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Generalized star packing problems

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Abstract

Results of Las Vergnas, Hell and Kirkpatrick imply that packing an undirected graph by a set of stars is polynomial if and only if this set is of type $\{S_1, S_2, \ldots, S_k\}$. That is, forbidding some stars from this 'sequential' set gives an NP-complete problem. This arises the question if it is possible to recover polynomiality by allowing some other non-star graphs to be components of the packing. This paper shows two types of graph sets which can be added to a non-sequential set of stars to maintain polynomiality. These new graphs are trees constructed from a star by replacing some of its leaves by forbidden stars of the packing. For both of these packing problems we show Edmonds-type algorithms implying Berge-type theorems, and the matroidality of the packings. In one of the Edmonds-type algorithms the alternating forest may overlap itself. We use reductions to the $\mathcal{H}$-factor problem of Lovász.

Keywords: graph packing, $\mathcal{H}$-factor

1 Introduction

Let $\mathcal{F}$ be a family of graphs. An *$\mathcal{F}$-packing* of a graph G is a subgraph of G, components of which are isomorphic to members of $\mathcal{F}$. An $\mathcal{F}$-packing is called *maximum* if it covers a maximum number of vertices of G and it is called *perfect* if it covers every vertex of G. Several authors (eg. [2, 3, 5, 6, 8, 9]) studied this kind of packing problem. Polynomial time algorithms for finding perfect or maximum $\mathcal{F}$-packings have been found for various special families $\mathcal{F}$ of graphs. For example, the $\mathcal{F}$-packing problem is polynomial if $\mathcal{F}$ consists of K_2 and a set of factor-critical graphs, see Cornuéjols, Hartvigsen, Pulleyblank [2, 3].

Las Vergnas proved that the $\{S_1, \ldots, S_k\}$-packing problem is polynomial [7], where S_i is an *i-star*, ie. a simple graph with a specified vertex, called *center*, whose deletion

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results in a graph consisting of i isolated vertices, called *leaves*. On the contrary, Hell and Kirkpatrick [6] proved that the $\mathcal{F} = \{S_i : i \in H\}$-packing problem is NP-complete whenever $H \subseteq \mathbb{N}$ is not of the form $\{1, 2, \ldots, k\}$. This arises a question: is it possible to add some other non-star graphs to $\mathcal{F} = \{S_i : i \in H\}$ to recover polynomiality?

There are some known results in this flavor. Eg. adding all trees with all degrees odd to $\mathcal{F} = \{S_1, S_3, S_5, \ldots\}$ results in the $\{1, 3, 5, \ldots\}$-factor problem, which is polynomial by an easy reduction to the matching problem. Another example is the NP-complete $\{S_k\}$-packing problem for $k \geq 2$. In this case the addition of all trees with highest degree exactly k results in a polynomial packing problem [4]. Finally, if $H = \{l, l+1, \ldots, u\}$ with $l \geq 2$ then by adding to $\mathcal{F}$ all trees with highest degree between l and u we obtain a polynomial packing problem [4].

An integer h is called a *gap* of $H \subseteq \mathbb{N}$ if $h \notin H$ but H contains an integer at least h and an integer at most h. Moreover, H is said to have *no two consecutive gaps* if $\min H \leq i \leq \max H$, $i \notin H$ implies $i + 1 \in H$.

In all of the above examples, an infinite set of new graphs is added to $\mathcal{F} = \{S_i : i \in H\}$. This paper shows that if $1 \in H$ and H has no two consecutive gaps, then certain finite sets of so-called 'superstars' can be added to $\mathcal{F}$ to maintain polynomiality. Note that $1 \in H$ ensures that $K_2 \in \mathcal{F}$ holds.

Definition 1.1. Let $H \subseteq \mathbb{N}$. The i-star is *forbidden* if i is a gap of H. For the integers $s \geq 0$, $t \geq 1$, a graph is said to be an (s, t)-*superstar* if it is constructed as follows: connect the center of an s-star to the centers of t forbidden stars. In a superstar every vertex inherits the notation center or leaf, except the center of the starting s-star, which is called the *supercenter* (or an (s, t)-*supercenter*).

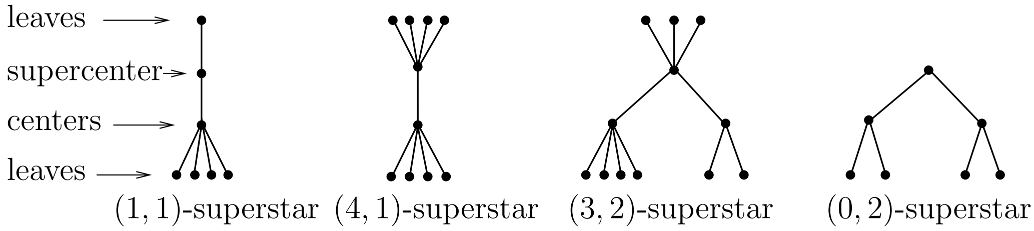


Figure 1: Examples of superstars, $H = \{1, 3, 5\}$

We introduce two types of sets consisting of superstars.

Definition 1.2. Let $H \subseteq \mathbb{N}$ and denote $u = \max H$. For $1 \leq b \leq u - 1$ let

$$\mathcal{S}_{H,b} = \{S_i : i \in H\} \cup \{(s,t)\text{-superstars}: 1 \leq s + t \leq u,\ 1 \leq t \leq b\}.$$

For $1 \leq b \in \mathbb{N}$ let

$$\mathcal{C}_{H,b} = \{S_i : i \in H\} \cup \{(s,t)\text{-superstars}: 0 \leq s \leq u,\ 1 \leq t \leq b\}.$$

For the case when $1 \in H \subseteq \mathbb{N}$ has no two consecutive gaps, we present polynomial Edmonds-type alternating forest algorithms for both of the $\mathcal{S}_{H,b}$ and $\mathcal{C}_{H,b}$-packing problems. These are direct generalizations of the matching algorithm of

Edmonds. Our algorithm solving the $\mathcal{C}_{H,b}$-packing problem allows '2-fold overlapping' trees, where a tree may cover a vertex twice. These algorithms imply Berge-type min-max formulae for the $\mathcal{S}_{H,b}$ and $\mathcal{C}_{H,b}$-packing problems. They also imply that these packing problems are matroidal (an $\mathcal{F}$-packing problem is matroidal if in every graph G the vertex sets coverable by $\mathcal{F}$-packings form a matroid).

Note that if $H = \{1, \ldots, u\}$ then $\mathcal{S}_{H,b} = \mathcal{C}_{H,b} = \{S_i : 1 \leq i \leq u\}$ yielding the star packing problem of Las Vergnas [7]. Hence both types of the superstar packing problems contain the classical matching problem. Observe that in both cases there are several classes of graphs (one for each b), which can be added to $\{S_i : i \in H\}$ to recover polynomiality.

Our alternating forest algorithms need a subroutine of deciding if certain special graphs has perfect $\mathcal{S}_{H,b}$ ($\mathcal{C}_{H,b}$)-packings. We do this by a reduction to the $\mathcal{H}$-factor problem in Sections 6 and 7.

The *$\mathcal{H}$-factor problem* was introduced by Lovász [10]. Let G be an undirected graph with degree-prescriptions $H_v \subseteq \mathbb{N}$ for all $v \in V(G)$. For a subgraph F of G define $\delta^F(v) = \min\{|\deg_F(v) - h| : h \in H_v\}$ and let $\delta^F = \sum\{\delta^F(v) : v \in V(G)\}$. The minimum δ^F among the subgraphs F is denoted by $\delta_{\mathcal{H}}(G)$. A subgraph F is called *$\mathcal{H}$-optimal* if $\delta^F = \delta_{\mathcal{H}}(G)$ and it is an *$\mathcal{H}$-factor* if $\delta^F = 0$, ie. if $\deg_F(v) \in H_v$ for all vertices $v \in V(G)$. The *$\mathcal{H}$-factor problem* is to determine the value of $\delta_{\mathcal{H}}(G)$. Lovász [10, 11] developed a structure theory for the $\mathcal{H}$-factor problem in case H_v has no two consecutive gaps for any $v \in V(G)$. He proved that the $\mathcal{H}$-factor problem is NP-complete without this restriction. Later, Cornuéjols [1] proved that the $\mathcal{H}$-factor problem is polynomial in case H_v has no two consecutive gaps for all $v \in V(G)$. The restriction that H has no two consecutive gaps in the $\mathcal{S}_{H,b}$ and $\mathcal{C}_{H,b}$-packing problems is related to these facts.

In Section 7 we show an alternative algorithm for the $\mathcal{C}_{H,b}$-packing problem by a direct reduction to the $\mathcal{H}$-factor problem using the Edmonds-Gallai decomposition of the graph. Also this approach implies matroidality.

Throughout the paper all graphs are simple and undirected. If G is a graph and $U \subseteq V(G)$ then *shrinking* the set U results in the graph G/U obtained from $G - U$ by joining a new vertex (called *shrunk* vertex) to each vertex of $V(G) - U$ which is adjacent to U in G. $\delta(U)$ denotes the set of edges $e \in E(G)$ joining U to $G - U$. A graph F is called factor-critical if it has no perfect matching but $F - v$ has one for all $v \in V(F)$.

For the rest of the paper, we assume H and b to be fixed.

2 Results

In this section we state Berge-type min-max theorems for the $\mathcal{S}_{H,b}$ and for the $\mathcal{C}_{H,b}$-packing problems. The proofs of these theorems depend on the algorithm and will be presented in Sections 4 and 5.

In our packing problems the role of factor-critical graphs is played by graphs we call hedgehogs.

