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A TDI Description of Restricted 2-Matching Polytopes

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Abstract

We give a TDI description for a class of polytopes which corresponds to a restricted 2-matching problem. The perfect matching polytope, triangle-free perfect 2-matching polytope and relaxations of the traveling salesman polytope are members of this class. For a class of restrictions G. Cornuéjols, D. Hartvigsen and W.R. Pulleyblank have shown that the unweighted problem is tractable; here we show that the weighted problems for these classes are also tractable.

1 Introduction

For combinatorial notions like matchings, 2-matchings or triangle-free 2-matchings polyhedral methods turned out to be effective in solving the weighted problems. TDI descriptions were given for these polytopes, in fact the structure of an optimum dual solution could be specified by blossoms for matchings, and triangle-clusters for triangle-free 2-matchings. We will give a description of a general class of polytopes which contains these examples, as well as other relaxations of the traveling salesman polytope.

It turns out that the class of polytopes examined in this paper has a slightly more complex description, in fact we need inequalities with large coefficients. This is a rare phenomenon, since most known TDI descriptions of combinatorial structures only have $0, \pm 1$ coefficients.

Let G be an undirected graph, with possible loops or parallel edges. Fix an arbitrary collection $\mathcal{H}$ of some (maybe none) odd length cycles in G. (A *cycle* is the edge set of a 2-regular, connected subgraph, the length of it is given by the number of edges. A loop is regarded as an odd cycle.) An $\mathcal{H}$-*matching* is the edge set of a subgraph the components of which are single edges and cycles in $\mathcal{H}$. An $\mathcal{H}$-matching is *perfect*

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if it covers all nodes. The $\mathbb{R}^E$-*incidence vector* x_M of an $\mathcal{H}$-matching M is given by $x_M(e) := 1$ if e is in a cycle component of M, $x_M(e) := 2$ if $\{e\}$ is a single edge component of M, all other entries of x_M are zero. In what follows, we will not make a difference between the $\mathcal{H}$-matching and its $\mathbb{R}^E$-incidence vector. The aim of the paper is to give a polyhedral description of the convex hull of perfect $\mathcal{H}$-matchings, denoted by $\mathcal{P}$.

There are some well-known classes of polytopes arising as $\mathcal{P}$ for a choice of $\mathcal{H}$. Let us first consider the choice $\mathcal{H} = \emptyset$. In this case, $\mathcal{P} = 2\mathcal{M}$, where $\mathcal{M}$ is the so-called perfect matching polytope. J. Edmonds gave a description of the perfect matching polytope [7], here we mention a stronger theorem due to W. Pulleyblank, J. Edmonds [10] and W.H. Cunningham, A.B. Marsh III [5]. A system of linear inequalities $Ax \le b$ is called *TDI* or *totally dual integral* if for any integer vector c, the dual program of maximizing cx subject to $Ax \le b$ has an integer optimum solution y, whenever the optimum value is finite (a definition due to J. Edmonds and R. Giles [8]).

Theorem 1.1 (Pulleyblank, Edmonds [10]; Cunningham, Marsh [5]). *For a graph $G = (V, E)$, the perfect matching polytope is determined by the TDI system*

$$x \ge 0 \tag{1}$$

$$x(d(v)) = 1 \qquad \text{if } v \in V \tag{2}$$

$$x(E[U]) \le (|U| - 1)/2 \qquad \text{if } U \subseteq V \text{ for which } G[U] \text{ is factor-critical.} \tag{3}$$

Moreover, for any integer vector c there is an integer optimum dual solution y_v, y_U such that $\{E[U] : y_U > 0\}$ is a nice family (defined in Sect. 3).

Consider choosing for $\mathcal{H}$ all odd cycles, except for the cycles of length three: in this case a perfect $\mathcal{H}$-matching is called a *triangle-free perfect 2-matching.* G. Cornuéjols and W. Pulleyblank gave the following description of the triangle-free perfect 2-matching polytope [2].

Theorem 1.2 (Cornuéjols, Pulleyblank [2]). *For an undirected graph $G = (V, E)$, the triangle-free perfect 2-matching polytope is determined by the TDI system*

$$x \ge 0 \tag{4}$$

$$\frac{1}{2}x(d(v)) = 1 \qquad \text{if } v \in V \tag{5}$$

$$x(E[a, b, c]) \le 2 \qquad \text{if } \{a, b, c\} \subseteq V(G) \tag{6}$$

For another example, consider the set $\mathcal{H}$ of all odd cycles of length at least 5, and possibly some additional cycles of length 3. In this case we only have a theorem describing $\mathcal{P}$ if G is simple:

Theorem 1.3 (Cornuéjols, Pulleyblank [2]). *If G is a simple graph, and $\mathcal{H}$ contains all odd cycles of length at least 5, then $\mathcal{P}$ is determined by the following TDI system*

$$x \ge 0 \tag{7}$$

$$\frac{1}{2}x(d(v)) = 1 \qquad \text{if } v \in V \tag{8}$$

$$x(ab) + x(bc) + x(ca) \le 2 \qquad \text{if } \{ab, bc, ca\} \subseteq E(G), \{ab, bc, ca\} \notin \mathcal{H} \tag{9}$$

In fact, it was an open problem whether there is a good description of this polyhedron if G is not simple. The paper gives a positive answer to a more general problem, but the description given here is slightly more complicated.

The last two systems do not have integer coefficients, but we get integer TDI systems if we replace (5), (8) by $x(d(v)) = 2$ and add $x(E[U]) \leq |U|$ for all sets $U \subseteq V$.

2 The Unweighted Theorem

For a basic introduction to matching theory, see [6]. In the sequel, the most important notion from matching theory is factor-criticality: An undirected graph is defined to be *factor-critical* if the deletion of any node leaves a graph with a perfect matching.

Let $\mathcal{H}'$ be a set of some odd cycles in $G = (V, E)$ and some nodes in V. If a node is in $\mathcal{H}'$ let us call it a *pseudo-node*. A $\mathcal{H}'$-matching is a node-disjoint collection of edges, pseudo-nodes and cycles in $\mathcal{H}'$, which is perfect if it covers V. The size of a $\mathcal{H}'$-matching is the number of covered nodes. Let $\nu^{\mathcal{H}'}(G)$ denote the maximum size of a $\mathcal{H}'$-matching. We define the graph G to be $\mathcal{H}'$-*critical* if it is factor-critical and there is no perfect $\mathcal{H}'$-matching in G. An induced subgraph $G[V']$ is $\mathcal{H}'$-*critical* if it is factor-critical and there is no perfect $\mathcal{H}'$-matching in $G[V']$.

We will make use of the following theorem, which is a special case of a theorem in [1] by G. Cornuéjols and D. Hartvigsen.

Theorem 2.1 (Cornuéjols, Hartvigsen, [1]). *Let $G = (V, E)$ and $\mathcal{H}'$ be as above. There is a perfect $\mathcal{H}'$-matching in G if and only if there is no set $X \subseteq V$ such that $c_G^{\mathcal{H}'}(X) > |X|$, where $c_G^{\mathcal{H}'}(X)$ denotes the number of $\mathcal{H}'$-critical components in $G - X$.*

A stronger version is:

Theorem 2.2 (Cornuéjols, Hartvigsen, [1]). *Let $G = (V, E)$ and $\mathcal{H}'$ be as above. Let $D = D_{\mathcal{H}'}(G)$ be the set of nodes $v \in V$ for which there is an $\mathcal{H}'$-matching of size $\nu^{\mathcal{H}'}(G)$ which exposes v. Let $A = A_{\mathcal{H}'}(G) = \Gamma_G(D)$, then $\nu^{\mathcal{H}'}(G) = \min_{X \subseteq V} |V| + |X| - c_G^{\mathcal{H}'}(X)$ and this minimum is attained by set $X = A$. Furthermore,*

1. *The components of $G[D]$ are $\mathcal{H}'$-critical and the other components of $G - A$ have a perfect $\mathcal{H}'$-matching, and*

2. *If a set $X \subseteq V$ is minimizing the minimum above,and Z is the union of $\mathcal{H}'$-critical components in $G - X$, then $D \subseteq Z$.*

3. *There is a maximum $\mathcal{H}'$-matching which covers each node a in A by an edge ad with $d \in D$.*

G. Cornuéjols and W.R. Pulleyblank [3] considered the case when $\mathcal{H}'$ is the set of odd cycles of length at least k, as a relaxation of the Hamiltonian cycle problem. They showed that finding a perfect $\mathcal{H}'$-matching is polynomially solvable for any k. Complexity issues depend on recognizing $\mathcal{H}'$-critical graphs. If a factor-critical graph has a perfect $\mathcal{H}'$-matching then there is a perfect $\mathcal{H}'$-matching using exactly one member of $\mathcal{H}'$. The state of the art is that a polynomial-time recognition algorithm could be given in the following cases:

1. There is a number k such that $\mathcal{H}'$ contains all odd cycles longer than k – and maybe some short cycles, too.

2. The members of $\mathcal{H}'$ can be listed in polynomial time. For example each has length less than a fixed number k.

The conclusion of this paper is that in both cases the weighted problem is also tractable. This settles an unpublished conjecture of W. Pulleyblank and M. Loebl [9]. They conjectured that the maximum weight $\mathcal{H}$-matching problem is polynomially solvable if $\mathcal{H}$ consists of triangles.

3 The Polyhedral Description

A family of sets is called *laminar* if any two of the sets are disjoint or one of them is a subset of the other. A laminar family of edge sets $\mathcal{L}$ is called a *nice family* if for all sets $F \in \mathcal{L}$ the subgraph $G(F) = (V(F), F)$ is factor-critical, the family $\{V(F) : F \in \mathcal{L}\}$ is laminar, and in addition it has one of the following two equivalent properties:

1. For any node $v \in V$ there is a v-exposed matching M such that for each $F \in \mathcal{F}$ we have $|M \cap F| = (|V(F)| - 1)/2$.

2. For an arbitrary member $F \in \mathcal{F}$, let $F_1, F_2, \ldots, F_k$ denote the maximal members of $\mathcal{F}$ which are proper subsets of F. Then the graph we get from $G(F)$ by contracting the edges in $\cup F_i$ is factor-critical.

