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A note on the directed source location algorithm

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Abstract

Recently Bárász, Becker and Frank gave a strongly polynomial time algorithm that solves the Directed Source Location Problem which is the following: given a directed graph $D = (V, A)$ and positive integers k and l, find a minimum cardinality set $R \subseteq V$ such that contracting R into a single node r results in graph which is (k, l)-edge-connected with respect to the root node r. They introduce the notion of *solid sets* and observe that the union of two intersecting solid sets is also solid. The bottleneck operation of their algorithm is the step of determining the hypergraph

$$\mathcal{H} = \{X : X \text{ is a maximal } s\text{-avoiding solid set for some } s \in V\}.$$

It is easy to see that $\mathcal{H}$ has at most $n(n-1)$ elements. The motivation of the present work was to prove that one can give a better bound on $|\mathcal{H}|$. Namely we prove here that $|\mathcal{H}| \leq 2(n-1)$.

1 Introduction and preliminaries

The motivation of discrete location problems comes from the application: one looks for an optimal placement of some facilities (warehouses, telecommunication centers, computer servers) in a network so as to satisfy certain customer demands. Typically it is the distance or the bandwidth that matters in defining the constraints and the objective functions. For an annotated bibliography of the topic, see the work of M. Labbe and F.V. Louveoux [3]. Source location is a new type of location problems where the flow-amount or connectivity rather than the distance between facilities and customers is taken into consideration. Source location may serve as a useful optimization framework for designing fault-tolerant telecommunication networks. For example, imagine such a network in which a subset R of nodes is considered a suitable source-set if there are k edge-disjoint paths from R to every node not in R and the objective is to compute a smallest source-set.

There are several versions of source location problems according to the connectivity type used in the constraints. Ito et al. [2] considered and analyzed the source location problem in directed graphs constrained with edge-connectivity or maximum flow-amounts. Their paper is a good overview of other models and results, as well, and it is the starting point of [1]. They proved a min-max theorem for the minimum cardinality of a subset R of nodes of an edge-capacitated digraph $D = (V, A)$ so that, for every node $v \in V - R$, the maximum flow-amount from R to v is at least k and from v to R is at least l. Based on this, they described an algorithm for computing such a minimum size set R. The running time of the algorithm depends polynomially on the size of D and exponentially on k and l. Throughout we will refer to this problem as the Flow-constrained Directed Source Location (FDSL) problem. For sake of simpler notation and explanation, we typically work with the uncapacitated case (when the capacity function is identically one). In this case the maximum flow-amount from s to t is equal to the maximum number of edge-disjoint paths from s to t. We will refer to this special case as the Directed Source Location Problem (or DSL for short).

Bárász, Becker and Frank in [1] describe an algorithm for solving the FDSL problem which has a running time polynomial in the size of D, k and l. The motivation of the present work was to improve the bound of the size of a hypergraph that has to be determined in this algorithm: this might help to reduce the time complexity of the algorithm.

1.1 Preliminaries

If V is a finite set and X, Y are subsets of V then we say that X and Y are **intersecting** if the sets $X - Y, Y - X$ and $X \cap Y$ are all nonempty. If furthermore $X \cup Y \neq V$ then X and Y are said to be **crossing**. We call a subset X-**avoiding** if it is disjoint from X. If $X = \{x\}$ is a singleton then we omit the braces and write x-**avoiding**. Sets X and Y are said to be **nesting** if $X \subseteq Y$ or $Y \subseteq X$.

Let $D = (V, A)$ be a digraph and $X \subseteq V$ a set of nodes. In D the in-degree $\varrho(X) = \varrho_D(X)$ denotes the number of edges entering X while the out-degree $\delta_D(X) = \delta(X)$ is the number of edges leaving X.