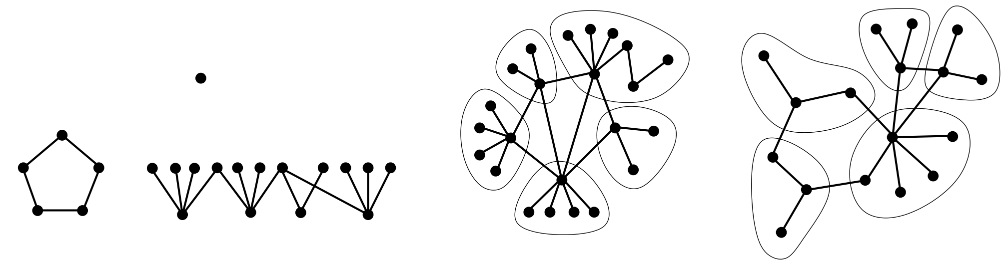
Definition 2.1. A tree is a *star-tree* if, viewed as a bipartite graph with color classes

A and D, the degree of each vertex in A is a gap of H. A connected graph W is a *small hedgehog* if either W is a star-tree or W is an odd cycle and $2 \notin H$.

One may easily find out that a small hedgehog has neither perfect $\mathcal{S}_{H,b}$-packing nor perfect $\mathcal{C}_{H,b}$-packing.

Definition 2.2. A graph W is a *factor-critical union of small hedgehogs* $W_1, \ldots, W_{2l+1}$ if W_i is an induced subgraph of W for $1 \leq i \leq 2l+1$, $V(W_1), \ldots, V(W_{2l+1})$ partition $V(W)$, and shrinking every W_i results in a factor-critical graph. $\mathcal{P} = \{W_1, ..., W_{2l+1}\}$ is a *decomposition of* W and W_i are the *small hedgehogs of* $\mathcal{P}$.

Definition 2.3. A factor-critical union of small hedgehogs is an $\mathcal{S}_{H,b}$*-hedgehog* if it has no perfect $\mathcal{S}_{H,b}$-packing, and it is a $\mathcal{C}_{H,b}$*-hedgehog* if it has no perfect $\mathcal{C}_{H,b}$-packing.



small hedgehogs, $H = \{1, 3, 5\}$ $\qquad$ $\mathcal{S}_{H,b}$ and $\mathcal{C}_{H,b}$-hedgehogs, $H = \{1, 3, 5\}$, $b = 1$

Figure 2: Examples of small hedgehogs and hedgehogs

The next two definitions are meant both for the $\mathcal{S}_{H,b}$ and for the $\mathcal{C}_{H,b}$-packing problems.

Definition 2.4. Let W be a hedgehog and $v \in V(W)$. If $W - v$ has a perfect packing then v is called *free*. If there exists a forbidden star subgraph S of W centered at v such that $W - S$ has a perfect packing then v is called *fixed*.

Lemma 3.7 and Corollary 3.12 will imply that given a vertex v of a hedgehog, we can decide in polynomial time if v is free and if v is fixed. Observe that in a small hedgehog all vertices of an odd cycle are both free and fixed, while in a star-tree the vertices of A are fixed and not free, while the vertices of D are free and not fixed. Proposition 3.6 implies that each vertex of a hedgehog is either fixed or free or both.

We prove the following Berge-type theorems.

Definition 2.5. For $D \subseteq V(G)$ let $h_{\mathcal{S}}(D)$ (resp. $h_{\mathcal{C}}(D)$) denote the number of $\mathcal{S}_{H,b}$ (resp. $\mathcal{C}_{H,b}$)-hedgehog components of $G[D]$. Furthermore, let $\Gamma^{fi}(D)$ (resp. $\Gamma^{fr}(D)$) denote those vertices in $V(G) - D$ which are adjacent to a fixed (resp. free) vertex of a hedgehog component of $G[D]$. Finally, let $def_{\mathcal{S}}(G)$ (resp. $def_{\mathcal{C}}(G)$) denote the minimum number of vertices uncovered by an $\mathcal{S}_{H,b}$ (resp. $\mathcal{C}_{H,b}$)-packing of G.

Theorem 2.6. *If G is a simple graph, $1 \in H$ has no two consecutive gaps and $1 \leq b \leq u - 1$ then*

$$def_{\mathcal{S}}(G) = \max_{D \subseteq V(G)} \left\{ h_{\mathcal{S}}(D) - b|\Gamma^{fi}(D) \setminus \Gamma^{fr}(D)| - u|\Gamma^{fr}(D)| \right\}.$$

Theorem 2.7. *If G is a simple graph, $1 \in H$ has no two consecutive gaps and $1 \leq b \in \mathbb{N}$ then*

$$def_{\mathcal{C}}(G) = \max_{D \subseteq V(G)} \left\{ h_{\mathcal{C}}(D) - b|\Gamma^{fi}(D)| - u|\Gamma^{fr}(D)| \right\}.$$

Unlike in the previous polynomial packing problems, deciding the existence of a perfect $\mathcal{S}_{H,b}$ ($\mathcal{C}_{H,b}$)-packing is not much easier for factor-critical unions of small hedgehogs than for general graphs, see Sections 6 and 7. In Thms. 4.3 and 7.4 we prove that both packing problems are matroidal.

3 Hedgehogs

In this section we simply use the terms 'hedgehog' and 'packing' because the results presented here apply both for the $\mathcal{S}_{H,b}$ and $\mathcal{C}_{H,b}$-packing problems.

In any packing we assume that for each vertex v it is determined if v is a leaf, a center or the supercenter of its component K. Ambiguity may occur in the following cases. If K is isomorphic to a star S_i with $i-1$ a gap of H then it is determined if K is viewed as an i-star or as a $(0,1)$-superstar with a distinguished one-degree vertex the supercenter. If K is a K_2 component then one specified vertex is the center and the other one is the leaf. Finally, let K be an $(s,1)$-superstar consisting of two stars S_s and S_j with an edge connecting their centers (we also call it a *bi-star* $B_{s,j}$). If s is a gap of H then it is determined which one of the two centers is the supercenter.

Whenever we make operations on a packing we use *cuttings*.

Definition 3.1.

1. *Cutting* a leaf v in an (s,t)-superstar with $t \geq 2$ means deleting v, and in case v was connected to a center c, deleting the edge joining c to the supercenter.
2. *Cutting* a leaf v in an (s,t)-superstar with $t = 1$ and $s \geq 1$ (a bi-star) means deleting v, and in case v was connected to a center c, deleting the edge joining c to the supercenter. However, there is one exception: if v is connected to a center c and s is a gap of H then delete v and appoint c to be the supercenter.
3. *Cutting* a leaf v in a star S_i means deleting v if $i-1 \in H$. If $i-1 \notin H$ then cutting v in S_i means deleting v and one more vertex of degree 1 (possibly the supercenter or the center when $i = 1$). In some applications it may occur that this is not allowed for some reasons. In this case appoint v to be the supercenter (to the center when $i = 1$).

In each of the above cases cutting a leaf edge means cutting its leaf vertex.

Definition 3.2. Let G be a graph and $D \subseteq V(G)$. A star and superstar packing Q of G is called *D-nice* if all leaves of all components of Q are contained in D.

Now we pay attention to the structure of packings in a hedgehog.

Definition 3.3. If W is a factor-critical union of small hedgehogs with decomposition $\mathcal{P}$ then a vertex $v \in V(W)$ is called *free* (resp. *fixed*) *in* $\mathcal{P}$ if it is free (resp. fixed) in a small hedgehog of $\mathcal{P}$. $D_{\mathcal{P}} \subseteq V(W)$ denotes the set of free vertices in $\mathcal{P}$. Let $A_{\mathcal{P}} = V(W) - D_{\mathcal{P}}$.

Observe that a vertex is either free in $\mathcal{P}$ or fixed in $\mathcal{P}$ or both.

Definition 3.4. A decomposition $\mathcal{P}$ of a hedgehog W is *standard* if $uv \notin E(W)$ whenever u and v are free in $\mathcal{P}$ and contained in different small hedgehogs of $\mathcal{P}$.

Lemmas 3.7 and 3.8 will imply that each hedgehog has a standard decomposition $\mathcal{P}$. A standard decomposition $\mathcal{P}$ has the property that $v \in V(W)$ is free (fixed) in $\mathcal{P}$ if and only if it is free (fixed), see Corollary 3.12. The following two statements are easy to check.

Proposition 3.5. *Connecting two small hedgehogs by an edge gives a graph with a perfect packing. This implies that connecting two hedgehogs by an edge gives a graph with a perfect packing.*

Proposition 3.6. *Let W be a hedgehog with decomposition $\mathcal{P}$ and let $v \in V(W)$. If v is free in $\mathcal{P}$ then $W - v$ has a $D_{\mathcal{P}}$-nice perfect packing. If v is fixed in $\mathcal{P}$ then W has a forbidden star subgraph S centered at v such that $W - S$ has a $D_{\mathcal{P}}$-nice perfect packing. Both packings may be chosen to consist of stars and bi-stars $B_{i,j}$ with i, j gaps of H.*

Lemma 3.7. *Let W be a factor-critical union of small hedgehogs with decomposition $\mathcal{P}$ and v_1, v_2 be two adjacent vertices which are free in $\mathcal{P}$, in two distinct small hedgehogs of $\mathcal{P}$. Then in polynomial time we may find either a perfect packing of W or a standard decomposition of W.*

Proof. For $i = 1, 2$, let Q_i be a perfect packing of $W - v_i$ guaranteed by Proposition 3.6. Q_i can clearly be chosen such that $\deg_Q(v_{3-i}) = 1$ holds. Observe that a perfect packing of W can be trivially constructed from Q_i unless $2 \notin H$ and Q_i covers v_{3-i} by a copy of K_2 for $i = 1, 2$. Consider a path P of maximal length starting in v_1 and containing alternately edges of $E(Q_2) \setminus E(Q_1)$ and $E(Q_1) \setminus E(Q_2)$ leading to vertices of degree one in Q_2 and Q_1, respectively. Algorithmically, if we traverse along such a walk P edge by edge, then we cannot cover a vertex more than once so P is really a path, which ends when the last vertex has no neighbor in Q_i of degree 1.

If P has odd length then its last vertex z is a leaf in a non-K_2 component K of Q_1 so Q_1 can be augmented to a perfect packing by swapping edges and non-edges along P and cutting z in K.

If P has even length then its last vertex z is either a leaf in a non-K_2 component K of Q_2 or it is uncovered by Q_2. If z is a leaf then Q_2 may be augmented to a perfect packing of W by swapping edges and non-edges along $v_2v_1 + P$ and cutting z in K.