By this definition, we may have a pair of subsets in $\mathcal{F}$ with $F_1 \subsetneq F_2$ and $V(F_1) = V(F_2)$. For a set $F \in \mathcal{L}$, let $\mathcal{L}_F$ denote the truncation of $\mathcal{L}$ to subsets of F, including F itself. The truncations to the maximal sets in $\mathcal{L}$ are called *components*.

A pair $\mathcal{F} = (\mathcal{L}, m)$ is called a *nice system* if $\mathcal{L}$ is a nice family, and $m : \mathcal{L} \to \mathbb{R}$ (with $m(F) \geq 0$ for any set $F \in \mathcal{L}$) is a non-negative "multiplicity function". Let $||\mathcal{F}|| := \sum_{\mathcal{L}} m(F)(|V(F)| - 1)$ and for a vector $x \in \mathbb{R}^E$ let $x(\mathcal{F}) := \sum_{\mathcal{L}} m(F)x(F)$. Let $\chi_{\mathcal{F}}(e) := \sum_{\mathcal{L}} m(F)|\{e\} \cap F|$ be the $\mathbb{R}^E$-characteristic vector of $\mathcal{F}$. Then $x(\mathcal{F}) = x \cdot \chi_{\mathcal{F}}$ follows. For a set $F \in \mathcal{L}$ let us denote by $\mathcal{F}_F$ the truncation $(\mathcal{L}_F, m|_{\mathcal{L}_F})$, the components of $\mathcal{F}$ are defined appropriately.

Definition 3.1. Let $G(\mathcal{F}) = (V(\mathcal{F}), E(\mathcal{F}))$ where $E(\mathcal{F}) := \cup\{F : F \in \mathcal{F}\}$ and $V(\mathcal{F}) := \cup\{V(F) : F \in \mathcal{F}\}$.

Lemma 3.2. *If $\mathcal{F}$ has one component, then for any node $v \in V(\mathcal{F})$ there is a 2-matching x_v with no cycle, $x_v(d(v)) = 0$, $x_v(d(u)) = 2$ for $u \in V(\mathcal{F}) - v$ (i.e. x_v is twice a matching in $G(\mathcal{F}) - v$), and $x_v(\mathcal{F}) = ||\mathcal{F}||$.*

Proof. The straightforward proof is left to the reader. □

Definition 3.3. Let $\mathcal{F}$ be a nice system. The inequality $x(\mathcal{F}) \leq ||\mathcal{F}||$ is called *valid* if it holds for any $\mathcal{H}$-matching x; in this case the system $\mathcal{F}$ will also be called valid.

It is easy to see that $\mathcal{F}$ is valid if and only if its components are valid. Keep in mind that in the definition of validity we considered $\mathcal{H}$-matchings, while the polytope $\mathcal{P}$ is the convex hull of perfect $\mathcal{H}$-matchings. However, it is easy to see the following.

Proposition 3.4. *If a nice system $\mathcal{F}$ with one component is not valid, then there is a perfect $\mathcal{H}$-matching x in $G(\mathcal{F}) = (V(\mathcal{F}), E(\mathcal{F}))$ for which $x(\mathcal{F}) > ||\mathcal{F}||$.*

The main theorem of the paper is the following polyhedral description.

Theorem 3.5. *Let $G = (V, E)$ and $\mathcal{H}$ be as above. Then $\mathcal{P}$ is determined by*

$$x \geq 0 \tag{10}$$

$$\frac{1}{2}x(d(v)) = 1 \qquad \text{if } v \in V \tag{11}$$

$$x(\mathcal{F}) \leq ||\mathcal{F}|| \qquad \text{if } \mathcal{F} \text{ is a valid system.} \tag{12}$$

There is an infinite number of inequalities for valid systems $\mathcal{F}$. To get a traditional polyhedral description, consider a fixed nice family $\mathcal{L}$. We define $C_{\mathcal{L}} = \{m : \mathcal{L} \to \mathbb{R}^+,$ $(\mathcal{L}, m)$ is valid$\}$, then

$$C_{\mathcal{L}} = \{m \geq 0\,, \tag{13}$$

$$\sum_{F \in \mathcal{L}} m(F)\,(|V(F)| - 1 - x(F)) \geq 0 \quad \text{for } \mathcal{H}\text{-matchings } x \text{ of } G\} \tag{14}$$

thus $C_{\mathcal{L}}$ is a polyhedral cone, let $\mathcal{G}_{\mathcal{L}}$ denote a finite generator. The set of solutions of (10)-(12) is the same as of

$$x \geq 0 \tag{15}$$

$$\frac{1}{2}x(d(v)) = 1 \qquad \text{if } v \in V \tag{16}$$

$$x(\mathcal{F}) \leq ||\mathcal{F}|| \qquad \text{if } \mathcal{F} = (\mathcal{L}, m) \text{ for some } m \in \mathcal{G}_{\mathcal{L}} \tag{17}$$

Notice that we have two complications in comparison with the Theorems 1.1 and 1.2. The general class of polytopes $\mathcal{P}$ can not be described by inequalities with 0, ± 1 coefficients, and we also need edge sets of subgraphs which are not induced by a node set. However, we will show in Sect. 7 that there is a TDI description with integer coefficients:

Theorem 3.6. *The following is a TDI description of $\mathcal{P}$ which has integer coefficients:*

$$x \geq 0 \tag{18}$$

$$x(d(v)) = 2 \qquad \text{if } v \in V \tag{19}$$

$$x(E[U]) \leq |U| \qquad \text{if } U \subseteq V \tag{20}$$

$$x(\mathcal{F}) \leq ||\mathcal{F}|| \qquad \text{if } \mathcal{F} \text{ is a valid system with integer multiplicity.} \tag{21}$$

We get a finite TDI description with integer coefficients if we only put the valid inequalities in (21) for $\mathcal{F}$ corresponding to a Hilbert base of some $C_{\mathcal{L}}$.

4 Decompositions of Valid Inequalities

Definition 4.1. A valid system $\mathcal{F}$ is of type (α) if it has one component, m is equal to 1 on the maximal set and zero elsewhere.

Definition 4.2. A valid system $\mathcal{F}$ is of type (β) if it has one component, and there is a perfect $\mathcal{H}$-matching x^* in $G(\mathcal{F})$ such that $x^*(\mathcal{F}) = ||\mathcal{F}||$.

To proceed with the proof we will need the following technical lemmas, Lemma 4.6 will be the most important in the next section.

Lemma 4.3. *If $\mathcal{F}$ is a valid system with one component, $m(F) > 0$ for all sets $F \in \mathcal{L}$, and if $F_1 \in \mathcal{L}$ is not the maximal set, then the truncation $\mathcal{F}_{F_1}$ is not of type (β).*

Proof. Suppose for contradiction, that x is a perfect $\mathcal{H}$-matching in $(V(\mathcal{F}_{F_1}), E(\mathcal{F}_{F_1}))$ and $x(\mathcal{F}_{F_1}) = ||\mathcal{F}_{F_1}||$. Then we construct x' by adding edges with weight 2 so that for all sets $F \in \mathcal{L}$, $F \not\subseteq F_1$ we have $x'(F) = |V(F)| - 1$ or $x'(F) = |V(F)|$, and for sets with $F \supseteq F_1$ only the second alternative holds. This x' can be constructed using property 1 for a $v \in V(F_1)$. Then the following calculation gives a contradiction with $\mathcal{F}$ being valid:

$$x'(\mathcal{F}) = x(\mathcal{F}_{F_1}) + \sum_{\mathcal{L}-\mathcal{L}_{F_1}} m(F)x'(F) > ||\mathcal{F}_{F_1}|| + \sum_{\mathcal{L}-\mathcal{L}_{F_1}} m(F)(|V(F)| - 1) = ||\mathcal{F}||$$

(The inequality holds since m is positive on the maximal set, which is in $\mathcal{L}-\mathcal{L}_{F_1}$.) □

Lemma 4.4. *Suppose $\mathcal{F}$ is a valid system with one component and with $m(F) > 0$ for each set $F \in \mathcal{L}$. Then for all sets $F \in \mathcal{L}$ which is not the maximal set in $\mathcal{L}$,*

a) the system $\mathcal{F}_F$ is valid, and

b) for any $\mathcal{H}$-matching x, the equality $x(\mathcal{F}_F) = ||\mathcal{F}_F||$ implies that $x(F') = |V(F')| - 1$ for all sets $F' \in \mathcal{L}_F$.

Proof. We prove the lemma by induction "from inside to outside". Suppose we have a set F_1, such that a) and b) holds for all sets F in $\mathcal{L}_{F_1} - F_1$.

Suppose F_1 is not the maximal set, and x is an $\mathcal{H}$-matching such that $x(\mathcal{F}_{F_1}) \geq ||\mathcal{F}_{F_1}||$. Then $x(F_1) = |V(F_1)|$ leads to a contradiction, since an $\mathcal{H}$-matching x' could be constructed as in Lemma 4.3 for which $x'(\mathcal{F}) > ||\mathcal{F}||$. Thus $x(F_1) \leq |V(F_1)| - 1$ and then we get statement a) for F_1 by

$$x(\mathcal{F}_{F_1}) = x(F_1)m(F_1) + \sum x(\mathcal{F}_{D_i}) \leq (|V(F_1)| - 1)m(F_1) + \sum ||\mathcal{F}_{D_i}|| = ||\mathcal{F}_{F_1}||$$

where D_i are the maximal sets in $\mathcal{L}_{F_1} - F_1$.

If for x we have $x(\mathcal{F}_{F_1}) = ||\mathcal{F}_{F_1}||$, then all equalities $x(\mathcal{F}_{D_i}) = ||\mathcal{F}_{D_i}||$ must hold. Then by induction we get that for each set $F' \in \mathcal{L}_{F_1} - F_1$ the equality $x(F') = |V(F')| - 1$ holds. This also implies $x(F_1) = |V(F_1)| - 1$ and then b) holds for F_1. □

Notice, that a) also holds for the maximal set, but b) does not hold necessarily.

