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Alternating paths revisited I: even factors

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Abstract

We give a new, algorithmic proof for the maximum even factor formula which can be converted into a polynomial time combinatorial algorithm to solve the maximum even factor problem. In several aspects, the approach is similar to Edmonds' Matching Algorithm, but there is a significant difference.

1 Introduction

W.T. Tutte [15] characterized the existence of a perfect matching in a graph, then C. Berge gave a min-max formula in [2] to determine the maximum size of a matching – the so-called Berge-Tutte formula. In [6] J. Edmonds gave an efficient algorithm to determine a maximum matching, which also implies a structural description – independently found by T. Gallai [7] – called the Gallai-Edmonds decomposition.

W.H. Cunningham and J.F. Geelen [4] investigated separation algorithms for the matchable set polytope, first studied by E. Balas and W. Pulleyblank in [1]. In [4] they concluded that a strongly polynomial time separation algorithm can be given, provided there is a polynomial time algorithm for solving a combinatorial problem which they called the optimal path-matching problem. They settled this by giving polynomial algorithms for the optimal path-matching problem, which is in fact a direct generalization of the matching problem. Tutte's original approach related matchings to the rank of a matrix over indeterminates – now known as the Tutte-matrix – Cunningham and Geelen noticed that the maximal value of a path-matching is equal to the rank of an analogue matrix. In general, it is an unsolved problem to determine the rank of a matrix A in polynomial time, if the entries of A are linear expressions of algebraically independent indeterminates. However, evaluations of the indeterminates with random integers yield a polynomial-time randomized algorithm, using the fact that the rank of a matrix over the integers can be determined in polynomial-time; see [9]. This provides a polynomial time randomized algorithm for maximum

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matching, optimal path-matching; Cunningham and Geelen gave two different deterministic algorithms. In [4] they showed a polyhedral description which admits polynomial time separation, then the optimal path-matching problem can be solved via the ellipsoid method. Later in [5] they gave an algorithm based on deterministic evaluations of the Tutte-matrix. Both algorithms return a maximum path-matching along with a so-called minimal "stable pair", this gives a min-max formula for optimal path-matchings.

As a direct extension of the Berge-Tutte formula, A. Frank and L. Szegő [8] gave a simplified min-max formula for optimal path-matchings. A Gallai-Edmonds-type structural description was given by B. Spille and L. Szegő [13]. Both results are formulated so that known results about matchings are direct special cases, in fact techniques useful for matchings are used in their proofs. So it was a challenge to develop an algorithm that follows lines similar to those in Edmonds' matching algorithm. B. Spille and R. Weismantel [14] constructed such an algorithm to find an optimal path-matching.

In [3] Cunningham introduced the so-called maximum even factor problem in digraphs, which is yet another generalization of maximum matching and optimal path-matching problems. The maximum even factor problem is NP-hard, but the problem is tractable when restricted to the class of odd-cycle-symmetric digraphs – this includes the path-matching problem as a special case. We refer to the maximum even factor problem as the restricted problem to odd-cycle-symmetric graphs. This problem may be represented by the rank of a Tutte-type matrix, the algorithm in [5] can be used to compute this rank.

The simplified formula of Frank and Szegő on path-matchings was extended to even factors by G. Pap and L. Szegő [12]. In fact, in [12] the formula is only formulated in a tighter class of graphs called "weakly symmetric", but the method also works for odd-cycle-symmetric graphs, this was already known by the authors at the time of publication. The aim of this paper is to give a new proof of the min-max formula presented in [12] for odd-cycle-symmetric graphs, this proof can be converted into a polynomial-time algorithm for the maximum even factor problem. Our approach is different from any approaches mentioned above. It bears some characteristics of Edmonds' matching algorithm, but there are significant differences, too. When we apply this algorithm to solve a maximum matching problem, we do not get back to Edmonds' algorithm.

Edmonds' algorithm operates in some contracted graphs, where it grows a so-called alternating forest: we maintain a contracted graph G_i with an alternating forest F_i which is alternating with respect to a matching M_i in G_i. We end up in one of four cases: either 1. the alternating forest may be enlarged, or 2. we find an M_i-alternating odd cycle which can be contracted or 3. maximality is attained, we return a verifying set or 4. we can give a larger matching, then we return to the original graph with a larger matching.

The proof and the algorithm in the present paper has a similar structure as Edmonds', but the concept of alternating forests is replaced by a more liberal substitute.