A digraph is called (k, l)-**edge-connected** with respect to a root node r if there are k edge-disjoint directed paths from r to every other node and there are l edge-disjoint directed paths from every node to r. It follows immediately from the directed edge-version of Menger's theorem that there are k edge-disjoint paths from r to v for every $v \in V$ (that is, D is $(k, 0)$-edge-connected) if and only if the in-degree of all nonempty subsets of $V - r$ is at least k, and an analogous characterization holds for $(0, l)$-edge-connectivity. Therefore (k, l)-edge-connectivity of a digraph is equivalent to requiring that the in-degree and out-degree of all nonempty subsets of $V - r$ is at least k and l, respectively. When $k = l$, this notion is equivalent to the k-edge-connectivity of D, while the case $l = 0$ corresponds to the rooted k-edge-connectivity of D.

A hypergraph $H = (V, \mathcal{E})$ is said to admit the **Helly property** or to be of **Helly-type** if any subset of pairwise intersecting hyperedges has a nonempty intersection. It will not cause any confusion to identify a hypergraph with its edge-set.

The Directed Source Location Problem is the following:

Problem 1.1. Given a directed graph $D = (V, A)$ and positive integers k and l. Find the minimum cardinality set $R \subseteq V$ such that contracting R into a single node r in the graph D results in graph which is (k, l)-edge-connected with respect to the root node r.

Bárász, Becker and Frank in [1] gave a strongly polynomial time algorithm that solves this problem. They introduce the following notions:

Definition: A nonempty set $X \subseteq V$ is called **in-solid** if $\varrho(Y) > \varrho(X)$ for every $Y \subsetneq X$ nonempty subset. A nonempty set $X \subseteq V$ is called **out-solid** if $\delta(Y) > \delta(X)$ for every $Y \subsetneq X$ nonempty subset. The set X is called **solid** if it is either in-solid or out-solid.

We mention that if one has to solve the $(k, 0)$-source location problem then it is enough to work with in-solid sets. Fortunately it turns out that an in-solid and an out-solid set can only be in a nice relation with each other, namely:

Lemma 1.2. *(Bárász, Becker, Frank [1]) If X is in-solid and Y is out-solid, then at least one of the subsets $A := X - Y$, $B := Y - X$, $C := X \cap Y$ is empty.*

Proof. Let $\alpha, \beta, \gamma, \gamma'$ denote, respectively, the number of edges from C to A, from B to C, from $V - (X \cup Y)$ to C, and from C to $V - (X \cup Y)$. If, indirectly, none of A, B, C is empty, then $\varrho(A) > \varrho(X)$ and $\delta(B) > \delta(Y)$. Therefore $\alpha > \beta + \gamma$ and $\beta > \alpha + \gamma'$ from which $0 > \gamma + \gamma'$ would follow, a contradiction. □

It is observed in [1] that for any node s of D the maximal s-avoiding in-solid sets are pairwise disjoint and so they form a partition of $V - s$ (since singletons are in-solid by the definition). We restate this proposition in Lemma 1.3 (the proof is different from the one in [1] and is based on Mihály Bárász's idea).

Lemma 1.3. *(Bárász, Becker, Frank [1]) If X and Y are solid and $X \cap Y \neq \emptyset$, then $X \cup Y$ is also solid.*

Proof. If one of X and Y is in-solid and the other is out-solid then by lemma 1.2 one of them is contained in the other and the result is immediate. So we can suppose that either both of them are in-solid or both are out-solid. We deal with the case when both are in-solid, the other case is analogous.

Let us suppose indirectly that there are two in-solid sets X and Y with a nonempty intersection such that their union is not in-solid (so X and Y have to be intersecting). Let us take two such sets with $\varrho(X) + \varrho(Y)$ minimum. Since the sets X and Y cannot be nesting we can notice that $\varrho(X) > \varrho(X \cup Y)$ and $\varrho(Y) > \varrho(X \cup Y)$ holds by the submodularity of ϱ. Since $X \cup Y$ is not in-solid, we can find an in-solid $Z \subsetneq X \cup Y$ with $\varrho(X \cup Y) \geq \varrho(Z)$. But $Z' = X \cup Z$ is in-solid, because X and Z are intersecting in-solid sets (since Z is not contained in Y) with $\varrho(X) + \varrho(Y) > \varrho(X) + \varrho(Z)$. Furthermore $\varrho(Z) \geq \varrho(Z')$ because $Z \subseteq Z' = X \cup Z$ and Z' is in-solid. Now Z' intersects Y and both of them are in-solid having $\varrho(X) + \varrho(Y) > \varrho(Y) + \varrho(Z')$ so $Y \cup Z' = X \cup Y$ is in-solid, a contradiction. □