Lemma 4.5. *Suppose $\mathcal{F}$ is a valid system with one component, with $m(F) > 0$ for each set $F \in \mathcal{L}$, and $\mathcal{F}$ is not of type (β). Then for each set $F \in \mathcal{L}$ there is a multiplicity m on $\mathcal{L}_F$ so that for the systems $\widehat{\mathcal{F}}_F := (\mathcal{L}_F, m_F)$*

1. *$\widehat{\mathcal{F}}_F \sim (\alpha)$, or*

2. *$\widehat{\mathcal{F}}_F \sim (\beta)$ and $m_F(F') > 0$ for all sets $F' \in \mathcal{L}_F$,*

and there are coefficients $\lambda_F > 0$ for which

$$\chi_{\mathcal{F}} = \sum_{F \in \mathcal{L}} \lambda_F \cdot \chi_{\widehat{\mathcal{F}}_F}. \tag{22}$$

Proof. The proof goes by induction on $|\mathcal{L}|$. For $|\mathcal{L}| = 1$ the statement holds, since $\mathcal{F}$ is a positive multiple of a system of type (α).

Let F_0 be the maximal set in $\mathcal{L}$. By Lemma 4.4, all truncations of $\mathcal{F}$ are valid, and by Lemma 4.3 no proper truncation of $\mathcal{F}$ is of type (β). Thus by induction one can give for each set $F \in \mathcal{L} - F_0$ a system $\widehat{\mathcal{F}}_F = (\mathcal{L}_F, m_F)$ as in 1. or 2. and coefficients $\lambda_F > 0$ for which

$$\chi_{\mathcal{F}} - m(F_0) \cdot \chi_{F_0} = \sum_{F \in \mathcal{L} - F_0} \lambda_F \cdot \chi_{\widehat{\mathcal{F}}_F} \tag{23}$$

holds. There are two cases. First, if there is no perfect $\mathcal{H}$-matching in $G(\mathcal{F})$ then $m_{F_0}(F_0) := 1$ and $m_{F_0}(F) := 0$ (for $F \neq F_0$) and $\lambda_{F_0} := m_{\mathcal{F}}(F_0) > 0$ give equality in (22).

Second, if there is at least one perfect $\mathcal{H}$-matching in $G(\mathcal{F})$. Since $\mathcal{F}$ is not of type (β), for each perfect $\mathcal{H}$-matching x in $(V(F_0), F_0)$ we have $x(\mathcal{F}) < ||\mathcal{F}||$. For a number $t > 0$ we let $\mathcal{F}^t := (\mathcal{L}, m^t)$ where $m^t(F) = m(F)$ if $F \neq F_0$, and $m^t(F_0) = t$. Let $t_0 := m(F_0)$, then $\mathcal{F}^{t_0} = \mathcal{F}$. There is a uniquely defined number T for which $\mathcal{F}^T$ is of type (β). Then $T > t_0$ holds, let $\widehat{\mathcal{F}}_{F_0} := \mathcal{F}^T$ and $\lambda'_{F_0} := t_0/T$, $\lambda'_F := (1 - t_0/T)\lambda_F$. This gives the desired decomposition 22. □

Lemma 4.6. *Suppose $\mathcal{F}$ is a valid system with one component, with $m(F) > 0$ for each set $F \in \mathcal{L}$, and $\mathcal{F}$ is not of type (β). If x is an $\mathcal{H}$-matching for which $x(\mathcal{F}) = ||\mathcal{F}||$, then $x(F') = |V(F')| - 1$ for all sets $F' \in \mathcal{L}$.*

Proof. Take the decomposition $\chi_{\mathcal{F}} = \sum_{F \in \mathcal{L}} \lambda_F \cdot \chi_{\widehat{\mathcal{F}}_F}$ in (22), then $||\mathcal{F}|| = \sum_{F \in \mathcal{L}} \lambda_F \cdot ||\widehat{\mathcal{F}}_F||$ holds, too. Since $\lambda_F > 0$, this implies $x(\widehat{\mathcal{F}}_F) = ||\widehat{\mathcal{F}}_F||$.

Suppose for $F' \in \mathcal{L}$ we have $x(F'') = |V(F'')| - 1$ for all sets $F'' \in \mathcal{L}$, $F'' \subsetneq F'$. Then $x(F') = |V(F')| - 1$ follows from $x(\widehat{\mathcal{F}}_F) = ||\widehat{\mathcal{F}}_F||$. □

Lemma 4.5 implies that $\mathcal{G}_{\mathcal{L}}$ can be chosen so that each member is a nice system of type (α) or (β). The valid systems of type (α) are determined by an edge set. The valid systems $\mathcal{F}$ of type (β) can be described as follows: Consider a fixed nice system

$\mathcal{L}$, and a perfect $\mathcal{H}$-matching x^* in $(V(\mathcal{L}), \mathcal{L})$. We define $C_{\mathcal{L},x^*} = \{m : \mathcal{L} \to \mathbb{R}^+$, $(\mathcal{L}, m)$ is valid, and $x^*(\mathcal{L}, m) = ||(\mathcal{L}, m)||\}$. Then

$$C_{\mathcal{L},x^*} = \{m \geq 0\,, \tag{24}$$

$$\sum_{F\in\mathcal{L}} m(F)\,(|V(F)| - 1 - x^*(F)) = 0\,, \tag{25}$$

$$\sum_{F\in\mathcal{L}} m(F)\,(|V(F)| - 1 - x(F)) \geq 0 \quad \text{for } \mathcal{H}\text{-matchings } x \text{ of } G\} \tag{26}$$

thus $C_{\mathcal{L},x^*}$ is a polyhedral cone. Let $\mathcal{G}(\alpha)$ and $\mathcal{G}(\beta)$ be the set of valid systems we get from the finite generators of these cones. Then Theorem 3.5 implies that $\mathcal{P}$ is determined by

$$x \geq 0 \tag{27}$$

$$\frac{1}{2}x(d(v)) = 1 \qquad \text{if } v \in V \tag{28}$$

$$x(\mathcal{F}) \leq ||\mathcal{F}|| \qquad \text{if } \mathcal{F} \in \mathcal{G}(\alpha) \cup \mathcal{G}(\beta). \tag{29}$$

5 Proof by a Primal-Dual Method

Instead of taking the dual program of (10)-(12), we consider a bunch of linear programs each of which corresponds to a nice family. There is a large, but finite number of nice families in any graph G. For a nice family $\mathcal{L}$, consider the following linear program in $\mathbb{R}^{V\cup\mathcal{L}}$:

$$\min \sum_{v\in V} y_v + \sum_{F\in\mathcal{L}} m_F \cdot (|V(F)| - 1) \tag{30}$$

$$\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv\in F\in\mathcal{L}} m(F) \geq c_{uv} \qquad \text{for } uv \in E - E(\mathcal{L}) \tag{31}$$

$$\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv\in F\in\mathcal{L}} m(F) = c_{uv} \qquad \text{for } uv \in E(\mathcal{L}) \tag{32}$$

$$m_F \geq 0 \qquad \text{for } F \in \mathcal{L} \tag{33}$$

$$\sum_{F\in\mathcal{L}} m(F)\,(|V(F)| - 1 - x(F)) \geq 0 \qquad \text{for } \mathcal{H}\text{-matchings } x \text{ of } G \tag{34}$$

Notice that the part (33)-(34) is equivalent to $(\mathcal{L}, m)$ being valid, and the objective in (30) is equal to $\sum_{v\in V} y_v + ||(\mathcal{L}, m)||$. Thus, for some solution (y, m) of (31)-(34) one can easily construct a dual solution of (10)-(12) of the same objective value. We abbreviate the objective in (30) by $(y, m) \cdot b$.

If there is at least one perfect $\mathcal{H}$-matching x in G, then $c \cdot x$ is a lower bound on $(y, m) \cdot b$ for a solution of some system (31)-(34). Choose $\mathcal{L}$ for which the minimum $(y, m) \cdot b$ in (30) is minimal, let (y, m) be a minimizing vector. The minimum is also attained by a pair $\mathcal{L}, (y, m)$ for which $m(F) > 0$ holds for all sets $F \in \mathcal{L}$ (the other sets can be eliminated from the laminar family). To prove the existence of a perfect $\mathcal{H}$-matching x for which $c \cdot x = (y, m) \cdot b$, we need these complementary slackness conditions:

$$x_{uv} > 0 \text{ implies } \frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F) = c_{uv} \tag{35}$$

$$x(\mathcal{L}, m) = ||(\mathcal{L}, m)|| \tag{36}$$

Let $E^= = E^=(\mathcal{L}, y, m)$ denote the set of *tight edges*, the edges uv for which equality holds in (31) or (32). Let $G^= = G^=(\mathcal{L}, y, m) = (V, E^=)$, and let $\overline{G} = \overline{G}(\mathcal{L}, y, m) = (\overline{V}, \overline{E})$ be the graph we get from $G^=$ by contracting all edges in $E[\mathcal{L}]$. Notice that this way we could get a lot of loops or parallel edges, and it is of great importance to keep them. For a set $\overline{U} \subseteq \overline{V}$ let U denote set of the corresponding nodes in V.

Let $\mathcal{L}_{(\beta)}$ denote the union of those components of $\mathcal{L}$ which is of type (β). We call a node in $\overline{V}$ a pseudo-node if it corresponds to a component of $\mathcal{L}_{(\beta)}$.

$$\mathcal{M} = \mathcal{M}(\mathcal{L}, y, m) := \{x : x \text{ is an } \mathcal{H}\text{-matching with } x(F) = |V(F)| - 1 \text{ for all sets } F \in \mathcal{L}\} \tag{37}$$

$$\mathcal{H}^\star = \mathcal{H}^\star(\mathcal{L}, y, m) := \{\text{odd cycles, which appear in some } x \in \mathcal{M}\}. \tag{38}$$

Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathcal{L}, y, m)$ denote the set of the pseudo-nodes, plus the odd cycles in $\overline{G}$ we get from $\mathcal{H}^\star$ by contracting the edges in $E(\mathcal{L})$. It follows from the definition of $\mathcal{M}$, that the contraction produces only cycles. For an $\mathcal{H}^\star$-matching x in $\mathcal{M}$ let $x/\mathcal{L}$ denote the resulting $\overline{\mathcal{H}}$-matching in $\overline{G}$ after contracting the edges in $E(\mathcal{L})$. It is easy to check, that the cycles in $\overline{\mathcal{H}}$ can be described in another way:

Claim 5.1. *Consider an odd cycle C in $\overline{G}$, let B be the set of nodes in V contracted to a node in $V(C)$, let $\mathcal{F}_B$ denote the truncation to the sets in $E[B]$. For a node $z \in V(C)$ let $\mathcal{F}_z$ denote the component of $\mathcal{F}$ contracted to z. Cycle C is in $\overline{\mathcal{H}}$ if and only if there is a perfect $\mathcal{H}$-factor x_C in $G[B]$ which has $x_C/\mathcal{L}_B = \chi(C)$ and $x_C(\mathcal{F}_B) = ||\mathcal{F}_B||$ (or equivalently, it has $x_C(\mathcal{F}_z) = ||\mathcal{F}_z||$ for each node $z \in V(C)$.)*

Claim 5.2. *If there is a perfect $\overline{\mathcal{H}}$-matching in $\overline{G}$, then there is a perfect $\mathcal{H}$-matching x in G for which $(y, m) \cdot b = c \cdot x$.*

Proof. In a perfect $\overline{\mathcal{H}}$-matching, each node $\overline{v} \in \overline{V}$ is covered by a pseudo-node, an odd cycle from $\overline{\mathcal{H}}$, or an edge.