The paper has the following structure: after the notation is introduced, we give a proof for the Berge-Tutte type Theorem 3.1; next, this is extended to a Gallai-

Edmonds type structural description; then we show how an algorithm can be constructed using the concept of the proof. In the end we make some concluding remarks on the concept of the paper, relating it to well-known concepts for matchings, accompanied by some examples.

2 Basic Definitions for even factors

In this section we summerize notions and notation used in the paper. Consider a digraph $D = (V, A)$ without loops, a *cycle (path)* is the arc-set of a closed (unclosed) directed walk without repetition of arcs or nodes. *A path-cycle-factor* is the arc-set $M \subseteq A$ of a subgraph of G which is a node disjoint union of paths and cycles. Equivalently, a path-cycle-factor is a set of arcs with in- and out-degree at most 1 in each node. We call an arc $e = uv \in A$ *symmetric* if $vu \in A$, otherwise e is *asymmetric*. A cycle or a path is *even (odd)* if it consists of an even (odd) number of arcs; a cycle is *asymmetric*, whenever it has at least one asymmetric arc. An *even factor* is a path-cycle-factor which has no odd cycle, let $\nu(D)$ denote the maximum cardinality of an even factor. An even factor is *perfect* if each node is covered with an even cycle.

A min-max formula was given in [12] by Pap and Szegő for $\nu(D)$ in a special class of digraphs D called weakly symmetric. A graph is called *weakly symmetric* (a definition introduced by Cunningham in [3]) if all cycles in the graph are symmetric – or equivalently, if all arcs inside a strong component are symmetric. The results and the method in [12] also work in a slightly larger class of graphs called odd-cycle-symmetric: D is called *odd-cycle-symmetric* if there is no asymmetric odd cycle – or equivalently, if each odd cycle in D is symmetric.

Throughout the paper we only consider odd-cycle-symmetric graphs, we show that determining $\nu(D)$ for odd-cycle-symmetric graphs is polynomially solvable. The restriction is in fact necessary, since for an arbitrary graph D it is NP-complete to determine $\nu(D)$ [3]. In some papers, we have used the terms "hardly symmetric graphs" for the notion called odd-cycle-symmetric graphs in this paper.

Some further definitions, notation: for a set $X \subseteq V$ let $\Gamma^+_D(X) := \{x \in V - X : \exists y \in X, yx \in A\}$ and $\varrho_D(X) = |\{uv \in A : u \in V - X, v \in X\}|$ and $\delta_D(X) = |\{uv \in A : u \in X, v \in V - X\}|$. For a path-cycle-factor M let $V^+(M) := \{v \in V : \delta_M(v) = 1\}$ and $V^-(M) := \{v \in V : \varrho_M(v) = 1\}$, so the set $V - V^-(M)$ consists of the *M-source-nodes* and $V - V^+(M)$ consists of the *M-sink-nodes*. For an arc set $F \subseteq A$ let $\overleftrightarrow{F} := F \cup \{uv : vu \in F\}$, hence if $C \subseteq A$ is an odd cycle then $\overleftrightarrow{C} \subseteq A$. A set $U \subseteq V$ *induces* the subgraph $D[U] = (U, A[U])$ where $A[U] = \{uv \in A : u, v \in U\}$. $D - U := D[V - U]$. For a set $U \subseteq V$ we denote the *contracted graph* by D/U having node set $V/U = V - U + \{U\}$ and arc-set A/U given by deleting the arcs in $A[U]$ and identifying the nodes in U by $\{U\}$ (we will use contractions only if $D[U]$ is connected, thus this definition is equivalent with the usual contraction of the arcs in $A[U]$). For $uv \in A$ we call v the *head* of arc uv, while u is called the *tail* of arc uv; *uv leaves u* and *enters v*.

An undirected graph $G = (V, E)$ is called *factor-critical* if for all $v \in V$ the graph $G - v$ has a perfect matching. (For a survey on matching theory, see L. Lovász and M.D. Plummer [10].) A digraph is called *symmetric-critical* if all its arcs are symmetric and the underlying undirected graph is factor-critical. In a digraph D a set $S \subseteq V$ is called a *source-component* if $D[S]$ is strongly connected and $\varrho_D(S) = 0$. Let $\sigma(D[X])$ denote the number of those source-components in $D[X]$ which are symmetric-critical. For example a source-node is a symmetric-critical source-component.