By an s-**avoiding in-solid** (**out-solid**) set Z we mean an in-solid (out-solid) subset of $V - s$. The adjective maximal is used if Z is not included in any other s-avoiding in-solid (out-solid) subset of $V - s$. By Lemma 1.3 the maximal s-avoiding in-solid sets are disjoint. Since each singleton is in-solid, the maximal s-avoiding in-solid sets

partition $V - s$. This will be called the **in-solid partition** of $V - s$. The out-solid partition of $V - s$ is defined analogously. It follows from Lemmas 1.3 and 1.2 that:
Consequence: The family of maximal s-avoiding solid sets is a partition of $V - s$. $\square\square$

We call this partition the **solid partition** of $V - s$.

Another interesting property of the hypergraph of solid sets is that it admits the Helly property. This is also proved in [1]: we restate this theorem here.

Theorem 1.4. *(Bárász, Becker, Frank [1]) The hypergraph of solid sets admits the Helly property, that is whenever we have pairwise intersecting solid sets $X_1, X_2, \dots X_t$ then $\bigcap_{i=1}^{t} X_i \neq \emptyset$.* $\square\square$

Proof. Suppose indirectly that it does not admit the Helly property. Then there is a smallest number $h \geq 3$ along with h solid sets $X_1, \dots, X_h$ such that any two of these sets intersect each other while the intersection $M = X_1 \cap \dots \cap X_h$ is empty. By Lemma 1.2 either the sets $X_1, \dots, X_h$ are all in-solid or they are all out-solid. By symmetry we may assume that every X_i is in-solid. Let $Y_i = X_1 \cap X_2 \cap \dots \cap X_{i-1} \cap X_{i+1} \cap \dots \cap X_h$ $(i = 1, \dots, h)$. By the minimal choice of h, $Y_i \neq \emptyset$, while $M = \emptyset$ implies that $Y_i \cap Y_j = \emptyset$ $(1 \leq i < j \leq h)$. If an edge enters one of the sets Y_i, then it enters at least one of the sets X_j. Therefore $\sum_i \varrho(Y_i) \leq \sum_i \varrho(X_i)$. On the other hand $\varrho(Y_i) > \varrho(X_{i+1})$ for each i as X_{i+1} is in-solid and $Y_i \subset X_{i+1}$. Hence $\sum_i \varrho(Y_i) > \sum_i \varrho(X_{i+1}) = \sum_i \varrho(X_i)$, a contradiction. $\square$

Bárász, Becker and Frank in [1] solve the Directed Source Location Problem by determining the maximal s-avoiding solid sets for every $s \in V$. Since these sets form a partition of $V - s$, there can be at most $n-1$ of them for a certain s (where $n = |V|$), so the hypergraph

$$\mathcal{H} = \{X : X \text{ is a maximal } s\text{-avoiding solid set for some } s \in V\}$$

(i.e. the union of solid partitions) cannot have more than $n(n-1)$ hyperedges. Our aim in this paper is to prove, that this bound can be improved, namely $|\mathcal{H}| \leq 2(n-1)$. This is interesting because the bottleneck operation of the algorithm in [1] is the determination of $\mathcal{H}$.

2 The main result

Now we are ready to state and prove our theorem, in a slightly more general form:

Theorem 2.1. *Let V be an n-element set (where $n \geq 2$) and suppose we are given a hypergraph with edge set $\mathcal{F}$ which satisfies the following condition*
$(*)$ *If X and Y are crossing elements of $\mathcal{F}$ then $X \cup Y$ is also in $\mathcal{F}$.*
Then the hypergraph

$$\mathcal{H} = \{X \subseteq V : X \textit{ is a maximal } s\textit{-avoiding set in } \mathcal{F} \textit{ for some } s \in V\}$$

has cardinality at most $2(n-1)$.