If a node $\overline{v}$ is covered by an edge $\overline{e} \in \overline{E}$, then there is an edge $uv \in E^=$ which is mapped onto $\overline{e}$ by the contraction. Let $\mathcal{F}_{\overline{v}}$ denote the component of $\mathcal{F}$ contracted to $\overline{v}$, then $v \in V(\mathcal{F}_{\overline{v}})$. We define x on the edges in $E(\mathcal{F}_{\overline{v}})$ by the vector x_v for $\mathcal{F}_{\overline{v}}$ from Lemma 3.2.

If a node $\overline{v}$ is covered by a pseudo-node, then we define x on the edges in $E(\mathcal{F}_{\overline{v}})$ by the vector x^* from the definition of (β).

If a node $\overline{v}$ is covered by an odd cycle $C \in \overline{\mathcal{H}}$, let B denote the set of nodes in V contracted to a node in $V(C)$. We define x on the edges in $E(\mathcal{F}_{\overline{v}})$ by the vector x_C from Claim 5.1.

The proof is completed by checking the complementary slackness conditions (35)-(36). Here (35) is due to using only the edges in $E^=$, (36) we get from the definitions of x^* and x_C. □

If there is no perfect $\overline{\mathcal{H}}$-matching in $\overline{G}$, then by Theorem 2.1 there is a set $\overline{X} \subseteq \overline{V}$ for which the number of $\overline{\mathcal{H}}$-critical components in $\overline{G} - \overline{X}$ is strictly greater than $|\overline{X}|$.

Let $\mathcal{F} := \mathcal{F}(\mathcal{L}, y, m)$ be the system with nice family $\mathcal{L}$ and multiplicity m. For $\overline{K} \subseteq \overline{V}$ we define $\mathcal{F}_{E[K]}$ to be the truncation of $\mathcal{F}$ to the subsets of $E[K]$.

Lemma 5.3. *If $\overline{G}[\overline{K}]$ is an $\overline{\mathcal{H}}$-critical component in $\overline{G} - \overline{X}$ for some $\overline{K} \subseteq \overline{V}$, then either*

a) $G^=[K]$ is $\mathcal{H}$-critical, or

b) for each perfect $\mathcal{H}$-matching x in $G^=[K]$ we have $x(\mathcal{F}_{E[K]}) < ||\mathcal{F}_{E[K]}||$.

Proof. Since $\overline{G}[\overline{K}]$ is $\overline{\mathcal{H}}$-critical, all components of $\mathcal{L}_{(\beta)}$ are node-disjoint from K. For each $\mathcal{H}$-matching x in $G^=[K]$ we have $x(\mathcal{F}_{E[K]}) \le ||\mathcal{F}_{E[K]}||$, since $\mathcal{F}$ is valid. Suppose for contradiction, that x is a perfect $\mathcal{H}$-matching in $G^=[K]$ for which $x(\mathcal{F}_{E[K]}) = ||\mathcal{F}_{E[K]}||$. By Lemma 4.6 we get that for each set $F \in \mathcal{F}_{E[K]}$ the equation $x(F) = |V(F)| - 1$ holds. Then it is easy to see, that $x/\mathcal{L}$ is a perfect $\overline{\mathcal{H}}$-matching in $\overline{G}[\overline{K}]$, a contradiction. □

Now, we have a set $\overline{X}$ at hand, which enables us to construct a solution of (31)-(34) for some other laminar family $\mathcal{L}'$ as follows. Let $\overline{K_i}$ $(i = 1, \cdots, k)$ be the set of $\overline{\mathcal{H}}$-critical components in $\overline{G} - \overline{X}$, here $k > |\overline{X}|$; furthermore we define $F_i := E^=[K_i]$. Let $X_1, \cdots, X_l$ denote the maximal sets (of edges) in $\mathcal{L}$ which correspond to a node in $\overline{X}$. Let $\mathcal{L}' := \mathcal{L} \cup \{F_1, \cdots, F_k\}$, and

$$\begin{aligned} m'(F_i) &:= m(F_i) + \varepsilon && \text{if } F_i \in \mathcal{L} \\ m'(F_i) &:= \varepsilon && \text{if } F_i \notin \mathcal{L} \\ m'(X_i) &:= m(X_i) - \varepsilon && \text{for } i = 1, \dots, l \\ m'(F) &:= m(F) && \text{otherwise} \\ y'(v) &:= y(v) && \text{if } v \in V - X - \cup K_i \\ y'(v) &:= y(v) + \varepsilon && \text{if } v \in X \\ y'(v) &:= y(v) - \varepsilon && \text{if } v \in \cup K_i \end{aligned} \tag{39}$$

Now we check for which value of ε we get a solution of (31)-(34). The dual change does not change $\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F)$ for $uv \in E(\mathcal{L}')$, and does not decrease $\frac{1}{2}y_u + \frac{1}{2}y_v + \sum_{uv \in F \in \mathcal{L}} m(F)$ for $uv \in E^=$, thus for any $\varepsilon \ge 0$ we get a solution of (32), and (31) for edges in $E^=$. To get a solution of (31) for edges not in $E^=$, notice that the left hand side decreases by ε if $u, v \in \cup K_i$, and decreases by $\varepsilon/2$ if $u \in \cup K_i$ and $v \in V - X - \cup K_i$. Thus we get the upper bound

$$\begin{aligned}\min\{2c'(uv) &\text{ for } uv \notin E^=, u \in \cup K_i, v \in V - X - \cup K_i \\ c'(uv) &\text{ for } uv \notin E^=, u, v \in \cup K_i\},\end{aligned}$$

where $c'(uv) := c(uv) - \frac{1}{2}y_u - \frac{1}{2}y_v - \sum_{uv\in F\in\mathcal{L}} m(F)$ is the "reduced" cost function. From (33) we get the upper bound

$$\min\{m(X_i) : i = 1, \dots, l\},$$

which is positive since $m > 0$.

We need to choose $\varepsilon > 0$ to get a solution of (34) for $\mathcal{L}', m', y'$. It is enough to check that if $\mathcal{L}''$ is a component of $\mathcal{L}'$, then $\sum_{F\in\mathcal{L}''} m'(F)\left(|V(F)| - 1 - x(F)\right) \geq 0$ holds for any $\mathcal{H}$-matching x of G. The components of $\mathcal{F}'$ are $\mathcal{F}'_{F_i}$ and $\mathcal{F}'_{X_i}$, or are identical with a component of $\mathcal{F}$, we only have to check the first two cases. Thus the set of inequalities (34) for $\mathcal{L}', m', y'$ is equivalent with $\mathcal{F}'_{F_i}$ and $\mathcal{F}'_{X_i}$ being valid.

Let X_i^j be the maximal sets in $\mathcal{L}_{X_i} - \{X_i\}$. Since

$$\chi_{\mathcal{F}'_{X_i}} = \left(1 - \frac{\varepsilon}{m(X_i)}\right) \cdot \chi_{\mathcal{F}_{X_i}} + \frac{\varepsilon}{m(X_i)} \cdot \sum_j \chi_{\mathcal{F}_{X_i^j}}$$

and since by Lemma 4.4 the systems $\mathcal{F}_{X_i}$ and $\mathcal{F}_{X_i^j}$ are valid, we get that the components $\mathcal{F}'_{X_i}$ are valid for each i and any $0 \leq \varepsilon \leq m(X_i)$. For a component $\mathcal{F}'_{F_i}$ and an $\mathcal{H}$-matching x we need:

$$\begin{aligned}0 \leq &\sum_{F\in\mathcal{L}'_{F_i}} m'(F)\left(|V(F)| - 1 - x(F)\right) = \\ = &\sum_{F\in\mathcal{L}'_{F_i}} m(F)\left(|V(F)| - 1 - x(F)\right) + \varepsilon \cdot \left(|V(F_i)| - 1 - x(F_i)\right) = \\ = &\,||\mathcal{F}_{F_i}|| - x(\mathcal{F}_{F_i}) + \varepsilon \cdot \left(|V(F_i)| - 1 - x(F_i)\right).\end{aligned} \tag{40}$$

The upper bound on ε is the minimum of the fraction $(||\mathcal{F}_{F_i}|| - x(\mathcal{F}_{F_i}))/(x(F_i) - |V(F_i)| + 1)$ on $\mathcal{H}$-matchings x for which $x(F_i) - |V(F_i)| + 1$ is positive. It is easy to see that $x(F_i) - |V(F_i)| + 1$ is positive if and only if x is a perfect $\mathcal{H}$-matching in $(V(F_i), F_i) = G^=[K_i]$. By Lemma 5.3, for these x's we have $||\mathcal{F}_{F_i}|| - x(\mathcal{F}_{F_i}) > 0$, the upper bound on ε will be $||\mathcal{F}_{F_i}|| - M(\mathcal{F}_{F_i})$, where the function M is defined by

$$M(\mathcal{F}) := \max\{x(\mathcal{F}) : x \text{ is a perfect } \mathcal{H}\text{-matching in } G(\mathcal{F})\}, \tag{41}$$

for an arbitrary nice system $\mathcal{F}$.