(!) Recall that in the well-known Berge-Tutte formula we use the number of odd components of $G - X$, here the definition of $\sigma(D[X])$ is concerned with $G[X]$. Notice, if we orient the edges of a factor-critical graph, we do not necessarily get a symmetric-critical digraph: it will only be symmetric-critical if all arcs are symmetric. **(!)**

Definition 2.1. The *deficiency* of an even factor M is $\mathrm{def}_D M := |V| - |M|$, which is non-negative, of course. The *deficiency* of a set $X \subseteq V$ is defined by $\mathrm{def}_D X := \sigma(D[X]) - |\Gamma^+_D(X)|$. Let us use the notation $\tau_D(X) := |V| + |\Gamma^+_D(X)| - \sigma(D[X]) = |V| - \mathrm{def}_D X$ for a node set $X \subseteq V$.

To prove the main theorem 3.1, in the next section we will use the following well-known statements about factor-critical undirected graphs:

Lemma 2.2. *If $G = (V, E)$ is an undirected graph, for some $U \subseteq V$ the induced subgraph $G[U]$ is factor-critical and G/U is factor-critical, then G is factor-critical, too.* □

Lemma 2.3. *If $G = (V, E)$ is a factor-critical undirected graph and $s, t \in V$ then*
a) if $s \neq t$ then there is an even path P from s to t and a perfect matching M in $G - V(P)$,
b) if $s = t$ then there is a perfect matching M in $G - s$ (a degenerate version of a)).

Proof. Let M_s, M_t be perfect matchings in $G - s$, $G - t$, respectively. Then $M_s \cup M_t$ consists of alternating cycles – which have even length – and one alternating s, t path P on an even number of edges – in case of $s = t$ P has no edges. In the alternating cycles let us choose the edges of M_s to form the perfect matching M on $V - V(P)$. □

3 The main theorem with a constructive proof

In this section, we give a simple, algorithmic proof of the following theorem. In Section 5 we show how a polynomial time algorithm can be constructed based on the same concept and using some claims proved in this section.

Theorem 3.1 (Pap, Szegő [12]). *If $D = (V, A)$ is an odd-cycle-symmetric digraph, then*

$$\nu(D) = \min_{X \subseteq V} |V| + |\Gamma^+_D(X)| - \sigma(D[X]). \tag{1}$$

The easy part of the proof is to see that the left hand side in (1) is at most the right hand side. To this end we show that for any even factor M and any set $X \subseteq V$

we have $|M| \leq \tau_D(X)$. Let

$$M_1 := \{vz \in M : v, z \in X\} = M[X], \tag{2}$$
$$M_2 := \{vz \in M : v \in X, z \in V - X\}, \tag{3}$$
$$M_3 := \{vz \in M : v \in V - X\}. \tag{4}$$

It is easy to see that M_1, M_2, M_3 is a partition of M.

Claim 3.2. *The following inequalities hold:*

$$|M_1| \leq |X| - \sigma(D[X]), \tag{5}$$
$$|M_2| \leq |\Gamma_D^+(X)|, \tag{6}$$
$$|M_3| \leq |V| - |X|. \tag{7}$$

Proof. First we show that if $S \subseteq X$ is a symmetric-critical component in $D[X]$ then there is a node s in S for which $\varrho_{M_1}(s) = 0$. Suppose for contradiction, for each node v in S there is an arc in M_1 with head v, then S would be covered by cycles in M_1 – and since a symmetric-critical graph has odd number of nodes – there would be an odd cycle in M. Thus there must be at least $\sigma(D[X])$ nodes s in X with $\varrho_{M_1}(s) = 0$, this proves inequality (5). Each arc in M_2 has head in $\Gamma_D^+(X)$, this proves inequality (6). Each arc in M_3 has tail in $V - X$, this proves inequality (7). □

The sum of inequalities (5)–(7) gives $|M| \leq \tau_D(X)$.

Definition 3.3. A set $X \subseteq V$ is a *verifying set for* an even factor M if $|M| = \tau_D(X)$, or equivalently $\mathrm{def}_D M = \mathrm{def}_D X$.

Claim 3.4. *If M is an even factor with a verifying set X – i.e. equality holds in (1) – then each of (5)-(7) holds with equality. In particular, for each node $v \in V - X$ we have $\delta_M(v) = 1$.*

Proof. Equality in (7) implies $\delta_M(v) = 1$ for the nodes v in $V - X$. □

As a preparation for the more difficult part of the proof we need to study some properties of symmetric-critical subgraphs and even factors.