If z is uncovered by Q_2 then $z = v_2$ and $C = P \cup \{v_1v_2\}$ is an odd cycle. Assume that there exists a non-K_2 component of Q_i intersecting C for some $i = 1, 2$. Let w be the last vertex of P which is a leaf in a non-K_2 component K of Q_i and let

$P[w, v_2]$ denote the section of P from w to v_2. If $i = 1$ (resp. $i = 2$) then cutting w in K and swapping edges and non-edges of Q_i along $P[w, v_2] + v_2v_1$ (resp. $P[w, v_2]$) transforms Q_i to a perfect packing of W. So we can assume that all components of Q_i intersecting C are copies of K_2. Suppose that there exists a small hedgehog W' of $\mathcal{P}$ intersecting both C and $W - C$. We may assume that $E(C) \cap E(Q_1)$ has an edge leaving W'. It is easy to see that Q_1 has an s-star component S with $s \leq u - 2$ such that $V(S) \subseteq V(W') \backslash V(C)$ and the center of S is adjacent to C. Now $Q_1 - V(S) - V(C)$ is a perfect packing of $W - V(S) - V(C)$. Moreover, $W[V(S) \cup V(C)]$ has a perfect packing with K_2-components and one $B_{s,2}$. If C is not induced in W then $W[V(C)]$ has a perfect $\{K_2, B_{1,2}\}$-packing. Otherwise, let C be a new small hedgehog in a new decomposition $\mathcal{P}'$. Iterate the above procedure till $\mathcal{P}'$ is standard. □

Lemma 3.8. *If $u \neq v$ are two non-adjacent vertices of a small hedgehog W then either $W + uv$ has a perfect packing (which can be found in polynomial time) or $W + uv$ is a factor-critical union of small hedgehogs (and a standard decomposition of it can be presented in polynomial time).*

Proof. If W is an odd cycle then $W + uv$ has a perfect packing using K_2-components and one bi-star $B_{1,2}$. If W is a star-tree then $W+uv$ contains a cycle C. Let $E' = E(C)$ if both u and v are free or both are fixed and let $E' = E(C) - uv$ otherwise. The graph $W' = (W + uv) - E'$ has an odd number of components, each of them either is a small hedgehog or has a perfect packing. It is easy to check that if a component of W' has a perfect packing then also $W + uv$ has one. Otherwise $W + uv$ is a factor-critical union of small hedgehogs. A standard decomposition of it can be found by Lemma 3.7. □

Definition 3.9. For $A \subseteq V(G)$ let D_A denote the set of vertices which are either isolated in $G - A$ or, in case $2 \notin H$, belong to an odd cycle component of $G - A$.

Observe that if $\mathcal{P}$ is a standard decomposition of a hedgehog then $D_{A_{\mathcal{P}}} = D_{\mathcal{P}}$.

Lemma 3.10. *If $A \subseteq V(G)$ in a simple graph G and Q is a packing of G then we can find in polynomial time a D_A-nice packing entering every component of $G[D_A]$ entered by Q.*

Proof. First delete all components of Q which are disjoint from D_A. Do the following cuttings till we get a D_A-nice packing. Let $v \notin D_A$ be a leaf in a component L of Q. First, let L be an i-star. If $i - 1 \in H$ then cutting v does not disconnect any component of $G[D_A]$ from A. If $i - 1 \notin H$ then there are two cases. If L has one more leaf $w \notin D_A$ then delete both v and w. Else all other leaves are in D_A, so appoint v to be the supercenter of L (to be the center if $i = 1$). Assume that L is an (s, t)-superstar with $s + t \geq 2$. Cutting $v \notin D_A$ may disconnect a component C of $G[D_A]$ from A only if $2 \notin H$, C is an odd cycle, and

1. either $t = 1$ and the supercenter z together with its (1 or 2) adjacent leaves are contained in C,
2. or a center of L, $c \in V(C)$ has exactly 2 adjacent leaves: $v \in A$ and $u \in V(C)$.

In case 1. cut a leaf neighbor of z, and in case 2. cut u. □

Lemma 3.11. *Let G be a simple graph, $A \subseteq V(G)$ and suppose that G has a matching M such that each vertex of A is joined by an edge of M to distinct components of $G[D_A]$. If G has a D_A-nice packing Q then we can find in polynomial time a packing Q' of G which covers*

- *A*
- *every component of $G[D_A]$ entered by Q and*
- *all but at most one vertex in every component of $G[D_A]$ not entered by Q.*

Moreover, $G - A - D_A$ contains only $(0,t)$-supercenters of Q'.

Proof. Delete all components of Q which do not meet both D_A and A.

We need to extend the operation of cutting. In the case $2 \notin H$ call an edge *bad* if it joins the supercenter z of an (s,t)-superstar S to a center c of S with $\deg_S(c) = 3$ such that c together with both of its leaves u_1, u_2 are contained in an odd cycle component C of $G[D_A]$. *Cutting* the bad edge cz in S means to delete c, u_1 and u_2. However, this is not allowed when $s \notin H$ and $t = 1$. If $s = 0$ then delete the whole S, while if s is a gap of H then delete u_1 and u_2 and appoint c to be the supercenter.

Let C be an odd cycle component of $G[D_A]$ and L be a component of Q meeting C. If L is a superstar with supercenter $z \in V(C)$ then delete all leaves of z. Otherwise try to cut the edges of $E(L) \cap \delta(C)$ without disconnecting any component of $G[D_A]$ from A in Q. Do this procedure for all components L of Q meeting an odd cycle C. After doing this we may assume that either C contains only $(0,t)$-supercenters or C is joined by only one edge of Q to A which is either a leaf or bad. This assumption trivially holds for single vertex components of $G[D_A]$.

Now we cover A. Do the next procedure until there is a vertex $v \in A \setminus V(Q)$. Let $e \in M$ be the edge joining v to a component C of $G[D_A]$. If C is joined by only one edge $f \in E(Q)$ to A then cut f. Next add e to Q. The addition of e either creates a K_2 component of Q or adjoins v as a leaf to a $(0,t)$-supercenter. Observe that $M \cap E(Q)$ strictly increases at each of these steps, so finally A will be covered by Q.

If C is a component of $G[D_A]$ not entered by Q then add a perfect matching of $C - v$ to Q for some $v \in V(C)$. Let C be an odd cycle component of $G[D_A]$ entered by Q. There are three possibilities.

- All vertices in $V(C) \cap V(Q)$ are $(0,t)$-supercenters and at most one $(1,t)$-supercenter. Choose $v \in V(C)$ to be the $(1,t)$-supercenter (if any).
- Only one vertex of C is covered by Q, a leaf v.
- C is entered by only one edge of Q, which is a bad edge with center $c \in V(C)$ and leaves u_1, $u_2 \in V(C)$. Choose $v = u_1$.

In all three cases add a perfect matching of $C - v$ to Q, moreover, in the first case delete those edges of the above matchings which glued the supercenters of two superstars. This is the packing Q'.

Finally, observe that the above procedures do not spoil the property of the original Q that $G - A - D_A$ contains only $(0, t)$-supercenters. □

Recall that if W is a factor-critical union of small hedgehogs with a standard decomposition $\mathcal{P}$ then $D_{A_\mathcal{P}} = D_\mathcal{P}$, and note that W has a matching M such that each vertex of $A_\mathcal{P}$ is joined by an edge of M to distinct components of $W[D_\mathcal{P}]$. Hence it is possible to apply Lemma 3.11 to W with the choice $A = A_\mathcal{P}$.

Corollary 3.12. *Let W be a hedgehog with a standard decomposition $\mathcal{P}$. $v \in V(W)$ is free (fixed) in $\mathcal{P}$ if and only if it is free (fixed).*

Proof. By Proposition 3.6, if v is free (fixed) in $\mathcal{P}$ then it is free (fixed). If v is non-free in $\mathcal{P}$ (ie. $v \in A_\mathcal{P}$) and $W - v$ has a perfect packing Q, then Lemmas 3.10 and 3.11 with the choice $A = A_\mathcal{P}$ give a perfect packing of W, which is impossible. Assume that v is non-fixed in $\mathcal{P}$ and there exists a forbidden star S centered at v such that $W - S$ has a perfect packing. Note that all neighbors of v in W are non-free in $\mathcal{P}$. Let u be a leaf of S. Now $W - u$ has a perfect packing contradicting to the previous part of the proof. □

Corollary 3.12 implies that all neighbors of a non-fixed vertex of a hedgehog are non-free. The following corollary of Lemmas 3.10 and 3.11 will be needed in the proofs of Theorems 2.6 and 2.7. def means both $def_\mathcal{S}$ and $def_\mathcal{C}$. Note that *2.* is implied by *1.*

Corollary 3.13. *Let W be a hedgehog and let either $X = \{y\}$ for a fixed vertex y or $X = V(T)$ for a forbidden star subgraph T of W centered at a non-fixed vertex.*

1. *If $def(W - X - v) < def(W - X)$ for $v \in V(W) - X$ then v is free in W.*
2. *If $def(W - X - S) < def(W - X)$ for a forbidden star subgraph S of $W - X$ then S is centered at a fixed vertex.*

4 An algorithm for the $\mathcal{S}_{H,b}$-packing problem

In this section we present an Edmonds-type algorithm for solving the $\mathcal{S}_{H,b}$-packing problem in the input graph G. The algorithm uses a subroutine of deciding if a factor-critical union of small hedgehogs has a perfect $\mathcal{S}_{H,b}$-packing. This recognition subroutine is contained in Section 6. We say that (S^+, S, Z) is an *alternating structure* if S^+ is a subgraph of G, S is an induced subgraph of S^+ and Z is an independent set of vertices of S, called the *odd vertices*. Besides, the following properties are satisfied:

- Every connected component of $S - Z$ is a hedgehog, called a *hedgehog of S*.
- The graph S' obtained from S by shrinking the hedgehogs of S is a forest.

- The odd vertices of S are of the following types.
  - A *fixed odd vertex* lies on exactly $b+1$ edges of S leading to fixed vertices of distinct hedgehogs of S.
  - A *free odd vertex* lies on exactly $u+1$ edges of S leading to distinct hedgehogs of S such that at most b of them lead to non-free vertices.
- Every component of $S^+ - S$ is a singleton joined by exactly one edge to S. These isolated vertices are called *outside vertices*. A fixed odd vertex is adjacent to at most $u - b$ outside vertices. The other vertices of $V(S)$ are adjacent to no outside vertices.