Consequence: In a directed graph $D = (V, A)$ with $|V| = n$ the cardinality of $\mathcal{H} = \{X : X \text{ is a maximal } s\text{-avoiding solid set for some } s \in V\}$ is not more than $2n-2$. We mention that quite many examples show that this bound is sharp. Consider for example the following graph: let the node set be indexed with the numbers $1, 2, \dots n$ and the edge set be the following: for every $i \in \{1, 2, \dots n-1\}$ there are i parallel edges from node i to node $i+1$ and there are $n-i$ parallel edges from node $i+1$ to node i. In this graph the in-solid sets are exactly the sets with nodes indexed by consecutive integers (subpaths of the path $1, 2, \dots n$) and the out-solid sets are the singletons.

Remark: We note that the proof is a little bit simpler if we also suppose that $\mathcal{F}$ has the *Helly property* which is true for the hypergraph of solid sets. In the proof below we will show these simplifications.

Proof of theorem 2.1: We use the following notations:

$$\mathcal{H}_s = \{X \subseteq V : X \text{ is a maximal } s\text{-avoiding set in } \mathcal{F}\}$$

where $s \in V$ is arbitrary. With this notation $\mathcal{H} = \cup_{s \in V} \mathcal{H}_s$. Because of property $(*)$ of $\mathcal{F}$, the family $\mathcal{H}_s$ is a subpartition of $V - s$.

We prove the theorem by induction on n. For $n = 2$ the theorem is trivially true. So we can suppose that $n > 2$.

If every set in $\mathcal{H}$ is a singleton then we are done since $|\mathcal{H}| \leq n$. So suppose that $\mathcal{H}$ has nontrivial sets as well (i.e. sets of size at least 2). Let $X \in \mathcal{H}$ be a minimal nontrivial set of $\mathcal{H}$, that is every $Y \in \mathcal{H}$ with $Y \subsetneq X$ is a singleton. Suppose that $X \in \mathcal{H}_s$. We divide $\mathcal{H}$ into two disjoint parts in the following manner:

$$\mathcal{H}^1 = \{Y \in \mathcal{H} : X \cap Y = \emptyset \text{ or } X \subseteq Y\}$$

$$\mathcal{H}^2 = \{Y \in \mathcal{H} : X \cap Y \neq \emptyset \text{ and } X \not\subseteq Y\}$$

Let us first count the elements of $\mathcal{H}^2$: we would like to prove that the cardinality of $\mathcal{H}^2$ is at most $2|X| - 2$. If $v \notin X$ then $\mathcal{H}_v \subseteq \mathcal{H}^1$, since X is in $\mathcal{F}$. If $v \in X$ then a set in $\mathcal{H}_v$ is

- either disjoint from X (these sets belong to $\mathcal{H}^1$)
- or contained in X (these sets must be singletons)
- or intersecting X (but not contained in X): such set must contain s because of property $(*)$ of $\mathcal{F}$ and so there can be at most one such set in $\mathcal{H}_v$ (again because of property $(*)$ of $\mathcal{F}$): we denote this set by Z_v if it exists.

Let us denote by $X_1 = \{v \in X : \ Z_v \text{ does not exist}\}$. We have the following cases:

- If $|X_1| \geq 2$ then we are done, since $\mathcal{H}^2$ contains at most $|X|$ singletons and at most $|X| - 2$ different Z_v-s in this case.

- If $X_1 = \{x\}$ then we can observe that $\{x\} \notin \mathcal{H}^2$, because if there was a $v \in X$ for which $\{x\} \in \mathcal{H}_v$ then Z_v would be a nontrivial x-avoiding set of $\mathcal{F}$ contradicting with $x \in X_1$. Again we are done, since $\mathcal{H}^2$ contains at most $|X|-1$ singletons and at most $|X|-1$ different Z_v-s.
- If $X_1 = \emptyset$ (i.e. Z_v exists for every $v \in X$) then let us consider the family $\{Z_v : v \in X\}$. We claim that there are at least two minimal sets in this family. Suppose otherwise and denote the unique minimal set by Z_a. Then $Z_a \subseteq Z_v$ for all $v \in X$. But for an element $v \in Z_a \cap X$ this can not be true, a contradiction.