We conclude that the exact bound which gives the maximum value of ε to get a dual change is given be the following formula.

Claim 5.4. *Suppose we have a dual solution $\mathcal{L}, y, m$ of (31)-(34), and a set $\overline{X}$ as defined above. Then the maximum value of ε for which $\mathcal{L}', y', m'$ defined by (39) is a solution of (31)-(34) is*

$$\begin{array}{ll}\min\{2c'(uv) & \text{for } uv \notin E^=, u \in \cup K_i, v \in V - X - \cup K_i \\ c'(uv) & \text{for } uv \notin E^=, u, v \in \cup K_i \\ m(X_i) & \text{for } i = 1, \dots, l \\ ||\mathcal{F}_{F_i}|| - M(\mathcal{F}_{F_i}) & \text{for } i = 1, \dots, k\}.\end{array} \tag{42}$$

If m is positive, then the formula gives a positive value for ε. The inequality $(y', m')b' = (y, m)b - (k - |\overline{X}|)\varepsilon < (y, m)b$ contradicts the choice of $\mathcal{L}, y, m$, this finishes the proof of the description (10)-(12) of $\mathcal{P}$. In fact, we have proved a little bit more:

Theorem 5.5. *For each vector c, there is a nice family $\mathcal{L}$ for which the optimum value of (30)-(34) is equal to $\max_{x \in \mathcal{P}} cx$.*

6 Complexity

Up to this point we have not discussed the complexity of the notions introduced. Recall the definition of $M(\mathcal{F})$, and let

$$M'(\mathcal{F}) := \max\{x(\mathcal{F}) : x \text{ is a perfect } \mathcal{H}\text{-matching in } G(\mathcal{F}) \text{ with exactly one odd cycle }\}. \tag{43}$$

The following lemmas are essential for proving results on complexity.

Lemma 6.1 (Cornuéjols, Pulleyblank, [2]). *If G is factor-critical and there is a perfect $\mathcal{H}$-matching in G, then there is a perfect $\mathcal{H}$-matching in G which has exactly one odd cycle.*

Cornuéjols and Pulleyblank gave a short proof of Lemma 6.1, a short proof of the following weighted version could not have been given, we postpone the proof of Lemma 6.2 to Sect. 8.

Lemma 6.2. *Suppose $\mathcal{F} = (\mathcal{L}, m)$ is a valid system with one component and there is at least one perfect $\mathcal{H}$-matching in $G(\mathcal{F})$. Then $M(\mathcal{F}) = M'(\mathcal{F})$, i.e. the maximum in $M(\mathcal{F})$ is attained by a perfect $\mathcal{H}$-matching x which has exactly one odd cycle.*

We show how Lemma 6.2 can be used to construct an an algorithm to check whether a nice system with one component $\mathcal{F}$ is valid and if so, to find the maximum $M(\mathcal{F})$. This is of our great interest, since function M is necessary to determine the value of ε from Lemma 5.4.

Theorem 6.3. *In cases 1 and 2 of Sect. 2 the maximum $M'(\mathcal{F})$ can be determined in polynomial time for any nice system $\mathcal{F}$ with one component.*

Proof. We use a simple reduction to weighted perfect matching using a principle of Cornuéjols, Pulleyblank [3]. In case 2 of Sect. 2 we try each cycle C in $\mathcal{H}$ which has $C \subseteq E(\mathcal{F})$. For each C we determine a maximum perfect matching in $G(\mathcal{F}) - V(C)$, and in the end we pick the maximum of all.

To determine $M'(\mathcal{F})$ for case 1 it is enough to find a maximum weight perfect 2-matching with only one cycle C and the length of C at least k, where k is a fixed odd number. This is equivalent with the following approach. We try each path $P \subseteq E(\mathcal{F})$ on k edges to find the maximum weight $M'(\mathcal{F}, P)$ of a perfect 2-matching with a cycle C for which $P \subseteq C$ – in the end we pick the maximum over all possible P's. For a fixed path P connecting nodes a and b, it is easy to see that $M'(\mathcal{F}, P)$ is the sum of the maximum weight of a perfect matching M_a in $G(\mathcal{F}) - (V(P) - a)$ and the maximum weight of a perfect matching M_b in $G(\mathcal{F}) - (V(P) - b)$. We perform a weighted perfect matching algorithm to find these maximums, the sum of matchings M_a and M_b gives the optimum $M'(\mathcal{F}, P)$ (after eliminating even cycles). □

Lemma 6.4. *A nice system $\mathcal{F} = (\mathcal{L}, m)$ with one component is valid if and only if its proper truncations are valid and $M'(\mathcal{F}) \leq ||\mathcal{F}||$.*

Proof. If $\mathcal{F}$ is valid, then all truncations of $\mathcal{F}$ are valid by Lemma 4.4 and $M'(\mathcal{F}) \leq ||\mathcal{F}||$ follows from the definition of validity.

Now, suppose the proper truncations of $\mathcal{F}$ are valid and $M'(\mathcal{F}) \leq ||\mathcal{F}||$. If $|\mathcal{L}| = 1$ then the validity of $\mathcal{F}$ follows from Lemma 6.1. If there is no perfect $\mathcal{H}$-matching in $G(\mathcal{F})$ then the validity of $\mathcal{F}$ follows merely from the validity of its truncations to the maximal proper truncations. Thus from now on we suppose $|\mathcal{L}| > 1$ and there is a perfect $\mathcal{H}$-matching in $G(\mathcal{F})$.

Let F_0 be the maximal set of $\mathcal{L}$ and let F_i ($i = 1, 2, \ldots, k$) be the family of maximal sets in $\mathcal{L} - F_0$, we will use the notation $\mathcal{F}_i := \mathcal{F}_{F_i}$. Consider the nice system $\mathcal{F}^0 = (\mathcal{L}, m_0)$ with $m_0(F_0) := 0$ and $m_0(F) := F$ for $F \neq F_0$. Since

$$\chi_{\mathcal{F}^0} = \sum_{i \geq 1} \chi_{\mathcal{F}_i}$$

and $\mathcal{F}_i$ are valid, we get that $\mathcal{F}^0$ is valid. Hence by Lemma 6.2 $M(\mathcal{F}^0) = M'(\mathcal{F}^0)$,

$$M(\mathcal{F}) = M(\mathcal{F}^0) + m(F_0)|V(F_0)| = M'(\mathcal{F}^0) + m(F_0)|V(F_0)| \leq M'(\mathcal{F}),$$

thus $M(\mathcal{F}) = M'(\mathcal{F})$. For an $\mathcal{H}$-matching x we have

$$\begin{aligned} x(\mathcal{F}) &= m(F_0)x(F_0) + \sum_{i \geq 1} x(\mathcal{F}_i) \leq \\ &\leq m(F_0)x(F_0) + \sum_{i \geq 1} ||\mathcal{F}_i|| = \\ &= m(F_0)x(F_0) + ||\mathcal{F}|| - m(F_0)(|V(F_0)| - 1), \end{aligned} \tag{44}$$

If $\mathcal{F}$ would not be valid, then for some $\mathcal{H}$-matching x we would have $x(\mathcal{F}) > ||\mathcal{F}||$. Then by (44) x must be a perfect $\mathcal{H}$-matching in $G(\mathcal{F})$. This implies $M'(\mathcal{F}) = M(\mathcal{F}) > ||\mathcal{F}||$, a contradiction. □

The correctness of the following algorithm follows from these lemmas. By Proposition 3.4, this algorithm is in fact a separation algorithm for $\mathcal{C}_{\mathcal{L}}$.

Algorithm 1. Checking validity and finding $M(\mathcal{F})$.

1. Determine $M'(\mathcal{F}_F)$ for each set $F \in \mathcal{F}$.
2. $\mathcal{F}$ is valid if and only if $M'(\mathcal{F}_F) \leq ||\mathcal{F}_F||$ holds for each $F \in \mathcal{F}$.
3. If $\mathcal{F}$ is valid, then $M(\mathcal{F}) = M'(\mathcal{F})$.

7 TDI Descriptions

We are in position to prove the following statement on dual integrality.

Theorem 7.1. *For each integer vector c, there is a nice family $\mathcal{L}$ for which the optimum value of (30)-(34) is equal to* $\max_{x \in \mathcal{P}} cx$ *and is attained by an integer vector (y, m).*

The proof goes by carefully reading Sect. 5, we use the dual change for an algorithm maintaining dual integrality. This can be done by choosing the set $\overline{X}$ in a special way.

We start with $\mathcal{L}$ empty and (y, m) some integer solution of (30)-(34). Then we iteratively change $\mathcal{L}$ and (y, m) as written in Sect. 5: we find the sets X and K_i, change $\mathcal{L}$ and (y, m), and if an entry of m' is 0 then we eliminate the set from $\mathcal{L}'$. The value of ε is defined by the formula (42). If the minimum was taken over an emptyset, then it is easy to see that set X is as in Theorem 2.1 which verifies that there is no perfect $\mathcal{H}$-matching. In this case the algorithm stops by announcing the polytope is empty and exhibiting the set X.

From this point on we suppose there is a perfect $\mathcal{H}$-matching in G. If there is a perfect $\overline{\mathcal{H}}$-matching in $\overline{G}$, then Lemma 5.2 gives an optimum $\mathcal{H}$-matching, the algorithm terminates.

If there is no perfect $\overline{\mathcal{H}}$-matching, to maintain integrality of (y, m) let us choose X and K_i according to the following rule. Consider the sets $\overline{D} = D_{\overline{\mathcal{H}}}(\overline{G})$ and $\overline{A} = A_{\overline{\mathcal{H}}}(\overline{G})$ from Theorem 2.2. Let $\overline{G}[\overline{A}_j \cup \overline{D}_j]$ be the components of $\overline{G}[\overline{D} \cup \overline{A}] - E[\overline{A}]$, where $\overline{A}_j \subseteq \overline{A}$ and $\overline{D}_j \subseteq \overline{D}$. Let K_{ij} $(i = 1, 2, \ldots, k_j)$ denote the components of $\overline{G}[\overline{D}]$ for which $\overline{K}_{ij} \subseteq \overline{D}_j$. Since $\overline{G}[\overline{D}]$ has $\overline{\mathcal{H}}$-critical components, each $\overline{G}[\overline{K}_{ij}]$ is a $\overline{\mathcal{H}}$-critical component in $\overline{G} - \overline{A}_j$.