Claim 3.5. *If $D = (V, A)$ is symmetric-critical and $s, t \in V$ are two not necessarily distinct nodes, then there is an even factor M_{st} for which*

1. *$|M_{st}| = |V| - 1$,*
2. *M_{st} has no arc entering s and no arc leaving t,*
3. *M_{st} consists of even cycles and an even s, t path P_{st} (in case of $s = t$ this path has length zero).*

Proof. Let us apply Lemma 2.3 for the underlying undirected graph. M_{st} is constructed by taking the edges in M in both directions and taking edges on the path P in direction s to t. □

Claim 3.6. *In an odd-cycle-symmetric digraph $D = (V, A)$, if $C \subseteq A$ is an odd cycle then $D[V(C)]$ is symmetric-critical.*

Proof. By definition, the arcs of C must be symmetric, thus $\overleftrightarrow{C} \subseteq A$. For any arc $ab \in A[V(C)]$ there is an even path $P \subseteq \overleftrightarrow{C}$ for which $P + ab$ is an odd cycle. Thus ab must be symmetric. □

Definition 3.7. For a node-set $U \subseteq V$, a directed path $P \subseteq A$ is called an *odd ear on U* if the first and last nodes of P are in U, the inner nodes of P are in $V - U$, and $|P|$ is odd.

Claim 3.8. *If $D = (V, A)$ is odd-cycle-symmetric and $D[U]$ is symmetric-critical for some $U \subseteq V$, then the contracted graph D/U is odd-cycle-symmetric.*

Proof. Odd cycles in D/U not incident with $\{U\}$ are symmetric, since $D/U - \{U\}$ is isomorfic to $D - U$. Consider an odd cycle C of D/U incident with $\{U\}$. The set $C' \subseteq A$ of arcs corresponding to the arcs of C must be an odd ear on U – say C' has first node t and last node s. By Claim 3.5 there must be an even path P_{st} from s to t in $D[U]$, then $C' \cup P_{st}$ is an odd cycle in D. So all arcs in C' must be symmetric, then all arcs in C must be symmetric in D/U, too. □

Claim 3.9. *Consider an odd-cycle-symmetric digraph D, suppose node-set $U \subseteq V$ induces a symmetric-critical subgraph $D[U]$, Y induces a symmetric-critical subgraph in D/U with $\{U\} \in Y$. Then the pre-image of Y, i.e. $Z = Y - \{U\} \cup U$ induces a symmetric-critical subgraph in D.*

Proof. By Lemma 2.2 we only need to show that each arc in $D[Z]$ is symmetric. Since all arcs in $D[U]$ and $D/U[Y]$ are symmetric, only arcs $ab \in A[Z]$ with $|\{a, b\} \cap U| = 1$ need to be checked. We show that ab is contained in an odd cycle of D. By Claim 3.5 there is an even path P_{ba} in $D/U[Y]$, then $C = P_{ba} + ab$ is an odd cycle in D/U. The set of arcs $C' \subseteq A$ corresponding to C give an odd ear on U, suppose this odd ear has first node t and last node s. By Claim 3.5 there is an even path P_{st} in $D[U]$, then $C' + P_{st}$ is an odd cycle in D. □

Definition 3.10. We say an even factor N *fits the odd cycle* C if $|N \cap C| = |V(C)| - 1$ and $\delta_N(V(C)) = 0$.

Observe, if N is an even factor, C is an odd cycle, and N_C is the set of arcs of N adjacent with a node in $V(C)$, then N fits C if and only if either $N_C = C - ab$ (for an arc $ab \in C$) or $N_C = C - ab + cb$ (for an arc $ab \in C$ and an arc $cb \in A$ with $c \notin V(C)$). Consider an even factor N which fits the odd cycle C, let us summerize the consequences of the above Claims on N and C: $V(C)$ induces a symmetric-critical subgraph, $D/V(C)$ is hardly symmetric, and N/C is an even factor in $D/V(C)$.

The following theorem describes the key property of the contraction of an odd cycle for even factors, which is the analogue of a known property of matchings, see Section 6.

