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On Kuhn's Hungarian Method – A tribute from Hungary

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Harold W. Kuhn, in his celebrated paper entitled *The Hungarian Method for the assignment problem,* [Naval Research Logistic Quarterly, 2 (1955), pp. 83-97] described an algorithm for constructing a maximum weight perfect matching in a bipartite graph. In his delightful reminescences [18], Kuhn explained how the works (from 1931) of two Hungarian mathematicians, D. Kőnig and E. Egerváry, had contributed to the invention of his algorithm, the reason why he named it the Hungarian Method. (For citations from Kuhn's account as well as for other invaluable historical notes on the subject, see A. Schrijver's monumental book [20].)

In this note I wish to pay tribute to Professor H.W. Kuhn by exhibiting the exact ralationship between his Hungarian Method and the achievements of Kőnig and Egerváry, and by outlining the fundamental influence of his algorithm on Combinatorial Optimization where it became the prototype of a great number of algorithms in areas such as network flows, matroids, and matching theory. And finally, as a Hungarian, I would also like to illustrate that not only did Kuhn make use of ideas of Hungarian mathematicians, but his extremely elegant method has had a great impact on the work of a next generation of Hungarian researchers.

1 Relationships

A little technicality: though both Egerváry and Kuhn used matrix terminology, here I follow Kőnig by working with the equivalent bipartite graph formulation.

Let us start with a quotation from Kuhn's paper [17]: 'One interesting aspect of the algorithm is the fact that it is latent in the work of D. König and E. Egerváry that predates the birth of linear programming by more than 15 years (hence the name of *Hungarian Method*)'. But what is the exact relationship of the algorithm arising from Egerváry's proof technique and Kuhn's method? In a paper [11], written in Hungarian, I exhibited in detail the achievements of Kőnig and Egerváry and Kuhn. The following section is an outline of some observations from [11].

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1.1 Kőnig's theorem and proof method

The starting point is Kőnig's matching theorem.

Theorem 1.1 (Kőnig). *In a bipartite graph $G = (S, T; E)$, the maximum cardinality $\nu = \nu(G)$ of a matching is equal to the minimum number $\tau = \tau(G)$ of nodes covering all the edges.*

It is useful to restate the theorem in an equivalent form.

Theorem 1.2. *In a bipartite graph $G = (S, T; E)$, the minimum number $\mu = \mu(G, S)$ of elements of S exposed by a matching is equal to the maximum of the deficit $h(X)$ over the subsets of S where $h(X) := |X| - |\Gamma(X)|$ and $\Gamma(X)$ denotes the set of elements of T having a neighbour in X. In particular, there is a matching covering S if and only if $|\Gamma(X)| \geq |X|$ holds for every subset $X \subseteq S$.*

The outline (in modern terms) of Kőnig's constructive proof for the non-trivial $\nu \geq \tau$ direction is as follows. By starting with any matching M, orient the edges in M from T to S and all other edges from S to T. Let R_S and R_T denote the set of nodes of S and of T, respectively, exposed by M. Let Z denote the set of nodes of the resulting directed graph which can be reached from R_S by a directed path. If $R_T \cap Z \neq \emptyset$, then we have a path P from R_S to R_T that alternates in M and then the symmetric difference of M and P is a matching M' with $|M'| = |M| + 1$. (Technically, one must simply reorient the edges of P in order to obtain the digraph corresponding to M'). If $R_T \cap Z = \emptyset$, then $L := (T \cap Z) \cup (S - Z)$ is a set of nodes covering all edges and $|M| = |L|$. In the alternative version of Kőnig's theorem above, $Z \cap S$ is a subset of S with maximum deficiency.

The following observation will be useful in estimating the efficiency of Egerváry's method. We call a subset $X \subseteq S$ **deficient** if $h(X) > 0$. Let $\mathcal{F} := \{X \subseteq S : h(X) = \mu(G, S)\}$, that is, $\mathcal{F}$ denotes the family of the subsets of S with maximum deficit. The members of $\mathcal{F}$ are called **max-deficient** sets. It can be shown that $\mathcal{F}$ is closed under union and intersection. Therefore, if $\mathcal{F}$ is not empty, there is a unique smallest max-deficient set, and in the constructive proof of Kőnig's theorem above, the max-deficient set $Z \cap S$, provided by the algorithm, itself is this unique smallest set.

1.2 Egerváry's theorem and proof method

In 1931 Egerváry [6] extended Kőnig's results to weighted bipartite matchings. His fundamental min-max result is as follows.