If T is a connected component of S then the corresponding component of S^+ is denoted by T^+. Due to Proposition 3.6, $T^+ - W$ has a perfect $\mathcal{S}_{H,b}$-packing for any hedgehog W of T. Thus $def_{\mathcal{S}}(S^+)$ is at most the number of components of S^+.

The algorithm maintains an alternating structure (S^+, S, Z), standard decompositions of the hedgehogs of S and a perfect $\mathcal{S}_{H,b}$-packing Q of $G - V(S^+)$. In each step, it either increases $E(S)$ (*growing*) or deletes a component of S^+ (and *augments* Q at the same time). When stopping, each hedgehog of S will be a connected component in $G - Z$ and each fixed odd vertex will be adjacent to only non-free vertices of hedgehogs of S. Thus with the notation $D = V(S) - Z$ the number of components of S^+ will be exactly $h_{\mathcal{S}}(D) - b|\Gamma^{fi}(D) \setminus \Gamma^{fr}(D)| - u|\Gamma^{fr}(D)|$. So Thm. 2.6 will imply that the following $\mathcal{S}_{H,b}$-packing is maximum in G: take Q together with perfect $\mathcal{S}_{H,b}$-packings of $T^+ - w$ for each component T^+ of S^+ where w is free in a hedgehog of T.

Algorithm for the $\mathcal{S}_{H,b}$-packing problem

Step 0 (Initialization). Start with any $\mathcal{S}_{H,b}$-packing Q of G. Go to step 1.

Step 1 (Optimality test). Let $V(S^+) = V(S) = V(G) - V(Q)$ and $E(S^+) = Z = \emptyset$. If Q is perfect, stop. Go to step 2.

Step 2 (Edge selection and augmentation or growing). Look for an edge xy, not of a hedgehog of S, such that x is free in a hedgehog W of S and y is not a free odd vertex, or x is non-free in a hedgehog W of S and $y \notin Z$. If no such edge exists, stop: each hedgehog of S is a connected component in $G - Z$ and each fixed odd vertex is adjacent to only non-free vertices of hedgehogs of S. Otherwise, denote the component of S containing W by T and distinguish the following cases.

Case 1. $y \in V(W)$. By Lemmas 3.7, 3.8 and by the recognition subroutine we can either obtain a perfect $\mathcal{S}_{H,b}$-packing of $W + xy$ or conclude that $W + xy$ is a hedgehog and obtain a standard decomposition of it. If $W + xy$ has a perfect $\mathcal{S}_{H,b}$-packing Q' then add Q' and a perfect $\mathcal{S}_{H,b}$-packing of $T^+ - W$ to Q, delete T^+ and go to step 1. Otherwise grow S by adding xy (the vertices free in W are also free in $W + xy$), and continue at step 2.

Case 2. y is in a hedgehog of T different from W. In this case, adding the edge xy creates a cycle C in the shrunk graph S'. Let T_C be the subgraph of $T + xy$ consisting of $Z \cap V(C)$ and of all hedgehogs and outside vertices of S^+ neighboring in S^+ to $Z \cap V(C)$. Note that $T^+ - T_C$ has a perfect $\mathcal{S}_{H,b}$-packing.

Every component of $T_C - E(C)$ is a hedgehog or consists of an odd vertex z connected by edges to hedgehogs and outside vertices. If such a latter component has a perfect $\mathcal{S}_{H,b}$-packing then T_C and hence $T^+ + xy$ has a perfect $\mathcal{S}_{H,b}$-packing so we may augment Q, delete T^+ and go to step 1.

Otherwise it is easy to see that every odd vertex z is connected to either i outside vertices with $i \notin H$ and $b = 1$ or to $u-1$ non-fixed vertices of neighboring hedgehogs with $u-1 \notin H$. So T_C is a factor-critical union of small hedgehogs. Take a standard decomposition of T_C by Lemmas 3.7 and 3.8. By the recognition subroutine we can decide if T_C has a perfect $\mathcal{S}_{H,b}$-packing. If it has then augment Q and go to step 1. If not then grow S by adding xy and updating $Z := Z \setminus V(T_C)$. Note that the free vertices of hedgehogs of $T_C - Z - xy$ are free in T_C by definition. Go to step 2.

Case 3. y is in a hedgehog W_y of a component $T_y \neq T$ of S. Augment Q by adding perfect $\mathcal{S}_{H,b}$-packings of $T - W$, $T_y - W_y$ and $W + xy + W_y$. Go to step 1.

Case 4. x is free in W and y is a fixed odd vertex of a component T_y of S. ($T_y = T$, and even $xy \in E(S)$ is possible.) Let $X \subseteq V(S^+)$ be the set of outside vertices of S^+ adjacent to y. If $|X| < u - b$ then delete x from T^+, join x as an outside vertex to y in S^+ and take a perfect $\mathcal{S}_{H,b}$-packing of the new component T^+. Go to step 1.

If $|X| = u - b$ then let v be a non-free neighbor of y in T_y such that the shrunk graph $(S - yv + xy)'$ is still a forest. Delete yv and add xy in S and S^+, and join in S the vertices of X to y. y becomes a free odd vertex of S. Go to step 2.

Case 5. y is an outside vertex of S. Delete y from S^+ and take a perfect $\mathcal{S}_{H,b}$-packing of $T^+ + xy$. Go to step 1.

Case 6. $y \notin V(S^+)$ is covered by component L of Q. For sake of simplicity, here an i-star component of Q with $i-1$ a gap of H is meant as a star and as a superstar at the same time. Similarly, the center of a K_2-component and the supercenter of a $B_{i,j}$-component with i, j gaps of H are not determined.

- x is free in W and y is neither the center of an i-star with $i+1 \notin H$, nor an (s,t)-supercenter with $s + t = u$. Delete T^+ from S^+ and replace L by perfect $\mathcal{S}_{H,b}$-packings of $T^+ - x$ and $L + xy$. Go to step 1.

- x is non-fixed in W and L is an i-star component centered at y with $i+1$ a gap of H. Grow S by adding $L + xy$ to W. Go to step 2.

- x is fixed in W and y is neither the center of a u-star, nor an (s,t)-supercenter such that either $s + t = u$ or $t = b$. Delete T^+ from S^+ and replace L by perfect $\mathcal{S}_{H,b}$-packings of $T^+ - W$ and $W + xy + L$. Go to step 1.

- x is non-free in W and y is an (s,t)-supercenter with $t = b$. Add $xy + L$ to S^+ and add y to Z as a fixed odd vertex. The isolated vertices of $L - y$ will be outside vertices. Go to step 2.

- y is either the center of a u-star or an (s,t)-supercenter with $s + t = u$ such that either $t < b$ or x is free in W. Add $xy + L$ to S and add y to Z as a free odd vertex. The components of $L - y$ will be hedgehogs of S. Go to step 2.

Now we prove the Berge-type theorem for the $\mathcal{S}_{H,b}$-packing problem which implies the validity of this algorithm. Note that $\Gamma(D) = \Gamma^{fi}(D) \cup \Gamma^{fr}(D)$ for $D \subseteq V(G)$ by Proposition 3.6. First we need Lemma 4.2.

Definition 4.1. For $D \subseteq V(G)$ let $s_G(D) = h_\mathcal{S}(D) - b\,|\Gamma^{fi}(D) \setminus \Gamma^{fr}(D)| - u\,|\Gamma^{fr}(D)|$. We say that G has the *Berge-property* if $def_\mathcal{S}(G) \geq \max\{s_G(D) :\ D \subseteq V(G)\}$.

Lemma 4.2. *Assume that all graphs with less vertices than G has the Berge-property. Let Q be an $\mathcal{S}_{H,b}$-packing of G, $D \subseteq V(G)$ be a vertex set and $x \in \Gamma(D)$ be a vertex adjacent in Q to more than b non-free vertices of distinct hedgehogs of D. Then there exist vertex sets $\emptyset \neq X \subseteq V(G)$ and $D' \subseteq V(G) - X$ such that $s_{G-X}(D') > s_G(D)$ and $def_\mathcal{S}(G - X) \leq |V(G) - V(Q)|$.*

Proof. At least one non-free neighbor y of x is a leaf in the component L of Q covering x. $L - y$ has a perfect $\mathcal{S}_{H,b}$-packing unless L is an i-star with $i - 1$ a gap of H. In this case choose another leaf y' of L such that y' is non-free in a different component of $G[D]$ than y, and apply the procedure below to y' as well. Since $L - y$ (or $L - y - y'$) has a perfect $\mathcal{S}_{H,b}$-packing, with the choice $X = \{y\}$ (or $X = \{y, y'\}$) we have $def_\mathcal{S}(G-X) \leq |V(G) - V(Q)|$.

Let the hedgehog containing y be W. Let i be minimal with the property that there exists an i-star subgraph S of W centered at y such that $W - S$ has a perfect $\mathcal{S}_{H,b}$-packing Q'. $i \geq 1$ by definition and W has no perfect $\mathcal{S}_{H,b}$-packing so even $i \geq 2$. Run the above algorithm on the $\mathcal{S}_{H,b}$-packing Q' of $W - y$. Since $W - y$ has no $\mathcal{S}_{H,b}$-packing P with $V(P) \supsetneq V(Q')$ it is easy to see that the algorithm outputs a set $D'' \subseteq V(W) - y$ with $s_{W-y}(D'') = |V(W) - y| - |V(Q')| \geq 2$. This implies that Q' is a maximum $\mathcal{S}_{H,b}$-packing of $W - y$ since $W - y$ has the Berge-property. Observe that $D' = D \cup D''$ would do if no vertex in $\Gamma^{fi}_G(D) \setminus \Gamma^{fr}_G(D)$ has a free neighbor in D''. Let v be a free vertex in a hedgehog of $W[D'']$. The alternating structure S^+ in the end of the algorithm (run on $W - y$) shows that $def_\mathcal{S}(W - y - v) < |V(W) - y| - |V(Q')| = def_\mathcal{S}(W - y)$. So v is already free in W by Lemma 3.13, *1*. □

Proof. (**Thm. 2.6**) If Q is a maximum $\mathcal{S}_{H,b}$-packing of G then the algorithm finds a set $D \subseteq V(G)$ such that $s_G(D) = |V(G) - V(Q)| = def_\mathcal{S}(G)$.