  Consider a minimal set Z_v of the family above. Then

  1. either $\{v\} \notin \mathcal{H}^2$
  2. or $\{v\} \in \mathcal{H}_w$ for some w, but then $w \in X - Z_v$ so $Z_w = Z_v$ (cannot be larger).

  In each cases we have that the number of different elements of $\mathcal{H}^2$ is at most $2|X|-2$ (using the fact that there are at least two minimal sets in the family $\{Z_v : v \in X\}$).

Remark: We note that if $\mathcal{F}$ has the Helly property, then the set X_1 is not empty, because if this was the case then $\emptyset = X \cap \bigcap_{v\in X} Z_v$ would give a contradiction with the Helly property of $\mathcal{F}$ and the fact that these sets are pairwise intersecting ($Z_v \cap X \neq \emptyset$ by the definition of Z_v and for $v_1, v_2 \in X$ both Z_{v_1} and Z_{v_2} contain s). Actually it is easy to see that $X_1 = \left(\bigcap_{v\in X-X_1} Z_v\right) \cap X$, but we do not need this here.

We want to count the cardinality of $\mathcal{H}^1$ by induction. For this sake we prove the following claim:

Claim:

$$\mathcal{H}^1 \subseteq \bigcup_{v\notin X} \mathcal{H}_v \cup \{\text{maximal } X\text{-avoiding sets in } \mathcal{F}\}$$

Proof of the claim: Take an element Z in $\mathcal{H}^1$. Then there is some $v \in V$ for which $Z \in \mathcal{H}_v$. If $v \notin X$ then we are done, so suppose $v \in X$ and hence Z is disjoint from X. Now Z is a maximal v-avoiding set in $\mathcal{F}$ so it is also a maximal X-avoiding set in $\mathcal{F}$ and hence an element of the right hand side. □

Remark: We note that equality is not necessarily true here: let $\mathcal{F}$ be the power set of the 3 element set ($\mathcal{F} = 2^{\{1,2,3\}}$) and X any nontrivial set of $\mathcal{H}$. This example also shows that X_1 can be empty (see above). But if $\mathcal{F}$ also has the Helly property, than we have equality. To show this, let us take an element Z of the right hand side. If $Z \in \bigcup_{v\notin X} \mathcal{H}_v$ then we know that $Z \in \mathcal{H}^1$ is also true, so suppose that Z is a maximal X-avoiding set of $\mathcal{F}$. But then $Z \in \mathcal{H}_v$ for an element $v \in X_1$ and so $Z \in \mathcal{H}^1$ is also true.

Now we are ready to finish our proof. Contract the set X into a single node and let $\mathcal{F}'$ be the set of the images of sets $Z \in \mathcal{F}$ that were either disjoint from X or containing X. We get a hypergraph on the $n - |X| + 1$ element set with edge set $\mathcal{F}'$ which has property $(*)$. So we can define $\mathcal{H}'$ with $\mathcal{F}'$ in a similar manner to that of $\mathcal{H}$ and notice that —by the above claim— $\mathcal{H}'$ contains the set of images of the elements of $\mathcal{H}^1$. So by induction we have

$$|\mathcal{H}^1| \leq |\mathcal{H}'| \leq 2(n - |X| + 1) - 2 = 2n - 2|X|.$$

This together with the bound for $|\mathcal{H}^2|$ gives the result. □

3 Concluding remarks

It turns out that theorem 2.1 is useful in other situations as well. For example Gabow in [4] gives a representation for a general intersecting (or crossing) set family, the so-called *tree-of-posets* representation which uses $O(n^2)$ space. Using theorem 2.1 we can give another representation for intersecting families (a crossing family can always be represented by representing two intersecting families) using the following easily proved fact:

Lemma 3.1. *Let $\mathcal{F}$ be an intersecting family over ground set V. Then*

$$\mathcal{H}^{\cap} \cup \{\emptyset, V\} = \mathcal{F} \cup \{\emptyset, V\}$$

where

$$\mathcal{H} = \{X \subseteq V : X \textit{ is a maximal } s\textit{-avoiding set in } \mathcal{F} \textit{ for some } s \in V\}$$

and

$$\mathcal{H}^{\cap} = \left\{ \bigcap_{j=1}^{t} X_j : \ t \in \mathbb{N}, \ X_1, X_2, \ldots X_t \textit{ are in } \mathcal{H} \right\} \square\square$$

So the representation is the following: we store the hypergraph $\mathcal{H}$ (which can obviously be done using at most $n(2n-2)$ bits, since $|\mathcal{H}| \leq 2n - 2$) plus in two more bits we tell whether any of V and $\emptyset$ are in $\mathcal{F}$ or not. Can't we hope for a better space bound representation for intersecting families? In the following section we show that we can not: even a subclass of intersecting families has $\theta(|V|^2)$ space complexity (this is also mentioned in [4], without a proof).

3.1 Representation of ring families

A ring family is a set family that is closed under intersection and union. It is easy to see that we can add $\emptyset$ or V to a ring family and it remains a ring family (though we can not always leave them out). For example if we are given a directed graph on node set V then the subsets $X \subseteq V$ having out degree 0 form a ring family, as can easily be checked (we call it *the ring family defined by D*). Fortunately, the converse is also true:

Lemma 3.2. *If $\mathcal{F}$ is a ring family over V containing $\emptyset$ and V then there exists a directed graph D for which*

$$X \in \mathcal{F} \iff \delta_D(X) = 0$$

Proof. Define the digraph as follows: for every $u \in V$ draw an edge (uv) going to every $v \in M(u)$, where $M(u)$ is the smallest set in $\mathcal{F}$ that contains u. Note that $M(u)$ exists as $V \in \mathcal{F}$ and it is unique. It is easy to see that this construction is good. □

Observe that if the transitive closure of graphs D_1 and D_2 is the same then they define the same ring family, that is for any $X \subseteq V$

$$\delta_{D_1}(X) = 0 \iff \delta_{D_2}(X) = 0.$$

The converse is also true, as stated in the next lemma (the proof is left to the reader).

Lemma 3.3. *Two digraphs on the same node set define the same ring family if and only if they have the same transitive closure.* □□

So we could use the digraph D to represent the ring family $\mathcal{F}$ (either in transitively closed or in any other form). D has size at most n^2, so we have developed a representation of size $O(n^2)$ for ring families. The following example shows that we can not hope for a better bound on a representation: we give $2^{n^2/4}$ different ring families over a ground set of size n (or $2^{(n-1)^2/4}$ if n is odd). So whatever representation one develops for ring families there will be a ring family that needs $n^2/4$ bits for its representation. So we prove the following lemma:

Lemma 3.4. *The space requirement for representing ring families over ground set V is $\theta(|V|^2)$.*

Proof. The example that proves the lower bound is the following: suppose n is even and divide the nodes of V into two sets X and Y, each of size $n/2$. Construct a digraph $D = (V, A)$ where $A = \{(xy) : x \in X,\ y \in Y\}$. Then D has $n^2/4$ edges and for every subset $A' \subseteq A$ the subgraph $D' = (V, A')$ is transitively closed, so by lemma 3.3 we have given $2^{n^2/4}$ different ring families over the n element ground set V. □

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References

[1] Mihály Bárász, Johanna Becker, András Frank: An algorithm for source location in directed graphs, Operation Research Letters, 33 (2005) pp. 221-230.

[2] Hiro Ito, Kazuhisa Makino, Kouji Arata, Shoji Honami, Yuichiro Itatsu, and Satoru Fujishige, Source location problem with flow requirements in directed networks, Optimization Methods and Software, Vol. 18, No. 4, August 2003, pp. 427-435.

[3] M. Labbe and F.V. Louveaux, Location problems, in: Annotated Bibliographies in Combinatorial Optimization, eds.: M. Dell' Amico, F. Maffioli, and S. Martello, Wiley, (1997) pp. 261-281.

[4] Harold N. Gabow: Centroids, Representations and Submodular Flows, Journal of Algorithms, 18 (1995) pp. 586-628.