Theorem 1.3. *Let $G = (S, T; E)$ be a complete bipartite graph with $|S| = |T|$ and let $c : E \to \mathbf{Z}_+$ be a nonnegative integer-valued weight function. The maximum weight of a perfect matching of G is equal to the minimum weight of a nonnegative, integer-valued, weighted-covering of c where a* weighted-covering *is a function $\pi : S \cup T \to \mathbf{R}$ for which $\pi(u) + \pi(v) \geq c(uv)$ for every edge $uv \in E$ and the weight of π is defined to be $\sum[\pi(v) : v \in S \cup T]$.*

This theorem seems to be the first appearence of the linear programming duality theorem for the case when the constraint matrix is the incidence matrix of a bipartite graph. The outline of Egerváry's proof is as follows. Let π be a nonnegative integer-valued weighted-covering of c with minimum weight. If there is a perfect matching M in the subgraph G_π of tight edges, where an edge uv is called **tight** if $\pi(u) + \pi(v) = c(uv)$, then M is a maximum weight perfect matching of G whose weigth is equal to the weight of π.

If there is no perfect matching in G_π, then Kőnig's theorem implies that there is a deficient set $X \subseteq S$ in G_π. Increase the π-value of each node in $\Gamma_{G_\pi}(X)$ by 1 and decrease the π-value of each node in X by 1. This way one obtains another weighted-covering π' of c whose weight is smaller than that of π. In case π' has negative (that is, -1) values, increase the π'-values on the elements of S by 1 and decrease the π'-values on the elements of T by 1. Since G is complete bipartite and $c \geq 0$, the resulting π'' is a nonnegative weighted-cover of c whose weight is smaller than that of π, in a contradiction with the minimum choice of π.

Egerváry noted that his theorem easily extends to rational weights (in the sense that the integrality of the weighted-covering is not required anymore), and, by continuity arguments, the theorem holds for real weight functions as well.

Egerváry, in his paper, did not speak on algorithms at all. But his proof above can easily be turned into an algorithm since it finds, starting with an arbitrary weighted-covering π, either a better weighted-covering or else a maximum weight perfect matching. A natural observation is that the revision of the current potential, as described in the proof above, may be done by $\min\{\pi(u) + \pi(v) - c(uv) : u \in X, v \in T - \Gamma_{G_\pi}(X)\}$, a value possibly larger than 1. Perhaps it is not unfair to call the algorithm described this way **Egerváry's algorithm**. This is clearly finite for integer or rational c. A. Jüttner [15], however, observed that Egerváry's algorithm in this generic form is not polynomial for integer-valued c and not necessarily finite for real-valued c, even if max-deficient sets are used throughout the run of the algorithm for the revision of the current π. It should be noted, however, that by appropriately specifying the choice of the deficient sets used for revising the current π, the algorithm can be made strongly polynommial. Namely, this is the case if the unique smallest max-deficient set is used throughout. This was proved in [11] directly but I am almost sure that a proof had appeared earlier in the literature. As mentioned above, the deficient set found by Kőnig's algorithm is the unique smallest max-deficient set. Therefore, the specific version of Egerváry's algorithm, when the deficient set found by Kőnig's alternating path technique, rather than just taking an arbitrary deficient set, is strongly polynomial.

This situation is analogous to the well-known case of maximum flows: for integer or rational capacities the max-flow min-cut algorithm of L.R. Ford and D.R. Fulkerson is finite though not polynomial, while for real capacities it is not even finite. On the other hand, if a shortest augmenting path is used at every augmentation step, which is actually automatic when breadth-first-search is applied to find an augmenting path, then the algorithm is strongly polynomial, as was proved by J. Edmonds and R.M. Karp [5] and by E.A. Dinits [1]. We stress however that in the maximum weight matching problem as well as in the maximum flow problem the proof of strong polynomiality is not at all trivial and certainly needs some work.

1.3 Kuhn's Hungarian Method

In light of Egerváry's proof technique, let us see the novelty of Kuhn's Hungarian Method. Egerváry used Kőnig's theorem as a black box or subroutine, and therefore the algorithm read off from his proof is not polynomial. The striking advantage of Kuhn's algorithm is that it is strongly polynomial, moreover this immediately follows from the description of the algorithm. The main idea of Kuhn's algorithm is that the two separate parts in Egerváry's proof (computing a deficient set and revising the current π) are combined into one.

In a general step, Kuhn's algorithm also has a weighted-covering π and considers the subgraph G_π of tight edges (on node set $S \cup T$). Let M be a matching in G_π. Orient the elements of M from T to S while all other edges of G_π from S to T. Let $R_S \subseteq S$ and $R_T \subseteq T$ denote the set of nodes exposed by M in S and in T, respectively. Let Z denote the set of nodes reachable in the resulting digraph from R_S by a directed path (that can be computed by a breadth-first search, for example).

