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on Combinatorial Optimization

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On the efficiency of Egerváry's perfect matching algorithm

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Abstract

This paper shows that the perfect matching algorithm implicitly given in Egerváry's original work is not polynomial in itself. First we show an example with integer weights that requires exponential number of steps. Then, another example shows that — enabling real number weights — it is also possible that the algorithms fails to find an optimal solution in finite number of steps.

Keywords: bipartite matching, strong polynomiality

1 Introduction

Let us consider a bipartite graph $G = (A, B, E)$ and for the sake of simplicity assume that $|A| = |B|$. A set $M \subseteq E$ is called *matching* if it covers each node at most once. M is called *perfect matching* if M is a matching and $|M| = |A|$.

For a set $X \subseteq A$ let $\Gamma_G(X) \subseteq B$ denote the neighbors of X in G, i.e. $\Gamma_G(X) := \{b \in B : \exists a \in X, (a, b) \in E\}$

Theorem 1.1 (Hall). *G has a perfect matching if and only if*

$$|\Gamma_G(X)| \geq |X| \tag{1}$$

holds for each $X \subseteq A$.

A set hurting the Equation (1) is called *deficient set*, and the amount $|X| - |\Gamma_G(X)|$ is called the *deficit* of X.

Now, assume that G has a perfect matching and let us given a weight function $w : E \longrightarrow \mathbb{R}$ on the edges.

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A function $\pi : A \cup B \longrightarrow \mathbb{R}$ is called *feasible potential* if $\pi(a)+\pi(b) \geq w(a,b)$ holds for every $(a,b) \in E$. An edge $(a,b) \in E$ is called π*-tight* if $\pi(a)+\pi(b) = w(a,b)$. Let E_π denote the set of π-tight edges.

If is easy to see that for any perfect matching M and any feasible potential π

$$w(M) \leq \sum_{v \in A \cup B} \pi(v) \tag{2}$$

holds, therefore we get that

$$\max\{w(M) : \text{M is perfect matching}\} \leq \min\{\sum_{v \in A \cup B} \pi(v) : \pi \text{ is a feasible pot.}\}. \tag{3}$$

In his famous paper [1], Egerváry showed that actually an equality holds in (3), i.e.

Theorem 1.2 (Egerváry). *A perfect matching M is optimal if and only if there exists a feasible potential π such that $M \subseteq E_\pi$.*

The proof of this theorem suggests the following natural algorithm to find a maximum weight matching.

First we find an initial feasible potential π. Then, we search a perfect matching in E_π. Theorem 1.2 shows that if there exists an $M \subseteq E_\pi$ perfect matching, then it must be optimal, so we can stop. Otherwise we get a deficient set X. Let

$$\alpha := \min\{\pi(a)+\pi(b)-w(a,b) : a \in X, b \notin \Gamma(X)\} \tag{4}$$

and let

$$\pi'(v) := \begin{cases} \pi(v)-\alpha & \text{if } v \in X, \\ \pi(v)+\alpha & \text{if } v \in \Gamma(X), \\ \pi(v)-\alpha & \text{otherwise.} \end{cases} \tag{5}$$

It is straightforward to see that π' is again a feasible potential and its value (i.e. $\sum_{v \in A \cup B} \pi(v)$) is strictly less then that of π. Thus, we repeat the steps above with this new potential until a(n optimal) perfect matching is found. It is clear that for integer weights it yields a finite algorithm for the maximum weight perfect matching problem.

Unfortunately this alone does not give an efficient algorithm, not even in the case when we modify the potential using a set X having maximum deficit.

This paper shows two counterexamples for this. The first one shows that in case of integer weights exponentially many steps might be necessary. The second example shows that if real number weights are enabled, then the value of the value of the potential will not even converge to the optimum value.

2 Integer weights

Let us see the graph shown in Figure 1. The numbers at the edges and at the nodes are the weights and the initial potentials, respectively.

Let $X_1 := \{x,a,c\}$ be the first chosen deficient set, then let $X_2 := \{x,a,b\}$ be the deficient set in the next iteration. In the followings we alternate these two sets. Is is easy to check that they all are of maximum deficit, and $\alpha = 1$ at each iteration. Therefore it takes $2^n - 2$ iterations for the edge (a,d) to become tight.

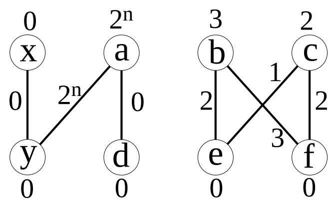

Figure 1: Counterexample with integer weights

3 Infinite running time

Let

$$f_n := \left(\frac{2}{1+\sqrt{5}}\right)^n \quad \forall n = 1, 2, \cdots . \tag{6}$$

It is easy to see that f_n has the following property.

Lemma 3.1. *$f_n - f_{n+1} = f_{n+2}$ holds for all $n = 0, 1, 2, \cdots$. Furthermore*

$$\sum_{n=1}^{\infty} f_n = \frac{\sqrt{5}+1}{2} < 2 \tag{7}$$

□

Now, let us consider the graph shown in Figure 2. The numbers at the edges and at the nodes are again the weights and the initial potentials, respectively.

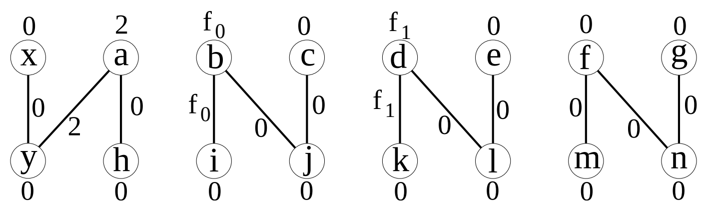

Figure 2: Counterexample featuring unconvergence

We choose the following deficient set at iteration $n = 1, 2, \cdots$.

$$X_n := \begin{cases} \{x, a, b, d, g\}, & \text{if } n = 3k+1 \\ \{x, a, b, e, f\}, & \text{if } n = 3k+2 \\ \{x, a, c, d, f\}, & \text{if } n = 3k \end{cases} \tag{8}$$

It is straightforward to check that in iteration i the set X_i is of maximum deficit and

$$\alpha_n = f_n. \tag{9}$$

Finally, Equation (7) shows that the edge (a, h) never becomes tight, therefore the algorithm will not stop in finite number of iterations and the series of the potentials fail to converge to an optimal solution.

4 A Sufficient Condition for Strong Polynomiality

The examples above show that improving the dual solution on arbitrary sets of maximum deficit does not necessarily give a polynomial (or even finite if real number weights are enabled) algorithm for the maximum weight perfect matching problem. On the other hand, the algorithm of Kuhn [2] finds deficient sets (with maximum deficit) that ensures the strong polynomiality. Based on this fact, a sufficient condition providing the strong polynomiality can also be given in a compact form as follows.

Claim 4.1. *If we modify the dual solution on the maximally deficient set having* maximum cardinality*, the algorithm terminates in strongly polynomial number of iterations.*

In fact, these are the sets found by Kuhn's method, so the above claim follows from its strong polynomiality. In addition, a direct proof not using alternating paths is also given in [3].

References

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