On the other hand, let G be the smallest graph in which there exists a set $D \subseteq V(G)$ such that $s_G(D) > def_\mathcal{S}(G)$. Let Q be a maximum $\mathcal{S}_{H,b}$-packing of G. Clearly, some $x \in \Gamma(D)$ is adjacent in Q to more than b non-free vertices of distinct hedgehogs of D. So applying Lemma 4.2 to Q, D and x gives a contradiction. □

Thm. 2.6 implies that our algorithm gives a maximum $\mathcal{S}_{H,b}$-packing of the input graph G. Note that Thm. 2.6 and the run of the algorithm imply also that if there exists no $\mathcal{S}_{H,b}$-packing P with $V(P) \supsetneq V(Q)$ then Q is a maximum $\mathcal{S}_{H,b}$-packing of G.

Theorem 4.3. *The $\mathcal{S}_{H,b}$-packing problem is matroidal.*

Proof. Run the algorithm on a maximum $\mathcal{S}_{H,b}$-packing Q of G. Let $D = V(S) - Z$ when terminating. Now $s_G(D) = |V(G) - V(Q)| = def_\mathcal{S}(G)$ holds. We prove that the following matroid $\mathcal{M}$ on $V(G)$ has bases exactly the vertex sets of the maximum $\mathcal{S}_{H,b}$-packings of G. First define a matroid $\mathcal{N}$ on the set $\mathcal{C}$ of components of $G[D]$. $\mathcal{N}$ is the sum of the transversal matroids of the following two bipartite graphs B_1, B_2. The color class of B_1 different from $\mathcal{C}$ is $\{v^i : 1 \leq i \leq b, v \in \Gamma(D)\}$ and $Cv^i \in E(B_1)$ if v is adjacent to $C \in \mathcal{C}$ in G. The color class of B_2 different from $\mathcal{C}$ is $\{v^i : 1 \leq i \leq u-b, v \in \Gamma^{fr}(D)\}$ and $Cv^i \in E(B_2)$ if v is adjacent to a free vertex of $C \in \mathcal{C}$. The final alternating structure of the algorithm shows that the rank of $\mathcal{N}$ is $b\,|\Gamma(D)| + (u-b)\,|\Gamma^{fr}(D)| = b\,|\Gamma^{fi}(D) \setminus \Gamma^{fr}(D)| + u\,|\Gamma^{fr}(D)|$. Now replace each element C of $\mathcal{N}$ by a series extension on the set of free vertices of C. Finally, add as bridges the vertices of $G-D$ and all non-free vertices of $G[D]$ resulting in the matroid $\mathcal{M}$.

It is clear that a base B in $\mathcal{M}$ arises as the vertex set of an $\mathcal{S}_{H,b}$-packing of G. On the other hand, let Q be a maximum $\mathcal{S}_{H,b}$-packing of G. If some $x \in \Gamma(D)$ is adjacent in Q to more than b non-free vertices of distinct hedgehogs of D then applying Lemma 4.2 to Q, D and x yields a counterexample to Thm. 2.6. Hence $V(Q) \in \mathcal{M}$. □

5 An algorithm for the $\mathcal{C}_{H,b}$-packing problem

In this section we show an Edmonds-type algorithm for the $\mathcal{C}_{H,b}$-packing problem. This algorithm is similar to that of the previous section, but here the alternating structure may overlap itself: it can happen that an odd vertex is covered twice. The algorithm uses a subroutine of deciding if a factor-critical union of small hedgehogs has a perfect $\mathcal{C}_{H,b}$-packing. Though this can be carried out similarly as in the $\mathcal{S}_{H,b}$-case in Section 6, we do not touch this question because the method of Section 7 decides the existence of a perfect packing in general graphs.

We say that (S_*^+, S_*, Z_*) is an *alternating structure* if S_* is an induced subgraph of S_*^+ (S_* is not necessarily a subgraph of G), Z_* is an independent set of vertices of S_*, called the *odd vertices* and the following properties are satisfied:

- Every connected component of $S_* - Z_*$ is a hedgehog (these graphs are called the *hedgehogs of* S_*).
- The graph S'_* obtained from S_* by shrinking the hedgehogs of S_* is a forest.
- The odd vertices of S_* are of the following types.
  - A *fixed odd vertex* lies on exactly $b+1$ edges of S_* leading to fixed vertices of distinct hedgehogs of S_*.
  - A *free odd vertex* lies on exactly $u+1$ edges of S_* leading to free vertices of distinct hedgehogs of S_*.
- The components of $S_*^+ - S_*$ are called *outside components*. Each outside component K is joined by exactly one edge to S_*, which is incident to some $v \in V(K)$. K is either a singleton $\{v\}$ (a *free outside component*), or a forbidden star centered at v (a *fixed outside component*). A vertex in $V(S_*) - Z_*$ is adjacent to

no outside component. A free odd vertex is adjacent to at most b fixed outside components and to no free outside components. A fixed odd vertex is adjacent to at most u free outside components and to no fixed outside components.

We make it clear what we mean under overlapping. There exist disjoint pairs $\{z_i^1, z_i^2\} \subseteq Z_*$ (called *odd-pairs*) with the following property. For all i, z_i^1 is a free and z_i^2 is a fixed odd vertex which are not adjacent to any outside components. Though S_*^+ is not a subgraph of G, shrinking each set $\{z_i^1, z_i^2\}$ to one vertex z_i gives a subgraph S^+ of G. The omission of the subscript $*$ corresponds to such subgraphs of G. Hence a vertex $z \in V(G)$ can be 'covered by S^+ twice', when it is both a free and a fixed odd vertex, in possibly the same component of S_*^+. In this case z is incident to no outside components.

If T_* is a connected component of S_* then the corresponding component of S_*^+ is denoted by T_*^+. We define how to *blow up* a component T_*. Let W be a hedgehog of T_*. Blowing up $T^+ - W$ (which is a subgraph of G) means taking a perfect $\mathcal{C}_{H,b}$-packing Q_* of $T_*^+ - W$ such that if $z \in Z_* \cap V(T_*^+)$ is a free odd vertex adjacent to t outside stars then z will be a (u,t)-supercenter in Q_* (or a center of a u-star if $t = 0$), and if $z \in Z_* \cap V(T_*^+)$ is a fixed odd vertex adjacent to s outside singletons then z will be an (s,b)-supercenter in Q_*. After identifying the odd pairs we get a $\mathcal{C}_{H,b}$-packing Q of G. If two odd vertices in $Z_* \cap V(T_*^+)$ are identified then their image is a (u,b)-supercenter in Q. One more step is in order: if $z \in V(G)$ is free odd in T_* and fixed odd in another component R_* of S_* then let the u-star component of Q centered at z be S. Now delete S from Q and add S to R_* such that the u leaves will be outside components of R_*. Do similarly if z is fixed odd in T_* and free odd in another component. Finally, break off all odd pairs which contain an odd vertex of T_*.

The algorithm maintains an alternating structure (S_*^+, S_*, Z_*) with the disjoint pairing $\{z_i^1, z_i^2\} \subseteq Z_*$, standard decompositions of the hedgehogs of S and a perfect $\mathcal{C}_{H,b}$-packing Q of $G - V(S^+)$. In each step, it either increases $E(S_*)$ (*growing*) or blows up a component of S_*^+ (and hence *augments* Q). When stopping, each hedgehog of S will be a connected component in $G - Z$ and if $v \in Z$ is not free odd (resp. not fixed odd) then v is adjacent to no free (resp. fixed) vertices of hedgehogs of S. Thus with the notation $D = V(S) - Z$ the number of components of S will be exactly $h_{\mathcal{S}}(D) - b|\Gamma^{fi}(D)| - u|\Gamma^{fr}(D)|$. So Thm. 2.7 will imply that the following $\mathcal{C}_{H,b}$-packing Q' is maximum in G. Blow up one by one $T^+ - w$ for each component T_*^+ of S_*^+ where w is free in a hedgehog of T, finally, add Q.

Algorithm for the $\mathcal{C}_{H,b}$-packing problem

Step 0 (Initialization). Start with any $\mathcal{C}_{H,b}$-packing Q of G. Go to step 1.

Step 1 (Optimality test). Let $V(S_*^+) = V(S_*) = V(G) - V(Q)$ and $E(S_*^+) = Z = \emptyset$. If Q is perfect, stop. Go to step 2.

Step 2 (Edge selection and augmentation or growing). Look for an edge xy, not of a hedgehog of S_*, such that either x is free in a hedgehog W of S_* and y is not a free odd vertex, or x is fixed in a hedgehog W of S_* and y is not a fixed odd vertex. If no such edge exists, stop: each hedgehog of S is a connected component in $G - Z$ and if

$v \in Z$ is not free odd (resp. not fixed odd) then v is adjacent to no free (resp. fixed) vertices of hedgehogs of S. Otherwise, denote the component of S_* containing W by T_* and distinguish the following cases.

Case 1. $y \in V(W)$. By Lemmas 3.7, 3.8 and by the recognition subroutine we can either obtain a perfect $\mathcal{C}_{H,b}$-packing of $W + xy$ or conclude that $W + xy$ is a hedgehog and obtain a standard decomposition of it. If $W + xy$ has a perfect $\mathcal{C}_{H,b}$-packing Q' then add Q' to Q, blow up $T_*^+ - W$ and go to step 1. Otherwise note that the vertices free (fixed) in W are also free (fixed) in $W + xy$. We grow S_* by adding xy and continue by step 2.

Case 2. y is in a hedgehog of T_* different from W. Adding the edge xy creates a cycle C_* in the shrunk graph S'_*. Let T_C be the subgraph of $T_*^+ + xy$ consisting of $Z_* \cap V(C_*)$ and of all hedgehogs and outside components neighboring to $Z_* \cap V(C_*)$. Note that $T_*^+ - T_C$ has a perfect $\mathcal{C}_{H,b}$-packing.