Claim 7.2. *For each j we have $k_j > |\overline{A}_j|$.*

Proof. If $k_1 \leq |\overline{A}_1|$ holds, then $X = \overline{A} - \overline{A}_1$ must be a minimizing set for $|V| + |X| - c_{\overline{G}}^{\overline{H}}(X)$, thus $k_1 = |\overline{A}_1|$. This implies that $\overline{G} - (\overline{A} - \overline{A}_1)$ cannot have any other $\overline{\mathcal{H}}$-critical than $\overline{K}_{ij}$ for $j \neq 1$, $i = 1, 2, \ldots, k_j$. Thus the union of $\overline{\mathcal{H}}$-critical components of $\overline{G} - (\overline{A} - \overline{A}_1)$ is strictly smaller than $\overline{D}$, a contradiction with part 2. of Theorem 2.2. □

Let us choose $\overline{X} := \overline{A}_1$ and $\overline{K}_i := \overline{K}_{i1}$! If (y, m) is integer, then the parity of y is the same on each node of a component of $G^=$. By the choice of $\overline{X}$, y_v has the same parity for the nodes v in K_i and X. We get that ε is integer, since the values in (42) are integer, only $c'(uv)$ has the fractional expression $\frac{1}{2}y_u + \frac{1}{2}y_v$. By the above observation on parity, this must be integer for $uv \notin E^=, u, v \in \cup K_i$. Hence, after the dual change, (y', m') is an integer vector. To complete the proof, we need to show the finiteness of the algorithm. In fact, finiteness follows from integrality, since each dual change decreases the dual objective by at least 1. What we will show is that rule implies that the number of dual changes is bounded by a polynomial of $|V|, |E|$.

Claim 7.3. *A dual change does not increase the deficiency* $\mathrm{def}(\mathcal{L}, y, m) := |\overline{V}| - \nu^{\overline{\mathcal{H}}}(\overline{G})$.

Proof. To see this, notice that edges which "get untight" by the dual change, i.e. edges in $\overline{G}$ but not in $\overline{G}(\mathcal{L}', y', m')$ must be induced by $\overline{X}$ or must be in $d(\overline{X}, \overline{V} - \overline{X} - \overline{D}_1)$. Consider a maximum $\overline{\mathcal{H}}$-matching $\overline{x}$ as in part 3. of Theorem 2.2, then $\overline{x}$ does not use any of these edges. Thus $\overline{x}' := \overline{x}/\{E^=[K_i] : i = 1, 2, \cdots, k_1\}$ is a $\overline{\mathcal{H}}(\mathcal{L}', y', m')$-matching in $\overline{G}(\mathcal{L}', y', m')$ of the same deficiency. If some sets with $m(X_i) = 0$ for a node in $\overline{X}$ are eliminated, that $\overline{x}'$ can be expanded using Lemma 3.2. □

Let $d(\mathcal{L}, y, m)$ denote the number of nodes of V corresponding to a node in $\overline{D}$.

Claim 7.4. *If* $\mathrm{def}(y, m) = \mathrm{def}(y', m')$ *then* $d(\mathcal{L}', y', m') \geq d(\mathcal{L}, y, m)$.

Proof. Let D' be the set of nodes in $\overline{G}(\mathcal{L}', y', m')$ which correspond to a node in $\overline{D} = \overline{D}(\mathcal{L}, y, m)$. We need to see that $D' \subseteq \overline{D}(\mathcal{L}', y', m')$, i.e. let $v' \in D'$.

Let K be the component of $\overline{G}[\overline{D}]$ which corresponds to v', consider an $\overline{\mathcal{H}}$-matching $\overline{x}$ in $\overline{G}$ exposing v' for which each node a in $\overline{A}$ is covered by an edge ad with $d \in \overline{D}$, and $\overline{x}$ has deficiency $\mathrm{def}(y, m)$ (as in part 3. of Theorem 2.2). Then $\overline{x}/\{E^=[K_i] : i = 1, 2, \cdots, k_1\}$ is an $\overline{\mathcal{H}}(\mathcal{L}', y', m')$-matching exposing v', thus $v' \in \overline{D}(\mathcal{L}', y', m')$. □

Let $n(\mathcal{L}, y, m)$ components of $\overline{G}[\overline{D} \cup \overline{A}] - E[\overline{A}]$. Let $\mathcal{L}(\overline{A}(y, m))$ denote the union of truncations of $\mathcal{L}$ to nodes in $\overline{A}$.

Claim 7.5. *If* $\mathrm{def}(\mathcal{L}, y, m) = \mathrm{def}(\mathcal{L}', y', m')$ *and* $d(\mathcal{L}', y', m') = d(\mathcal{L}, y, m)$*, then* $n(\mathcal{L}', y', m') \leq n(\mathcal{L}, y, m)$ *and* $|\mathcal{L}(\overline{X}(y', m'))| \leq |\mathcal{L}(\overline{X}(y, m))|$.

Proof. The first follows from the observation that each edge in $\overline{G}[\overline{D} \cup \overline{A}] - E[\overline{A}]$ stays tight. The second follows, since new members $\mathcal{L}'$ are in $\mathcal{D}(\mathcal{L}', y', m')$. □

Claim 7.6. *If we have* $\mathrm{def}(\mathcal{L}, y, m) = \mathrm{def}(\mathcal{L}', y', m')$ *and* $d(\mathcal{L}', y', m') = d(\mathcal{L}, y, m)$ *and* $n(\mathcal{L}, y, m) = n(\mathcal{L}', y', m')$ *and* $|\mathcal{L}'(\overline{A}'(y', m'))| = |\mathcal{L}(\overline{A}(y, m))|$*, then the number of tight edges in $G - A$ increases (where A is the set of nodes in V corresponding to a node of $\overline{A}$).*

Proof. The premises imply that an edge induced in a component $\overline{G}[\overline{K}_i]$ is determining the minimum (42), this edge is is getting tight for $\mathcal{L}', y', m'$. Only edges incident with a node of $\overline{X} \subseteq \overline{A}$ can get untight. □

We conclude, that the special choice of X assures that either $\text{def}(\mathcal{L}, y, m)$ decreases or $d(\mathcal{L}, y, m)$ increases or $n(\mathcal{L}, y, m) + |\mathcal{L}(\overline{A}(y, m))|$ decreases or the number of tight edges in $G - A$ increases. Thus the number of dual changes needed is at most $|V|^2 \cdot |E|^2$. This provides an algorithm for finding a maximum weight perfect $\mathcal{H}$-matching together with an integer dual optimum, and proves that (15)-(17) is TDI.

The algorithm is polynomial if we have an oracle to solve the unweighted problem in each contracted graph, and we have an oracle to find the maximum $M(\mathcal{F})$ for any nice family with one component. In cases 1 and 2 of Sect. 2 the set $\overline{\mathcal{H}}$ for the contracted graph is also of the same case with the same k, which gives us the first oracle. The second oracle is given by Lemma 6.3 and Algorithm 1.

We will use an elementary technique to derive a TDI description with integer coefficients.

Proof. (of Theorem 3.6) For some fixed integer vector c, consider an integer optimum solution (y, m) of (30)-(34). Let $y'_v := \lfloor \frac{1}{2} y_v \rfloor$, let U_1 be the set of nodes v with y_v odd, and $z'_U := 1$ if $U = U_1$, otherwise $z'_U := 0$. Let ω' be a 0-1 vector with the only 1 in the entry for $\mathcal{F} = (\mathcal{L}, m)$. Then (y', z', ω') is an optimum dual solution of (18)-(21). □

8 Proof of Lemma 6.2

To prove 6.2 we will use the following observations on the matching polytope.

Proposition 8.1. *Suppose $G = (V, E)$ is a graph with a fixed node $t \in V$ and $|V|$ is even. Then the following system is a description of the perfect matching polytope for G.*

$$\begin{aligned} x &\geq 0 \\ x(d(v)) &= 1 \quad \textit{for each node } v \in V \\ x(d(U)) &\geq 1 \quad \textit{for each set } U \subseteq V \textit{ with } G[U] \textit{ factor-critical and } t \notin U \end{aligned} \tag{45}$$

Proof. (A refinement of the proof in [11].) Suppose the polytope Q determined by (45) has a vertex x which is not in the convex hull of perfect matchings of G. We choose a counterexample with $|E| + |V|$ minimal. Hence $0 < x_e < 1$ for each $e \in E$, each degree in G is at least 2. If $|E| = |V|$, then G is an even cycle, thus no counterexample. So we must have $|E| > |V|$. As x is a vertex of Q, there are $|E|$ linearly independent constraints in (45) satisfied with equality. Since $|E| > |V|$, there is a set $U \subseteq V$, $|U| \geq 3$ with $G[U]$ factor-critical and $t \notin U$ for which equality $x(d(U)) = 1$ holds.

Consider the projections x' and x'' of x to the edges of the graphs G/U and $G/(V - U)$. It is easy to see, that these vectors satisfy (45) for $t' := t$ and $t'' := \{V - U\}$, respectively. By the minimality of $|E| + |V|$, x' and x'' must be in the perfect matching polytope for G/U and $G/(V - U)$. As in [11] one can show that x must also be in the perfect matching polytope of G, a contradiction. □

Using routine transformations and an uncrossing technique in [11] one can also prove the following extension of Proposition 8.1. A different proof of Proposition 8.2 can be given by observing that in a dual-changing algorithm for maximum weight perfect matching we can keep a dual solution which has no blossom containing t.

Proposition 8.2. *Suppose $G = (V, E)$ is a graph with a fixed node $t \in V$ and $|V|$ is even. Then*

$$\begin{aligned} x &\geq 0 & & \\ x(d(v)) &= 1 & &\text{for } v \in V \\ x(E[U]) &\leq (|U| - 1)/2 & &\text{for } U \subseteq V, G[U] \text{ factor-critical and } t \notin U. \end{aligned} \tag{46}$$

is a description of the perfect matching polytope for G and for any vector c there is an optimum dual solution y_v, y_U of (46) for which $\{E[U] : y_U > 0\}$ is a nice family.