Lemma 3.11. *Suppose $D = (V, A)$ is odd-cycle-symmetric, N is an even factor in D which fits the odd cycle C; define $D' = D/V(C)$ and $N' = N/V(C)$.*

1. *If N is maximum, then N' is maximum.*

2. *If X' is a verifying set for N', then the pre-image X is a verifying set for N.*

It is easy to see that the first assertion in this theorem follows from Claim 3.12, while the second follows from Claim 3.13. For technical reasons in the algorithm, these claims are formulated a little bit stronger than needed for Lemma 3.11.

Claim 3.12. *Suppose $D = (V, A)$ is odd-cycle-symmetric, C is an odd cycle in D, and M is an even factor in $D/V(C)$. Then*

1. *there is an even factor of size $|M| + |V(C)| - 1$ in D,*

2. *$\nu(D) \geq \nu(D/V(C)) + |V(C)| - 1$.*

Proof. The second statement follows from the first, we need to prove the first. Let M' denote the set of arcs in D corresponding to arcs in M. Then M' contains at most one arc entering U, let this arc be $s's$ if any, otherwise choose $s \in V(C)$ arbitrarily. Similarly, M' contains at most one arc leaving $V(C)$, let this arc be tt' if any, otherwise choose $t \in V(C)$ arbitrarily. Let M_{st} be the even factor in $D[V(C)]$ as in Claim 3.5, we show that $M' \cup M_{st}$ is an even factor of cardinality $|M| + |V(C)| - 1$.

The cardinality is as indicated, since M' and M_{st} are disjoint, it is also easy to see that $M' \cup M_{st}$ is a path-cycle-factor. The only way we could get an odd cycle would be using the arcs $s's$ and tt'. Let C be the cycle in $M' \cup M_{st}$ using these arcs, then C/P_{st} is a cycle in M and P_{st} is even, thus C must be even, too. □

Claim 3.13. *Suppose N is an even factor in D which fits some odd cycle C; define $D' = D/V(C)$ and $N' = N/V(C)$. As already observed, N' is an even factor in D'. If X' is a verifying set in D' for N', then $\{C\} \in X'$ and the pre-image $X := X' - \{C\} \cup V(C)$ is a verifying set for N.*

Proof. Since X' is a verifying set for N', N' is a maximum even factor. By definition $\delta_{N'}(\{C\}) = 0$, thus by Claim 3.4 we get that $\{C\} \in X'$. Consider a symmetric-critical source-component Q' in $D'[X']$, if $\{C\} \notin Q'$ then Q' is a symmetric-critical source-component in $D[X]$, too. If $\{C\} \in Q'$ then we claim that $Q := Q' - \{C\} \cup V(C)$ induces a symmetric-critical source-component in $D[X]$, which can be seen as follows: Since $D'[Q']$ is a source component in $D'[X']$, $D[Q]$ must be a source component in $D[X]$; moreover $D'[Q']$ is symmetric-critical, $D[V(C)]$ is symmetric-critical, then by Claim 3.9 $D[Q]$ is symmetric-critical.

Thus $D[X]$ has at least as many symmetric-critical source-components as $D'[X']$, that is $\sigma(D[X]) \geq \sigma(D'[X'])$. Furthermore $\{C\} \in X'$ implies $\Gamma^+_D(X) = \Gamma^+_{D'}(X')$ and $|X| = |X'| - |V(C)| + 1$.

$$
\begin{aligned}
|N| = |N'| + |V(C)| - 1 = \nu(D') + |V(C)| - 1 = \\
= |V/C| + |\Gamma^+_{D'}(X')| - \sigma(D'[X']) + |V(C)| - 1 \geq \\
\geq |V| + |\Gamma^+_D(X)| - \sigma(D[X]) \geq |N|.
\end{aligned}
$$

□

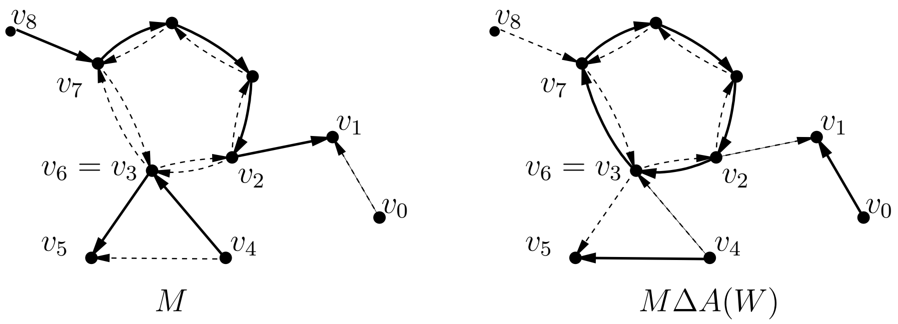

Figure 1: A special M-alternating walk $W = (v_0, \cdots, v_8)$ with $q = v_0$ and the symmetric difference $M\Delta A(W)$.