Assume that $z^1 \in Z_* \cap V(C_*)$ and $z^2 \in Z_*$ form an odd pair. z^2 may be in T_* as well. If z^1 is free and z^2 is fixed then denote by N the set of $u-1$ free neighbors of z^1 which are not in a hedgehog of C_*. Delete $N \cup \{z^1\}$ from T_*, join the vertices of N as outside components to z^2 and blow up the new T_*. Similarly, if z^1 is fixed and z^2 is free then rejoin the $b-1$ forbidden star neighbors of z^1 to z^2 as outside components, delete z^1 and blow up the new T_*.

So we assume that T_C has no odd vertex covered twice and hence that T_C is a subgraph of G. A non-hedgehog component of $T_C - E(C_*)$ consists of an odd vertex z connected by edges to hedgehogs and outside singletons and stars. If such a component has a perfect $\mathcal{C}_{H,b}$-packing then add a perfect $\mathcal{C}_{H,b}$-packing of T_C to Q and blow up $T_*^+ - T_C$.

Otherwise every odd vertex $z \in Z \cap V(T_C)$ is connected to either i outside singletons with $i \notin H$ and $b = 1$ or to exactly $u-1$ non-fixed vertices of neighboring hedgehogs with $u - 1 \notin H$. So T_C is a factor-critical union of small hedgehogs. Take a standard decomposition of T_C by Lemmas 3.7 and 3.8. Note that the free (fixed) vertices of hedgehogs of $T_C - Z - xy$ are free (fixed) in T_C as well. By the recognition subroutine we can decide if T_C has a perfect $\mathcal{C}_{H,b}$-packing. If yes then add a perfect $\mathcal{C}_{H,b}$-packing of T_C to Q and blow up $T_*^+ - T_C$. Otherwise add xy to S_*^+, update $Z_* := Z_* \setminus V(T_C)$ and go to step 2.

Case 3. y is in a hedgehog W^y of a component $T_*^y \neq T_*$ of S_*. Blow up $T_* - W$, then blow up $T_*^y - W^y$ and add a perfect $\mathcal{C}_{H,b}$-packing of $W + xy + W^y$. Go to step 1.

Case 4a. x is free in W and y is a non-free odd vertex, possibly in T_* (even $xy \in E(S)$ is possible). Let X be the set of outside singletons adjacent to y. If $|X| < u$ then delete x from T_*^+, join x to y as an outside singleton, and blow up the new T_*^+. Go to step 1. If $|X| = u$ then delete X from S_*^+, join a new free odd vertex y^2 to x in T_*^+ and join the singletons of X to y^2 as hedgehogs. y and y^2 will be an odd-pair. Go to step 2.

Case 4b. x is fixed in W and y is a non-fixed odd vertex, possibly in T_* (even $xy \in E(S)$ is possible). Let X be the set of outside stars adjacent to y. If $|X| < u$ then delete a forbidden star subgraph S of W centered at x from T_*^+, join S to y as an outside star and blow up $T_*^+ - S$. Go to step 1. If $|X| = u$ then delete the stars of

X from S_*^+, join a new fixed odd vertex y^2 to x in T_*^+ and join the centers of the stars of X to y^2. The stars of X will be hedgehogs of S_*, y and y^2 will be an odd-pair. Go to step 2.

Case 5. y is contained in an outside component K of $S_*^+ - S_*$. Delete K from S_*^+, blow up $T_*^+ - x$ and add a perfect $\mathcal{C}_{H,b}$-packing of $K + xy$. Go to step 1.

Case 6. $y \notin V(S_*^+)$ is covered by a component L of Q. We use the convention mentioned in *Case 6.* of the $\mathcal{S}_{H,b}$-algorithm.

- x is non-fixed in W and y is neither the center of an i-star with $i+1 \notin H$, nor an (s,t)-supercenter with $s = u$. Blow up $T_*^+ - x$ and replace L by a perfect $\mathcal{C}_{H,b}$-packing of $L + xy$. Go to step 1.
- x is non-fixed in W and L is an i-star component centered at y with $i+1$ a gap of H. Grow S_* by adding $L + xy$ to W. Go to step 2.
- x is non-fixed in W and y is a (u,t)-supercenter. Add $L+xy$ to S_*^+ and add y to Z as a free odd vertex. The forbidden stars of $L-y$ will be outside components. Go to step 2.
- x is fixed in W and y is not an (s,b)-supercenter. Blow up $T_*^+ - W$ and replace L by a perfect $\mathcal{C}_{H,b}$-packing of $W + xy + L$. Go to step 1.
- x is fixed in W and y is an (s,b)-supercenter. Add $L+xy$ to S^+ and add y to Z as a fixed odd vertex. The isolated vertices of $L-y$ will be outside components. Go to step 2.

Now we prove the Berge-type theorem for the $\mathcal{C}_{H,b}$-packing problem which also yields the correctness of the above algorithm.

Definition 5.1. For $D \subseteq V(G)$ let $c_G(D) = h_{\mathcal{C}}(D) - b\,|\Gamma_G^{fi}(D)| - u\,|\Gamma_G^{fr}(D)|$. We say that G has the *Berge-property* if $def_{\mathcal{C}}(G) \geq \max\{c_G(D) :\ D \subseteq V(G)\}$.

Lemma 5.2. *Assume that all graphs with less vertices than G has the Berge-property. Let Q be a $\mathcal{C}_{H,b}$-packing of G, $D \subseteq V(G)$ and $x \in \Gamma(D)$ be a vertex adjacent in Q either to more than b non-free vertices of distinct hedgehogs of D or to more than u non-fixed vertices of distinct hedgehogs of D. Then there exist vertex sets $\emptyset \neq X \subseteq V(G)$ and $D' \subseteq V(G) - X$ such that $c_{G-X}(D') > c_G(D)$ and $def_{\mathcal{C}}(G - X) \leq |V(G) - V(Q)|$.*

Proof. If x has more than b non-free neighbors then exactly as in the $\mathcal{S}_{H,b}$-case, we can choose $D' = D \cup D''$ because each vertex free (fixed) in a hedgehog of $W[D'']$ is free (fixed) in W. This is implied by the alternating structure S_*^+ in the end of the algorithm together with Lemma 3.13, *1.* and *2.*

Assume that x has more than u non-fixed neighbors. At least one non-fixed neighbor y of x is a center in the component L of Q covering x. If y has a leaf $u \in \Gamma(D)$ then we are done by $D' = D$ and $X = \{u\}$. Otherwise let X consist of y and its adjacent leaves. $X \subseteq V(W)$ for the hedgehog W of y. $L - X \in \mathcal{C}_{H,b}$ so $def_{\mathcal{C}}(G-X) \leq |V(G) - V(Q)|$. We already know that $def_{\mathcal{C}}(W - X) \geq 2$ because $def_{\mathcal{C}}(W - u) \geq 2$ for a leaf u of

X by the proof of Lemma 4.2. Running the above algorithm on $W - X$ we get a set $D'' \subseteq V(W) - X$ with $c_{W-X}(D'') \geq 2$. Here as well, each vertex free (fixed) in a hedgehog of $W[D'']$ is free (fixed) in W by Lemma 3.13 so we are done by $D' = D \cup D''$. □

Thm. 2.7 can be proved exactly as Thm. 2.6 in page 12. Thus our algorithm gives a maximum $\mathcal{C}_{H,b}$-packing of the input graph G.

Also the $\mathcal{C}_{H,b}$-packing problem is matroidal. We do not go into details because this assertion will be proved in Section 7 using a different method. We only describe the matroid $\mathcal{N}$ on the set $\mathcal{C}$ of components of $G[D]$, and refer to the proof of Thm. 4.3. $\mathcal{N}$ is the sum of the transversal matroids of the following two bipartite graphs B_1, B_2. The color class of B_1 different from $\mathcal{C}$ is $\{v^i : 1 \leq i \leq u, v \in \Gamma^{fr}(D)\}$ and $Cv^i \in E(B_1)$ if v is adjacent to a free vertex of $C \in \mathcal{C}$ in G. The color class of B_2 different from $\mathcal{C}$ is $\{v^i : 1 \leq i \leq b, v \in \Gamma^{fi}(D)\}$ and $Cv^i \in E(B_2)$ if v is adjacent to a fixed vertex of $C \in \mathcal{C}$ in G.

6 The recognition of $\mathcal{S}_{H,b}$-hedgehogs

In this section we show a procedure for finding a perfect $\mathcal{S}_{H,b}$-packing in a factor-critical union of small hedgehogs W given by a standard decomposition $\mathcal{P}$, if it exists. We prove that if W has a perfect $\mathcal{S}_{H,b}$-packing then it has one with a special structure, called a *good* packing. If there exists a good packing then using the $\mathcal{H}$-factor algorithm of Cornuéjols [1] we can find a $D_{\mathcal{P}}$-nice $\mathcal{S}_{H,b}$-packing of W which enters each component of $W[D_{\mathcal{P}}]$. Hence Lemma 3.11 gives a perfect $\mathcal{S}_{H,b}$-packing of W.

Definition 6.1. A star and superstar packing Q is called *special* if each component of Q is either a star (with no supercenter), or an (s,t)-superstar with either $t = 1$ and s a gap of H or $s + t = u$.

If W is a factor-critical union of small hedgehogs with standard decomposition $\mathcal{P}$ then a $D_{\mathcal{P}}$-nice $\mathcal{S}_{H,b}$-packing Q is *good* if Q enters each component of $W[D_{\mathcal{P}}]$ and Q has at most one component K (called *exceptional*) such that $Q - K$ is special and K is either

1. an (s,t)-superstar with either $t = 1$ and $s \in H$ or $t = 2$ and s a gap of H,
2. or an (s,t)-superstar with $s + t = u - 1$.

Lemma 6.2. *Let W be a factor-critical union of small hedgehogs with standard decomposition $\mathcal{P}$. If W has a perfect $\mathcal{S}_{H,b}$-packing Q then it has a good packing.*

Proof. Let R be a subgraph of W which is either

(a) a $D_{\mathcal{P}}$-nice special $\mathcal{S}_{H,b}$-packing entering all components of $W[D_{\mathcal{P}}]$ except one component J, or

(b) a $D_{\mathcal{P}}$-nice special $\mathcal{S}_{H,b} \cup \{S_g\}$-packing entering all components of $W[D_{\mathcal{P}}]$, containing exactly one forbidden g-star component S such that the center of S is non-free and incident to an edge of $E(Q) \setminus E(S)$.