Let $G = (V, E)$ be an arbitrary graph with real weights c_e on the edges $e \in E$. Fix a node $s \in V$. Suppose x is twice the characteristic vector of a perfect matching M in $G - s$ of maximum weight $c \cdot x$. We call an odd cycle $Q \subseteq E$ *alternating* if $s \in V(Q)$ and $x(Q) = |V(Q)| - 1$.

Proposition 8.3. *Using the above notations, either*
a) there is an alternating odd cycle Q for which $c(Q) \geq c|_Q \cdot x|_Q$, or
b) there is a vector $y \in \mathbb{R}^V$ and a nice system $\mathcal{F}$ for which $s \notin V(\mathcal{F})$ and $y_s < 0$ and $c_{uv} \leq \frac{1}{2}y_u + \frac{1}{2}y_v + \chi_{\mathcal{F}}(uv)$ for each edge $uv \in E$ and $c \cdot x = \sum\{y_v : v \in V - s\} + ||\mathcal{F}||$.

Proof. Suppose c is integer. We construct an auxiliary graph $G' = (V + t, E')$ where $E' := E \cup \{tv :$ for each edge $sv \in E\} + st$ with weight-function $c'_{uv} := c_{uv}$ and $c'_{tv} := c_{sv}$ if $u, v \in V - s$ and $c'_{st} := -1$. Let x' be twice the characteristic vector of the perfect matching $M + st$ in G', its weight is $c \cdot x - 2$. If the maximum weight of twice a perfect matching in G' is at least $c \cdot x$ then it is attained by some x' for which $x'(st) = 0$. Thus there is an alternating even cycle for x and x' which contains st, when we delete st from this cycle and identify t with s we get the alternating cycle Q as required in a).

Suppose the maximum weight of twice a perfect matching in G' is $c \cdot x - 2$. Using Proposition 8.2 with the fixed node t we get a vector $y' \in \mathbb{R}^{V+t}$ and a nice system $\mathcal{F}' = (\mathcal{L}', m')$ for which $t \notin V(\mathcal{F}')$ and $c'_{uv} \leq \frac{1}{2}y'_u + \frac{1}{2}y'_v + \chi_{\mathcal{F}'}(uv)$ for each edge $uv \in E'$ and $c \cdot x - 2 = \sum\{y''_v : v \in V + t\} + ||\mathcal{F}'||$. Let $\mathcal{F}$ be the truncation of $\mathcal{F}'$ to the sets not containing s and let

$$\begin{aligned} y''_s = y''_t &:= \frac{1}{2}(y'_s + y'_t) \\ y''_v &:= y'_v + \sum\{m'(F) : F \in \mathcal{L}' \text{ and } s \in V(F')\} \quad \text{for } v \in V - s \end{aligned}$$

For $y'', \mathcal{F}$ we also have $c'_{uv} \leq \frac{1}{2}y''_u + \frac{1}{2}y''_v + \chi_{\mathcal{F}}(uv)$ for $uv \in E'$ and $c \cdot x - 2 = \sum\{y''_v : v \in V + t\} + ||\mathcal{F}||$. Now $s, t \notin V(\mathcal{F})$ and since $M + st$ is optimal, by slackness condition we get that $\frac{1}{2}y''_s + \frac{1}{2}y''_t = -1$, thus $y''_s = y''_t = -1$. We have $c \cdot x = \sum\{y''_v : v \in V - s\} + ||\mathcal{F}||$, let us construct y as follows. $y_v := y''_v$ for each node v in $V - s$ and $y_s := -1$, then y and $\mathcal{F}$ is as required in b). □

Proof. (of Lemma 6.2) Consider a valid system $\mathcal{F}$ with one component, suppose x is a perfect $\mathcal{H}$-matching in $G = (V, E) := (V(\mathcal{F}), E(\mathcal{F}))$, and $x(\mathcal{F}) = M(\mathcal{F})$ – i.e. x has maximum weight for weight-function $c := \chi_{\mathcal{F}}$. We choose x which has a number

k of odd cycles, namely $C_1, \dots, C_k$. We suppose $\mathcal{H} = \{C_1, \dots, C_k\}$, it will be easy to see that this does not affect any part of the proof, since in what follows we show that there is a maximum vector x using at most one of these cycles.

Suppose for contradiction that $k \geq 3$ and there is no maximizing x with less number of odd cycles. Consider $G^{C_i} = G[V(C_i)] = (V(C_i), E^{C_i})$ for some $i \leq k$, let x^{C_i}, c^{C_i} be the truncation to the edges $E^{C_i} = E[V(C_i)]$. Then x^{C_i} must be a maximizing perfect $\mathcal{H}$-matching in G_{C_i} with weight c^{C_i}. Thus, by applying the Theorem, there must be a vector $y^{C_i} \in \mathbb{R}^{V(C_i)}$ and a valid system $\mathcal{F}^{C_i} = (\mathcal{L}^{C_i}, m^{C_i})$ in G^{C_i} for which the following hold.

$$\frac{1}{2}y_u^{C_i} + \frac{1}{2}y_v^{C_i} + \sum_{uv \in F \in \mathcal{L}^{C_i}} m^{C_i}(F) \geq c_{uv} \qquad \text{for } uv \in E^{C_i} \tag{47}$$

$$\frac{1}{2}y_u^{C_i} + \frac{1}{2}y_v^{C_i} + \sum_{uv \in F \in \mathcal{L}^{C_i}} m^{C_i}(F) = c_{uv} \qquad \text{for } uv \in E(\mathcal{L}^{C_i}) \tag{48}$$

$$\sum_{v \in V(C_i)} y_v^{C_i} + ||\mathcal{F}^{C_i}|| = c^{C_i} \cdot x^{C_i} \tag{49}$$

A sub-partition V^i of V will be called *special*, if there are vectors $y^i \in \mathbb{R}^{V^i}$ and valid systems $\mathcal{F}^i = (\mathcal{L}^i, m^i)$ in $G^i = G[V^i] = (V^i, E^i)$ for which $V(C_i) \subseteq V^i$ and the following hold.

$$\frac{1}{2}y_u^{i} + \frac{1}{2}y_v^{i} + \sum_{uv \in F \in \mathcal{L}^{i}} m(F) \geq c_{uv} \qquad \text{for } uv \in E^i - E(\mathcal{L}^i) \tag{50}$$

$$\frac{1}{2}y_u^{i} + \frac{1}{2}y_v^{i} + \sum_{uv \in F \in \mathcal{L}^{i}} m^i(F) = c_{uv} \qquad \text{for } uv \in E(\mathcal{L}^i) \tag{51}$$

$$\sum_{v \in V^i} y_v^{i} + ||\mathcal{F}^{i}|| = c^{i} \cdot x^{i} \tag{52}$$

(here c^i, x^i denote the truncation to E^i) and furthermore, $\mathcal{F}^i$ has one component with maximal set F^i for which $V(F^i) = V^i$, and x^i is a perfect $\mathcal{H}$-matching in V^i.

There is at least one special sub-partition, namely $V^i = V(C_i)$, consider a special sub-partition V^i of V for which $\cup V^i$ is maximal. Let $\mathcal{F}^*, y^*$ be the merging of $\mathcal{F}^i, y^i$. We construct an auxiliary graph $G' := (V', E')$ by $V' := V - \cup V^i + s$ as the new node set and let us construct the new edge set by identifying the nodes in $\cup V^i$ by s:

$$\begin{aligned} E' := E[V - \cup V^i] \cup \{sv :\ & \text{the copy of an edge } uv \text{ with } u \in \cup V^i, v \notin \cup V^i\} \cup \\ & \cup \{e :\ \text{a loop on } s \text{ as the copy of } uv \text{ with } u \in V^i, v \in V^j \text{ for some } i \neq j\}. \end{aligned}$$

The auxiliary weight function is defined as follows:

$$\begin{aligned} c'_{uv} &:= c_{uv} && \text{if } uv \in E[V - \cup V^i], u \in \cup V^i, v \notin \cup V^i \\ c'_{sv} &:= c_{uv} - \frac{1}{2}y_u^* && \text{if } sv \in E' \text{ is the copy of } uv \in E \\ c'_{e} &:= c_{uv} - \frac{1}{2}y_u^* - \frac{1}{2}y_u^* && \text{if } e \in E' \text{ is the copy of an edge } uv \text{ with} \\ & && \quad u \in V^i, v \in V^j \text{ for some } i \neq j \end{aligned}$$

Let x' be the truncation of x to E', then x' is twice the characteristic vector of a perfect matching M in $G[V - \cup V^i] = G' - s$. Apply Proposition 8.3 to graph G' with s, x' and c'.

First we will prove, that case b) leads to a contradiction. Suppose there is a vector $y' \in \mathbb{R}^{V'}$ and a nice system $\mathcal{F}'$ for which $s \notin V(\mathcal{F}')$ and $y'_s < 0$ and $c'(uv) \leq \frac{1}{2}y'_u + \frac{1}{2}y'_v + \chi_{\mathcal{F}'}(uv)$ for each edge uv and $c' \cdot x' = \sum y'_v + ||\mathcal{F}'||$. Merging together y^*, y' and deleting the entry of s we get y^0. Merging together $\mathcal{F}^*, \mathcal{F}'$ we get $\mathcal{F}^0 = (\mathcal{L}^0, m^0)$ which is clearly valid, since $\mathcal{H} = \{C_1, \dots, C_k\}$. It is easy to see, that

$$\frac{1}{2}y^0_u + \frac{1}{2}y^0_v + \sum_{uv \in F \in \mathcal{L}^0} m^0(F) \geq c_{uv} \qquad \text{for } uv \in E - E(\mathcal{L}^0) \tag{53}$$

$$\frac{1}{2}y^0_u + \frac{1}{2}y^0_v + \sum_{uv \in F \in \mathcal{L}^0} m^0(F) = c_{uv} \qquad \text{for } uv \in E(\mathcal{L}^0) \tag{54}$$

$$\sum_{v \in V} y^0_v + ||\mathcal{F}^0|| = c \cdot x \tag{55}$$

Furthermore, since the entry of s was negative we get that each edge uv with $u \in V^i$ and $v \notin V^i$ (for some i) must not be tight in (53). This means that for each i and each edge $uv \in E$ with $u \in V^i$ and $v \notin V^i$ we have $\frac{1}{2}y^0_u + \frac{1}{2}y^0_v + \sum_{uv \in F \in \mathcal{L}^0} m^0(F) > c_{uv}$ holds. Since $k \geq 3$, there must be an $ab \in E$ edge that is not in $E(\mathcal{L}^0)$. By $0 \leq c_{ab} \leq \frac{1}{2}y^0_a + \frac{1}{2}y^0_b$ we get that at least one of y^0_a and y^0_b must be non-negative, suppose $y^0_a \geq 0$. Consider a characteristic vector z of twice a maximum weight perfect matching in $G - a$, clearly it has weight $c \cdot z = ||\mathcal{F}|| \geq c \cdot x$. On the other hand, $c \cdot z \leq \sum_{v \in V - a} y^0_v + ||\mathcal{F}^0|| = c \cdot x - y^0_a \leq c \cdot x$. By slackness conditions z can only be positive on tight edges, thus z would have to leave at least one node exposed in each V^i. This is a contradiction.