The following definitions present the tool of the proof and of the algorithm needed to find N fitting an odd cycle C.

Definition 3.14. Let M be a fixed even factor in $D = (V, A)$, recall the definition $V^+(M) := \{v \in V : \delta_M(v) = 1\}$, let $K^+ := V - V^+(M)$ – i.e. K^+ is the set of M-sink-nodes. A sequence $W = (v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n)$ is called an *M-alternating walk* if $v_0 \in K^+$, $e_i = v_i v_{i+1} \in A$ if i is even, $e_i = v_{i+1} v_i \in M$ if i is odd. Here n is called the *length* of W, v_0 is the *first node* of W and v_n is the *last node* of W. W is called *even/odd* by the parity of its length. Let $A(W) = \{e_i : 0 \leq i \leq n-1\}$ denote the *set of arcs in* W. A node v_i with i even/odd is called an *even/odd node* of W.

Definition 3.15. An M-alternating walk $W = (v_0, e_0 v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n)$ is called *special* if its even nodes are pairwise distinct, and its odd nodes are pairwise distinct. The *starting segment* of a walk $(v_0, e_0, v_1, e_1, \ldots, v_n)$ of length k is $(v_0, e_0, v_1, e_1, \ldots, v_k)$. Notice, the starting segment of a special M-alternating walk is a special M-alternating walk.

Claim 3.16. *If for some nodes $w, z \in V$ there is an even M-alternating walk from w to z, then there is a special even M-alternating walk from w to z, too.*

Proof. If an M-alternating walk W is not special, then $v_i = v_j$ (for some $i < j$, $i \equiv j \mod 2$), so a shorter M-alternating walk can be constructed by deleting the section from v_i to v_j. So it is easy to see, that a shortest M-alternating walk of even length from v_0 to v_n must be special. □

Definition 3.17. Let $L = L(D, M)$ be the set of nodes in V which arise as a last node of some even alternating walk.

In the algorithmic section we will discuss that given an even factor M in a digraph D, the set $L(D, M)$ can be determined quite easily – by using an ordinary breadth first search in an auxiliary bipartite graph. The following Claim follows from the definition of a special even M-alternating walk.

Claim 3.18. *For a special even M-alternating walk W, $M\Delta A(W)$ is a path-cycle-factor with $V^-(M\Delta A(W)) = V^-(M)$.* □

Notice, that $M\Delta A(W)$ is not necessarily an even factor, see Figure 1 for example. On the left hand side of Figure 1 we have an odd-cycle-symmetric graph where the bold lines form an even factor M, and $v_0, v_1, \ldots, v_8$ is a special M-alternating walk. On the right hand side the bold lines form a path-cycle-factor $M\Delta A(W)$, illustrating Claim 3.18.

Proof of Theorem 3.1. We prove Theorem 3.1 by induction on $|V|$. Consider an odd-cycle-symmetric digraph $D = (V, A)$, let $M \subseteq A$ be a maximum even factor. We need to show that there is a verifying set X giving equality $|M| = \tau_D(X)$. Recall the definition of $V^-(M) := \{z \in V : \varrho_M(z) = 1\}$, so $V - V^-(M)$ the set of M-source-nodes.

Case I. Suppose there is an arc $ab = e \in A$ with $a \in L = L(D, M)$ and $b \in V - V^-(M)$. In this case we will find a maximum even factor N which fits an odd cycle, the proof will be completed using Lemma 3.11. By Claim 3.16 there is a special even M-alternating walk W with last node a, suppose $q = v_0 \in K^+$ is the first node, i.e.

$$W = (q = v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n = a).$$

Let $n = 2l$ be the length of W, let W_i be the starting segments of W length $2i$ (for $i = 0, \ldots, l$), of course each W_i is a special even M-alternating walk. By Claim 3.18 $M_i := M\Delta A(W_i)$ are path-cycle-factors, here $M_0 = M$. It is easy to see that for $i \le n - 1$

$$M_{i+1} = M_i + v_{2i}v_{2i+1} - v_{2i+2}v_{2i+1}, \tag{8}$$

i.e. M_{i+1} is obtained from M_i by replacing an arc entering v_{2i+1} by a different arc entering v_{2i+1}. (Figure 2 illustrates the sequence of M_i's for the example of D, W given in Figure 1.)