Note that Proposition 3.6 implies the existence of such a subgraph R of type (a). We prove that W has a good packing, by induction on $\text{dist}(R) = |(E(R)\Delta E(Q)) \setminus E(W[D_\mathcal{P}])|$ such that in case (a) we prefer more $\sum_{v\in V(J)} \deg_Q(v)$. For sake of simplicity here the supercenter of a bi-star is not determined.

First assume that R is of type (a). Delete the components of R contained in J. Choose an edge $vy \in E(Q)$ with $v \in V(J)$, $y \in A_\mathcal{P}$ minimizing $\deg_Q(y)$. $R + vy$ has a good packing unless

- *y is the center of an i-star component L of R with $i+1 \notin H$.* If there exists an edge $e \in E(L) \setminus E(Q)$ then let $R' = R + vy - e$. Note that this is always the case if $i = u$. Now $\text{dist}(R') < \text{dist}(R)$ so we are done by induction. Otherwise if y is incident to an edge $E(Q) \setminus (E(L) + vy)$ then we are done by $R' = R + vy$ which is of type (b). Otherwise, if J is an odd cycle then let the two neighbors of v in J be v_1, v_2. Now $R+vy+vv_1+vv_2$ is a good packing. If J is a singleton then let the component of Q containing L be L'. $\deg_{L'}(y) = i+1 \notin H$ so y is the supercenter of L'. v is a leaf of y because y was chosen to minimize $\deg_Q(y)$. Let $R' = R + vy - yz$ where z is a center of L'. Now $\text{dist}(R') = \text{dist}(R)$ but $\sum_{v\in V(J)} \deg_Q(v)$ increased.

- *y is the supercenter of an (s,t)-superstar component L of R with $s+t = u$.* If there exists an edge $yz \in E(L) \setminus E(Q)$ where z is a leaf of L then let $R' = R+vy-yz$. Otherwise $yc \notin E(Q)$ for some center c of L. Let z be an arbitrary leaf neighbor of c in L. Choose $R' = R + vy - yc - cz$.

Second, assume that R is of type (b). If S has an edge $e \notin E(Q)$ then let $R' = R-e$ so assume that $E(S) \subseteq E(Q)$. Let the center of S be v and let $vy \in E(Q) \setminus E(S)$. $R + vy$ has a good packing unless

- *$y \in D_\mathcal{P}$ is covered by a K_2-component L of R.* Now $R - L + vy$ is good.

- *y is the center of a u-star component L of R.* There exists an edge $yz \in E(L) \setminus E(Q)$ so let $R' = R + vy - yz$.

- *$b = 1$ and y is a leaf of an i-star component L of R with $i-1$ a gap of H, or y is the supercenter of an $(s,1)$-superstar L of R.* Let the center of L be c. $b = 1$ implies that there exists an edge $cz \in E(L-y)$ such that $\{yc, cz\} \setminus E(Q) \neq \emptyset$. We are done by $R' = R + vy - yc - cz$.

- *y is the supercenter of an (s,t)-superstar L with $s+t = u$.* If there exists a center c and a neighboring leaf z in L such that $\{yc, cz\} \setminus E(Q) \neq \emptyset$ then we are done by $R' = R + vy - yc - cz$. Otherwise there is an edge $yz \in E(L) \setminus E(Q)$ where z is a leaf of L. Now let $R' = R + vy - yz$.

□

We define $2|A_\mathcal{P}|+1$ $\mathcal{H}$-factor problems in the figures. Let $(G^0, \mathcal{H}^0)$ be the $\mathcal{H}$-factor problem we get when replacing each component of $W[D_\mathcal{P}]$ by a graph shown in Figure

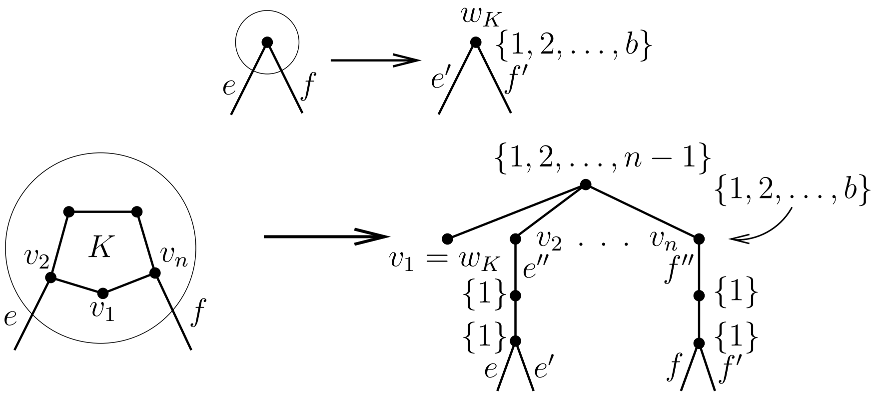


Figure 3: Gadgets replacing a singleton and an odd cycle component C of $W[D_{\mathcal{P}}]$

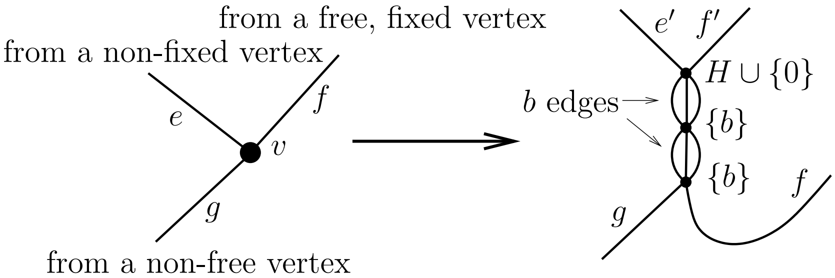


Figure 4: A gadget replacing $v \in A_{\mathcal{P}}$

3 and each vertex of $A_{\mathcal{P}}$ by a gadget shown in Figure 4. For $v \in V(W)$, let $(G^1_v, \mathcal{H}^1_v)$ (resp. $(G^2_v, \mathcal{H}^2_v)$) what we get from $(G^0, \mathcal{H}^0)$ when replacing the gadget of $v \in A_{\mathcal{P}}$ by F^1_v, see Figure 5 (resp. F^2_v in Figure 6).

Observe that if Q is good then $\deg_Q(v) \in H \cup \{0\}$ for each $v \in A_{\mathcal{P}}$ except possibly the supercenter of the exceptional component. So if a good packing Q has an exceptional component of type 1. (resp. type 2.) with supercenter s then G^1_s (resp. G^2_s) has a perfect $\mathcal{H}^1_s$-factor (resp. perfect $\mathcal{H}^2_s$-factor), and if a good packing Q has no exceptional component then G^0 has a perfect $\mathcal{H}^0$-factor. Indeed, the following edge set E' can be extended to a perfect $\mathcal{H}$-factor in the appropriate $\mathcal{H}$-factor problem. For $e \in E(W - D_{\mathcal{P}}) \cap E(Q)$ let $e \in E'$. Moreover, for $e \in \delta(D_{\mathcal{P}}) \cap E(Q)$ let $e \in E'$ if e joins a center c to $A_{\mathcal{P}}$ such that c is contained in an odd cycle component of $W[D_{\mathcal{P}}]$ and has two neighboring leaves in Q. For other edges $e \in \delta(D_{\mathcal{P}}) \cap E(Q)$ let $e' \in E'$.

On the other hand, let F be an $\mathcal{H}$-factor in any of the above $2|A_{\mathcal{P}}| + 1$ $\mathcal{H}$-factor problems. Let P be the subgraph of W with the following edge set $E \subseteq E(W)$ (P contains no singletons). Let $e \in E$ if $e' \in E(F)$ (an *upper* edge) and let $e \in E$ if $e \in E(F)$ (a *lower* edge). Note that P satisfies property (*) below.

> For $v \in A_{\mathcal{P}}$ let t denote the number of lower edges of P incident to v. Now $t \leq b$, if $t > 0$ then $0 \leq \deg_P(v) \leq u$, and if $t = 0$ then $\deg_P(v) \in H \cup \{0\}$.

Call $e \in E$ *dangerous* if the deletion of e would disconnect a component of $W[D_{\mathcal{P}}]$ from $A_{\mathcal{P}}$ in P. Let $e \in E$ be a non-dangerous edge incident to $v \in A_{\mathcal{P}}$. If $P - e$ still