Suppose we have case a) for the instance G', s, x', c' of the Proposition 8.3. Let Q be the edge set in G corresponding the M-alternating cycle Q'. Then Q is either an M-alternating odd ear on some V^i, or Q is an M-alternating odd path between V^i and V^j for some $i \neq j$.

In the first case we prove that if we replace V_i by $V^Q := V_i \cup V(Q)$ we get a special sub-partition, which contradicts the choice of the sub-partition. We only have to show the existence of a vector y^Q and a valid system $\mathcal{F}^Q = (\mathcal{L}^Q, m^Q)$ in $G^Q := G[V^Q] = (V^Q, E^Q)$ which is a solution of

$$\frac{1}{2}y^Q_u + \frac{1}{2}y^Q_v + \sum_{uv \in F \in \mathcal{L}^Q} m^Q(F) \geq c_{uv} \qquad \text{for } uv \in E^Q - E(\mathcal{L}^Q) \tag{56}$$

$$\frac{1}{2}y^Q_u + \frac{1}{2}y^Q_v + \sum_{uv \in F \in \mathcal{L}^Q} m^Q(F) = c_{uv} \qquad \text{for } uv \in E(\mathcal{L}^Q) \tag{57}$$

$$\sum_{v \in V^Q} y^Q_v + ||\mathcal{F}^Q|| = c^Q \cdot x^Q \tag{58}$$

and furthermore, $\mathcal{F}^Q$ must have one component with maximal set F^Q for which $V(F^Q) = V^Q$. Since x^Q is a maximum perfect $\mathcal{H}$-matching in G^Q, by the Theorem there must be a vector y^Q and a valid system $\mathcal{F}^Q$ in $G^Q := G[V^Q] = (V^Q, E^Q)$

which is a solution of (56)-(58). Consider the graph $\overline{G^Q}$ which we get from the tight edges and the contraction of the maximal sets in $\mathcal{L}^Q$. In case $\overline{G^Q}$ is factor-critical, then we add the set E^Q to $\mathcal{F}^Q$ with multiplicity zero to get a valid system as required.

Suppose for contradiction that $\overline{G^Q}$ is not factor-critical. Let $\overline{D}, \overline{A}, \overline{C}$ be the Gallai-Edmonds decomposition of $\overline{G^Q}$, let D, A, C be the sets corresponding to them in V. Since $D \neq V$ and x^Q has only one odd cycle, there must be an edge uv with $x^Q(uv) = 2$, $uv \notin E(\mathcal{L}^Q)$ and $v \notin D$. Since x^Q is maximum, edge uv must be tight, that is

$$\frac{1}{2}y_u^Q + \frac{1}{2}y_v^Q = \frac{1}{2}y_u^Q + \frac{1}{2}y_v^Q + \sum_{uv \in F \in \mathcal{L}^Q} m^Q(F) = c_{uv} \tag{59}$$

Consider the maximum weight "twice a perfect matching" problem in the graphs $G^Q - u$ and $G^Q - v$, with maximum values w_u, w_v. It is easy to see

$$w_u \leq \sum_{z \in V^Q - u} y_z^Q + ||\mathcal{F}^Q|| = c^Q \cdot x^Q - y_u^Q. \tag{60}$$

Since $v \notin D$ we get that

$$w_v < \sum_{z \in V^Q - v} y_z^Q + ||\mathcal{F}^Q|| = c^Q \cdot x^Q - y_v^Q. \tag{61}$$

By (59), (60) and (61) we get

$$w_u + w_v < 2c^Q \cdot x^Q - 2c_{uv}. \tag{62}$$

If $uv \in E^i$, then we give a bound on w_u, w_v using $y^i, \mathcal{F}^i$ which satisfies (50)-(52). Let w'_u, w'_v denote the maximum weight of "twice a perfect matching" in the graphs $G^i - u$ and $G^i - v$. Let $M \cap Q$ denote the edges of x_Q on the ear Q. Clearly $w'_u + 2c(M \cap Q) \leq w_u$ and $w'_v + 2c(M \cap Q) \leq w_v$.

$$w'_u = ||\mathcal{F}^i|| + \sum_{z \in V^i - u} y_z^i = c^i \cdot x^i - y_u^i = c^Q \cdot x^Q - 2c(M \cap Q) - y_u^i,$$

$$w'_v = ||\mathcal{F}^i|| + \sum_{z \in V^i - v} y_z^i = c^i \cdot x^i - y_v^i = c^Q \cdot x^Q - 2c(M \cap Q) - y_v^i.$$

Then by (62) we get

$$\frac{1}{2}y_u^i + \frac{1}{2}y_v^i > c_{uv}$$

which is a contradiction, since $m^i \geq 0$ and the edge uv must be tight for $y^i, \mathcal{F}^i$.

If $uv \notin E^i$, then $uv \in M \cap Q$ is an edge on the ear Q. We will use a similar argument as in the last paragraph to get to a contradiction. Suppose the ear

$$Q = \{q_l r_l : l = 0, \cdots, t\} \cup \{r_l q_{l+1} : l = 0, \cdots, t-1\}$$

is connecting $q = q_0 \in V^i$ and $r = r_t \in V^i$. Then $V(Q) = \{q_l : l = 0, \cdots, t\} \cup \{r_l : l = 0, \cdots, t\}$ and $M \cap Q = \{r_l q_{l+1} : l = 0, \cdots, t-1\}$.

Suppose $u = r_j$ and $v = q_{j+1}$. By part a) in Proposition 8.3 we know that

$$\frac{1}{2}y_q^i + \frac{1}{2}y_r^i \leq c(\{q_l r_l : l = 0, \cdots, t\}) - c(\{r_l q_{l+1} : l = 0, \cdots, t-1\}). \tag{63}$$

Since the edges in $E(\mathcal{L}^i)$ are tight, it is easy to see

$$\begin{aligned} w_u \geq & ||\mathcal{F}^i|| + \sum_{z \in V^i - r} y_z^i + 2c(\{r_l q_{l+1} : l = 0, \cdots, j-1\}) + \\ & + 2c(\{q_l r_l : l = j+1, \cdots, t\}) \geq \\ \geq & c^i \cdot x^i - y_r^i + 2c(\{r_l q_{l+1} : l = 0, \cdots, j-1\}) + \\ & + 2c(\{q_l r_l : l = j+1, \cdots, t\}) \end{aligned} \tag{64}$$

and similarly we get

$$\begin{aligned} w_v \geq & ||\mathcal{F}^i|| + \sum_{z \in V^i - q} y_z^i + 2c(\{r_l q_{l+1} : l = j+1, \cdots, t\}) + \\ & + 2c(\{q_l r_l : l = 0, \cdots, j\}) = \\ \geq & c^i \cdot x^i - y_q^i + 2c(\{r_l q_{l+1} : l = j+1, \cdots, t\}) + 2c(\{q_l r_l : l = 0, \cdots, j\}) \end{aligned} \tag{65}$$

hence by (63), (64) and (65)

$$\begin{aligned} w_u + w_v \geq & 2c^i \cdot x^i - y_r^i - y_q^i + 2c(\{q_l r_l : l = 0, \cdots, t\}) + \\ & + 2c(\{r_l q_{l+1} : l = 0, \cdots, t-1\}) - 2c(q_j r_{j+1}) = \\ = & 2c^i \cdot x^i - y_r^i - y_q^i + 4c(M \cap Q) + 2c(\{q_l r_l : l = 0, \cdots, t\}) - \\ & - 2c(\{r_l q_{l+1} : l = 0, \cdots, t-1\}) - 2c_{uv} \geq \\ \geq & 2c^i \cdot x^i + 4c(M \cap Q) - 2c_{uv} = 2c^Q \cdot x^Q - 2c_{uv} \end{aligned} \tag{66}$$

which is a contradiction with (62).

Now we get to the case when Q is an M-alternating odd path between $r \in V^i$ and $q \in V^j$ for some $i \neq j$. By part a) in Proposition 8.3

$$\frac{1}{2}y_q^i + \frac{1}{2}y_r^i \leq c(\{q_l r_l : l = 0, \cdots, t\}) - c(\{r_l q_{l+1} : l = 0, \cdots, t-1\}). \tag{67}$$

Then we construct x' by changing the entries in $E^i \cup E^j \cup Q$ as follows. We take twice a perfect matching in $G^i - q$ and $G^j - r$, and we add twice the matching $Q - M$. Since the edges in $E(\mathcal{L}^i) \cup E(\mathcal{L}^j)$ are tight and for the maximal sets $F^{i,j}$ we have $V^i = V(F^i)$, $V^j = V(F^j)$, we can choose the perfect matchings in $V_i - q$ and $V_j - r$ to have value $||\mathcal{F}^i|| + \sum\{y_z^i : z \in V_i - r\}$ and $||\mathcal{F}^j|| + \sum\{y_z^j : z \in V_j - q\}$. By (52) these values are equal to $c^i \cdot x^i - y_r^i$ and $c^j \cdot x^j - y_q^j$. Thus the change of value on the edges in $E^i \cup E^j$ is at least $-y_r^i - y_q^j$. By (67) the change in the value on the edges of Q is at least $y_r^i + y_q^j$. This means, x' is an $\mathcal{H}$-factor having less number of odd cycles and $c \cdot x \leq c \cdot x'$. □

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