Subcase Ia. Suppose M_l is an even factor. It is easy to see that M_l has no arc leaving a; by assumption $b \in V - V^-(M)$, thus M_l has no arc entering b; hence $M_l + ab$ is a path-cycle-factor. $M_l + ab$ is not an even factor, since $|M_l + ab| > |M|$. As M_l is an even factor, there must be a unique odd cycle $C \subseteq M_l + ab$, and for this cycle $ab \in C$. So $N := M_l$ fits C.

Subcase Ib. Suppose M_l is not an even factor. M_0 is an even factor, consider the smallest $0 \le i < l$ for which M_i is an even factor, but M_{i+1} is not an even factor. Then by (8) there is a unique odd cycle C in M_{i+1}, it is easy to see that it contains $v_{2i}v_{2i+1}$, so $N := M_i$ fits C. (In Figure 2 M_3 fits a cycle of length 5 occurring in M_4.)

Now we have a maximum even factor N which fits an odd cycle C. By part 1 of Lemma 3.11 $N' = N/V(C)$ is a maximum even factor in $D' = D/V(C)$. By induction, there is a verifying set X' for N' in D'. By part 2 of Lemma 3.11 there is a verifying set for N in D, which completes the proof in case I.

Case II. Suppose there is no arc $ab = e \in A$ with $a \in L = L(D, M)$ and $b \in V - V^-(M)$.