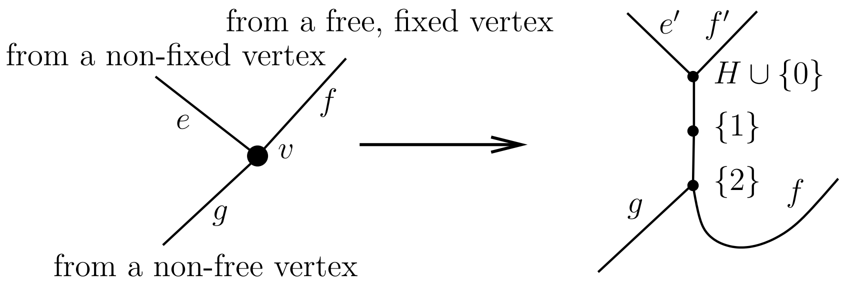


Figure 5: The gadget F_v^1

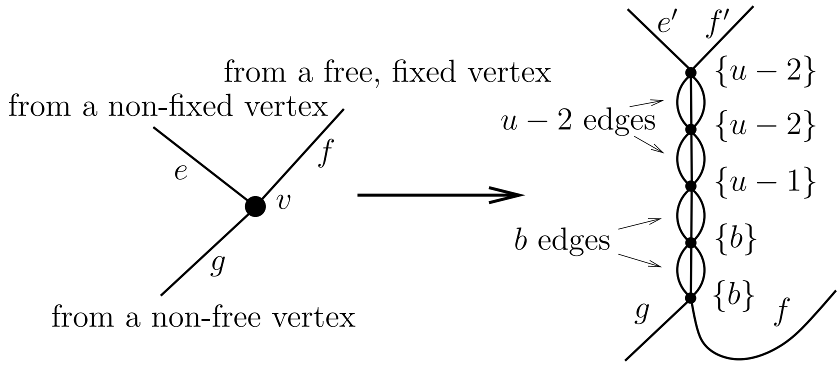


Figure 6: The gadget F_v^2

satisfies property (*) then delete e. Otherwise if $f \in \delta(v) \cap (E-e)$ is non-dangerous then $P-e-f$ satisfies property (*) so delete both e and f. In the end of this procedure, for all non-dangerous edges zc some end-node, say c, has the property that $c \in A_{\mathcal{P}}$, $\deg_P(c)-1$ is a gap of H and $\delta(c) \cap (E-e)$ contains only upper dangerous edges. Thus if a component K of P contains a non-dangerous edge zc then $K \in \mathcal{S}_{H,b}$ with supercenter z. If K contains only dangerous edges then if $e \in E(K)$ is a lower edge joining $v \in V(C)$ to $A_{\mathcal{P}}$ for an odd cycle component C of $W[D_{\mathcal{P}}]$ then add the two incident edges of v in C to K. Now the new K is in $\mathcal{S}_{H,b}$. Hence the new P is a $D_{\mathcal{P}}$-nice $\mathcal{S}_{H,b}$-packing of W which enters each component of $W[D_{\mathcal{P}}]$. So Lemma 3.11 gives a perfect $\mathcal{S}_{H,b}$-packing of W.

7 Reducing the $\mathcal{C}_{H,b}$-packing problem to the $\mathcal{H}$-factor problem

In this section we show an alternative method solving the $\mathcal{C}_{H,b}$-packing problem in general graphs by a similar but simpler reduction to the $\mathcal{H}$-factor problem as in Section 6. This reduction also yields the matroidality of the problem. From the previous results we use only Lemmas 3.10 and 3.11.

Let $V(G) = D \dot\cup A \dot\cup C$ denote the Edmonds-Gallai decomposition of the graph G and let D_c denote the set of vertices of D which belong to either singletons of $G[D]$ or to odd cycle components of $G[D]$ in the case $2 \notin H$. Let $A_c = \Gamma(D_c)$. Note that

$D_{A_c} = D_c$. Observe that D_c consists of the components of $G[D]$ which have no perfect $\mathcal{C}_{H,b}$-packing.

Lemma 7.1. *If Q is a D_c-nice $\mathcal{C}_{H,b}$-packing of G then we can find in polynomial time a $\mathcal{C}_{H,b}$-packing of G covering each vertex of G except at most one in each component of $G[D_c]$ not entered by Q.*

Proof. Note that by the properties of the Edmonds-Gallai decomposition, A_c can be matched into the components of $G[D_c]$. Let Q' be the packing guaranteed by Lemma 3.11 applied to $A_c \subseteq V(G)$. Observe that $A - A_c$ can be matched into the components of $G[D - D_c]$ by a matching M_A. Add M_A to Q' together with a perfect matching M_C of $G[C]$. If C is a component of $G[D - D_c]$ not entered by the new Q' then add a perfect $\mathcal{C}_{H,b}$-packing of C to Q'. Finally, let C be a component of $G[D - D_c]$ entered by the new Q'. Choose $w_C \in V(C) \cap V(Q')$ such that w_C is covered by M_A, if possible. Let M_D denote the union of perfect matchings of $C - w_C$ for the components C of $G[D - D_c]$ entered by Q'. Now delete those edges of $M_D \cup M_A \cup M_C$ which glue the supercenters of two superstars. □

Lemmas 3.10 and 7.1 imply the following corollary on the structure of coverable vertex sets.

Lemma 7.2. *A maximum $\mathcal{C}_{H,b}$-packing Q of G covers each vertex of G except one vertex in $def_{\mathcal{C}}(G)$ components of $G[D_c]$.*

Lemmas 3.10 and 7.1 imply also that we only have to find a D_c-nice $\mathcal{C}_{H,b}$-packing entering a maximum number of components of $G[D_c]$. This is reduced to the $\mathcal{H}$-factor problem as follows. Each component of $G[D_c]$ must be replaced by a gadget shown in Figure 3, each vertex of A_c by a gadget shown in Figure 7, let $H_v = \{0, 1, \ldots, b\}$ for each $v \in G - D_c - A_c$ and finally, delete $E(G - D_c - A_c)$. Note that we completely ignore the edges induced by $V(G) - A_c$.

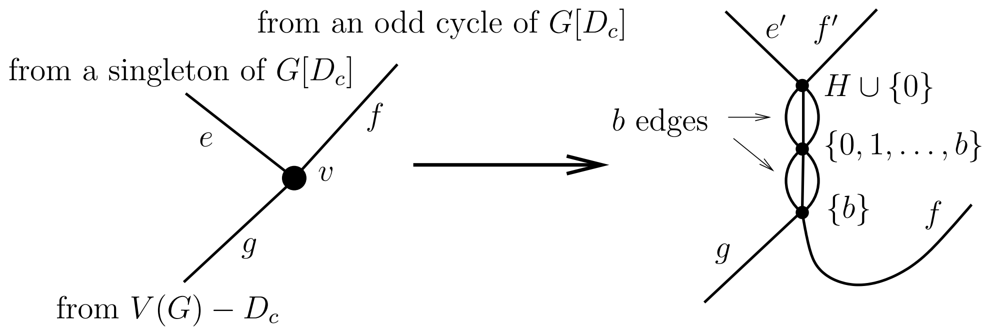


Figure 7: A gadget replacing $v \in A_c$

Let us denote the new graph by G_{aux} with degree-prescription $\mathcal{H}$. Let F be an $\mathcal{H}$-optimal subgraph of G_{aux}, F may have singleton components. It is easy to see that we can assume that $\deg_F(v) \in H_v$ for all vertices outside $\{w_C : C$ is a component of $G[D_c]\}$. As in the previous section, let P be the subgraph of G with the following edge set $E \subseteq E(G)$ (P contains no singletons). Let $e \in E$ if $e' \in E(F)$ (an *upper* edge) and let $e \in E$ if $e \in E(F)$ (a *lower* edge). Now P satisfies property (*) below.

For $v \in A_c$ let s (resp. t) denote the number of upper (resp. lower) edges of P incident to v. Now $s \leq u$, $t \leq b$ and if $t = 0$ then $\deg_P(v) \in H \cup \{0\}$.

Call $e \in E$ *dangerous* if the deletion of e would disconnect a component of $G[D_c]$ from A_c in P. Let $e \in E$ be a non-dangerous edge incident to $v \in A_c$. Exactly as in Section 6, the deletion of appropriate non-dangerous edges leads to a situation when for all non-dangerous edges zc some end-node, say c, has the property that $c \in A_c$, $\deg_P(c) - 1$ is a gap of H and $\delta(c) \cap (E - e)$ contains only upper dangerous edges. Thus if a component K of P contains a non-dangerous edge zc then $K \in \mathcal{C}_{H,b}$ with supercenter z. If K contains only dangerous edges then if $e \in E(K)$ is a lower edge joining $v \in V(C)$ to A_c for an odd cycle component C of $G[D_c]$ then add the two incident edges of v in C to K. The new K is now in $\mathcal{C}_{H,b}$. Hence the new P is a D_c-nice $\mathcal{C}_{H,b}$-packing of G which enters each component C for which $w_C \in V(F)$. So $def_{\mathcal{C}}(G) \leq \delta_{\mathcal{H}}(G_{\text{aux}})$ by Lemma 7.1.

On the other hand, if Q' is a maximum $\mathcal{C}_{H,b}$-packing of G then Lemma 3.10 by the choice $A = A_c$ gives a D_c-nice packing Q entering every component of $G[D_c]$ entered by Q'. Define $E' \subseteq G_{\text{aux}}$ as follows. For $e \in E(G - D_c) \cap E(Q)$ let $e \in E'$. Moreover, for $e \in \delta(D_c) \cap E(Q)$ let $e \in E'$ if e joins a center c to A_c such that c is contained in an odd cycle component of $G[D_c]$ and has two neighboring leaves in Q. For other edges $e \in \delta(D_c) \cap E(Q)$ let $e' \in E'$. Now E' can clearly be extend to a perfect $\mathcal{H}$-factor F of G_{aux} with $\delta_F \leq def_{\mathcal{C}}(G)$. Thus $def_{\mathcal{C}}(G) = \delta_{\mathcal{H}}(G_{\text{aux}})$.

Summarizing, from an $\mathcal{H}$-optimal subgraph of G_{aux} one can construct in polynomial time a maximum $\mathcal{C}_{H,b}$-packing of G. These considerations also yield the matroidality of the $\mathcal{C}_{H,b}$-packing problem. We use the following statement [12]. A subgraph G' of G_{aux} is called an $\mathcal{H}$-*subgraph* if $\deg_{G'}(v) \in H_v$ for all $v \in V(G')$.

Theorem 7.3. *Suppose $1 \in H_v$ and H_v has no two consecutive gaps for all $v \in V(G_{aux})$. Then those vertex sets which can be covered by $\mathcal{H}$-subgraphs form a matroid.*

Theorem 7.4. *The vertex sets of G which can be covered by a $\mathcal{C}_{H,b}$-packing form a matroid.*

Proof. Denote by $\mathcal{M}$ the matroid of the vertex sets of $\mathcal{H}$-subgraphs of G_{aux}. Contract $V(G_{\text{aux}}) - \{w_C : C \text{ is a component of } G[D_c]\}$ in $\mathcal{M}$. Now take a series extension of w_C on $V(C)$ for each component C. Finally, direct sum the elements of $V(G) - D_c$ as bridges. The above considerations imply that the independent sets of the resulting matroid are exactly the vertex sets of G which can be covered by a $\mathcal{C}_{H,b}$-packing. □

We mention that the superstar packing problem can clearly be solved in its *local* version instead of the above *global* version. In the local version each vertex $v \in V(G)$ has an own prescription H_v, b_v. We only have to replace H, b to H_v, b_v when speaking about a vertex v.

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