $c(sa_6) \neq c(sa_4)$ so $\delta_{G''}(A'_1 \cup A'_3) \cap \delta_{G''}(s) = P_{l'}$ in (4) for $\mathcal{A}'$ for some l' with $|P_{l'}| \geq |P'_{l'}| = \frac{d_{G'}(s)}{2} = \frac{d_G(s)}{2} - 1$. In particular, $c(sa_2) = c(sa_5) = l'$. By (4) for $\mathcal{A}$, $l = c(sa_3) \neq c(sa_2) = l'$ thus, by Claim 3.7(c), $e = e' = sa_1, f = sa_5, f' = sa_3$. Since $\{e, f\}$ and $\{e', f'\}$ are allowed, $l \neq 1 \neq l'$. Then, by the maximality of P_1, $|P_1| \geq |P_l| \geq \frac{d_G(s)}{2} - 1$. $d_G(s) \geq |P_1| + |P_l| + |P_{l'}| \geq 3(\frac{d_G(s)}{2} - 1)$, that is $d_G(s) \leq 6$. Then $d_G(s) = 6$ and $|P_1| = |P_l| = |P_{l'}| = 2$, namely $P_1 = \{sa_1, sa_4\}, P_l = \{sa_3, sa_6\}, P_{l'} = \{sa_2, sa_5\}$, so (7) and (8) are satisfied. We have already seen that (5) is satisfied. By (1) and (2) for $\mathcal{A}'$ and for $\mathcal{A}$, Claim 3.7(b) and (7), $d_G(a_5, a_4) = \frac{k-1}{2} = d_G(a_3, a_4)$. Then, by Claim 3.7(b), (6) is satisfied. □

Case b: If there exists an edge $g = sa_3$ with $c(g) \neq c(e)$ then let $e' := e, f' := g$. Otherwise, since $\mathcal{A}$ is not a C_4-obstacle in G, there is an edge $h = sa_1$ with $c(h) \neq c(e)$ and then let $e' := sa_3, f' := h$.

Claim 3.8. *$\{e', f'\}$ is admissible (and hence allowed).*

Proof. Suppose not. Then, by Claim 2.1(a), there exists $x', y' \in X \neq V$ and $d_G(X) \leq k+1$. By Claim 2.4, $G[X]$ is connected so, by (2), $\exists i \in \{2, 4\}$ $X = \{a_1, a_3, a_i\}$. Then, by (3), (4) and Claim 2.6, $d_G(s, X) \geq d_{G'}(s, X) + 1 = d_{G'}(s, a_1 \cup a_3) + d_{G'}(s, a_i) + 1 \geq (\frac{d_G(s)}{2} - 1) + 1 + 1$. Then, by Claim 2.3, $1 \geq d_G(X) - k \geq 2d_G(s, X) - d_G(s) \geq 2$, contradiction. □

Since $c(e') = c(e)$, G'' satisfies (9). Suppose that G'' does not satisfy (10). Then, since $|V| = 4$, G'' contains a C_4-obstacle $\mathcal{A}' := \{A'_1, A'_2, A'_3, A'_4\}$. Since $a_1a_3 \in E(G'')$, wlog. $A'_1 \cup A'_2 = a_1 \cup a_3$ and $A'_3 \cup A'_4 = a_2 \cup a_4$. By Claim 2.6, $d_{G''}(s, A'_i) \geq 1$ so $d_G(s, A'_i) \geq 2$ $i \in \{1, 2\}$. By (3) and (4) for $\mathcal{A}$ in G', there exist $1 \leq l \leq r$ and $j \in \{1, 2\}$ such that for every edge $sd \in E(G')$ with $d \in A_j \cup A_{j+2}$, $c(sd) = l$. Then there exist $sd_1, sd_2 \in E(G'')$ with $d_1 \in A_j, d_2 \in A_{j+2}$ and $c(sd_1) = c(sd_2) = l$. This contradicts (4) for $\mathcal{A}'$. $\square$

By Lemmas 2.8, 3.1 and 3.4, there exists a complete allowed splitting off and Theorem 1.1 is proved. $\square$

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