If $R_T \cap Z$ is non-empty, then we have obtained a path P consisting of tight edges that alternates in M. The symmetric difference of P and M is a matching M' of G_π consisting of one more edge than M does. The procedure is then iterated with this M'. If $R_T \cap Z$ is empty, then revise π as follows. Let $\Delta := \min\{\pi(u) + \pi(v) - c(uv) : u \in Z \cap S, v \in T - Z\}$. Decrease (increase, respectively) the π-value of the elements of $S \cap Z$ (of $T \cap Z$, resp.) by Δ. The resulting π' is also a weighted-covering. Construct the subgraph of $G_{\pi'}$ and iterate the procedure with π' and with the unchanged M.

The wonderful thing is that Kuhn's algorithm can be seen with no effort to be strongly polynomial. Indeed, observe first that there may be at most $|S|$ cases of matching augmentation. Second, in a phase when the current matching M is unchanged, the set of nodes reachable from R_S in G_π is properly included in the set of nodes reachable from R_S in $G_{\pi'}$. Hence, this situation may occur at most $|S|$ times, that is, after at most $|S|$ consecutive changes of the weighted-covering, a matching augmentation must follow. Since a breadth-first-search needs $O(|E|)$ steps, the overall complexity of Kuhn's Hungarian Method may be bounded by $O(|E||S|^2)$.

At that time, complexity consideration was not an issue beyond finiteness and therefore it is not surprising that Kuhn was content with proving the finiteness of his algorithm. We stress that the foregoing proof of strong polynomiality is basically automatic.

2 Influence

The main merit of Kuhn's Hungarian Method is that in the past half a century it has became the starting point of a fast developing area of efficient combinatorial algorithms, now called Combinatorial Optimization. Its seminal ideas, developed originally for the weighted bipartite matching problem (that is, the assignment problem) have been applied by L.R. Ford and D.R. Fulkerson to the transportation problem and, more generally, to minimum cost flows, as well (see, in [7]). In all of these cases, as A.J. Hoffman and J.B. Kruskal [14] discovered, the integrality of the optimal solutions is due to the total unimodularity of the underlying constraint matrix: the incidence matrix of a bipartite graph or a digraph. In 1965, J. Edmonds [2] was able to generalize the approach of the Hungarian Method to non-bipartite matchings, as well, a much more complex situation where the constraint matrix is not totally unimodular. Edmonds' weighted matroid intersection algorithm [3] was another fundamental breakthrough of similar vein where the spirit of the Hungarian Method was used and extended.

Harold Kuhn could use ideas of Hungarian mathematicians. A next generation of Hungarian researchers, in turn, highly profited from his method and achieved important results in Combinatorial Optimization. For example, É. Tardos [22] was the first to construct a strongly polynomial algorithm for the minimum cost circulation problem. A. Sebő [21] found fundamental structural results on edge-weighted undirected graphs with no negative cycles. L. Lovász's deep theory on matroid parity [19] was also affected by the Hungarian Method.

Finally, I would like to make some personal remarks. The Hungarian Method caught my heart and imagination very early. I have been teaching it in regular courses for decades, and I am still fascinated at every occasion by its clean elegance and beauty. The method has had a great impact on my research, too. For example, [9] describes a weighted matroid intersection algorithm that may be considered as a straight extension of the original algorithm of Kuhn because, instead of working with dual variables assigned to subsets of the ground-set, as earlier matroid intersection algorithms did, it uses only node-numbers, just as Kuhn's algorithm does. The same idea could be carried over in [10] to submodular flows, a wonderfully general and flexible framework, due to J. Edmonds and R. Giles [4]. The theoretical and practical efficiency and the wide range of applicability of the Hungarian Method are only one side of its far-reaching effect. Another one is that the method is an effective proof technique. For example, the version of Kuhn's Hungarian Method developed by Ford and Fulkerson (see, in [7]) for solving the min-cost flow problem could be used in [8] to prove a common generalization of a theorem of C. Greene [12] and a theorem of C. Greene and D. Kleitman [13] on maximum chain and antichain families of a partially ordered set. These theorems are deep generalizations of Dilworth's classical chain-covering theorem. Based on ideas of the Hungarian Method, one can compute a maximum cardinality subset of a poset that is the union of k chains (or k antichains).

In 2001, encouraged by these precedents, young and senior researchers in Budapest, the city of Kőnig and Egerváry, felt obliged to establish the Egerváry Research Group, supported by the Hungarian Academy of Sciences. (For its homepage, see 'http://www.cs.elte.hu/egres'). Our main goal has been to work on combinatorial algorithms and structures in the spirit of Kuhn's Hungarian Method and of the min-max theorems of Kőnig and Egerváry.

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