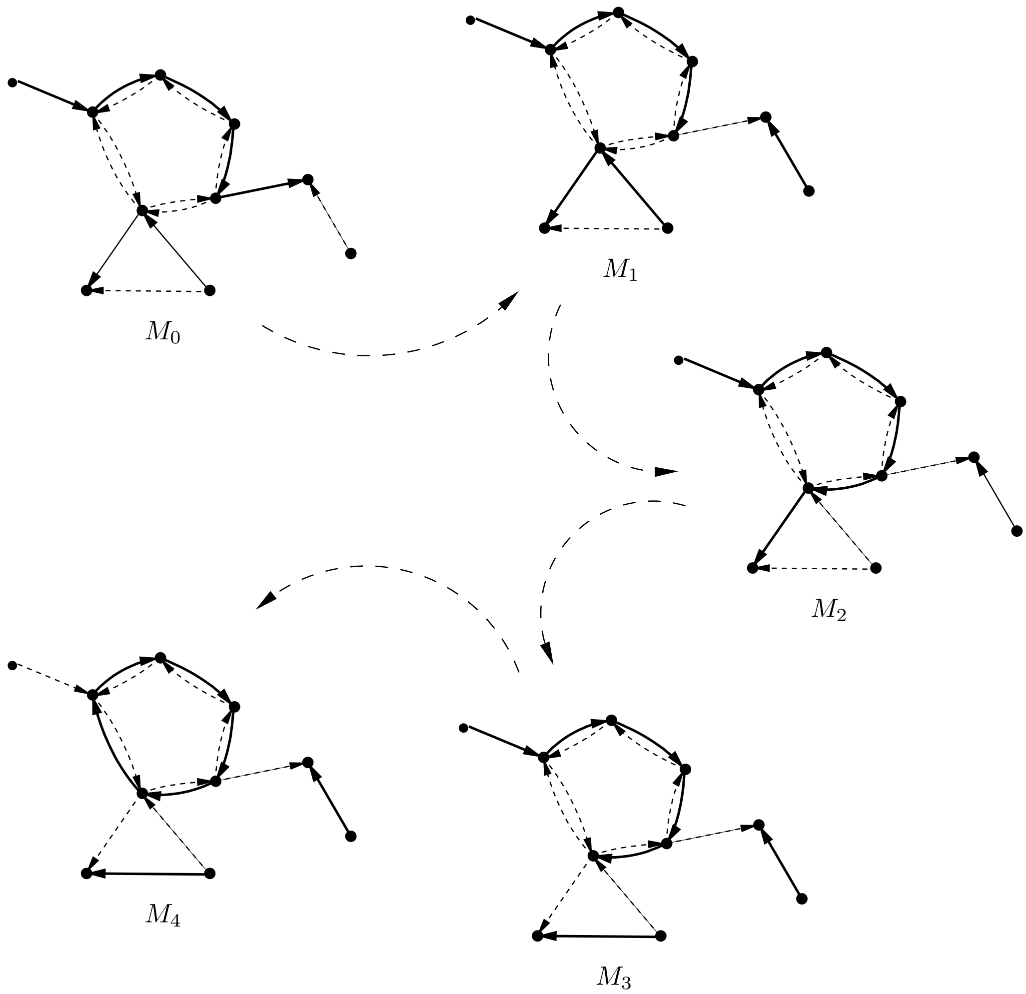

Figure 2: The sequence of M_i's in Case I for W indicated in Figure 1.

We will prove that L is a verifying set. Let $M_1 := M[L] = \{vz \in M : v \in L, z \in L\}$, $M_2 := \{vz \in M : v \in L, z \in V - L\}$ and $M_3 := \{vz \in M : v \in V - L\}$. Let S be the set of source-nodes in $D[L]$.

Claim 3.19. *In Case II we have* $|M_3| = |V| - |L|$, $|M_2| = |\Gamma_D^+(L)|$ *and* $|M_1| = |L| - |S|$.

Proof. The first equality follows from $K^+ \subseteq L$.

Consider a node b in $\Gamma_D^+(L)$, then there must be a node a in L for which $ab \in A$. $b \in V^-(M)$ follows from the assumption of case II, thus there must be an arc $cb \in M$. By the definition of L there is an even M-alternating walk W with last node a, the extension of W by arcs ab and cb gives an even M-alternating walk with last node c, thus $c \in L$. We get that each node in $b \in \Gamma_D^+(L)$ is covered by an arc $cb \in M_2$ with $c \in L$, which implies the second equality.

For the third equality, consider a node $b \in L - S$; then there is an arc $ab \in A$ with $a \in L$. From the assumption of case II we get that b must be in $V^-(M)$, i.e. there must be an arc $cb \in M$. There is an even M-alternating walk W with last node a, the extension of W by arcs ab and cb gives an even M-alternating walk with last node c, thus $c \in L$. We get that each node in $b \in L - S$ is covered by an arc $cb \in M_3$, which implies the third equality. □

The proof of Theorem 3.1 is completed by the following calculation:

$$|M| = |M_1| + |M_2| + |M_3| = |V| + |\Gamma_D^+(L)| - |S| \geq |V| + |\Gamma_D^+(L)| - \sigma(D[L]) \geq |M|.$$

□□

4 A Gallai-Edmonds-type structure

A structural description was given by Edmonds [6] and Gallai [7] on the structure of maximum matchings in a graph. The properties of the Gallai-Edmonds decomposition are extended to even factors in the following theorem, a proof was given in [12] for weakly symmetric digraphs, which also works for odd-cycle-symmetric digraphs. Here we give a different proof.

Theorem 4.1 (Pap, Szegő, [12]). *Suppose* $D = (V, E)$ *is odd-cycle-symmetric, let* L_D *be the set of nodes* $v \in V$ *for which there is a maximum even factor* M *with* $\delta_M(v) = 0$. *Then the following assertions hold.*

1. $\nu(D) = |V| + |\Gamma_D^+(L_D)| - \sigma(D[L_D])$
2. *For any verifying set* X *we have* $L_D \subseteq X$.
3. *All source-component in* $D[L_D]$ *is symmetric-critical.*
4. M *is a maximum even factor if and only if it fulfills the inequalities (5)–(7) for* $X = L_D$ *with equality.*

Proof. Assertion 2 also follows from assertion 1 by Claim 3.4, assertion 3 follows from assertion 2, assertion 4 also follows from assertion 1. We prove the first assertion by induction on $|V|$. Consider a maximum even factor M, in Section 3 we have shown that (Case I) there is a maximum even factor N which fits an odd cycle C or (Case II) $L(D, M)$ is a verifying set.

First suppose $L(D, M)$ is a verifying set. By Claim 3.4 $L_D \subseteq L(D, M)$, if $L_D = L(D, M)$ then we are done. Otherwise consider a node $v \in L(D, M) - L_D$, by definition there is a special even M-alternating walk W with last node v. $M \Delta A(W)$ is a path-cycle-factor with no arc leaving v, so by $v \notin L_D$ we get that $M \Delta A(W)$ is no even factor. By the argument in Subcase Ib we get a maximum even factor N which fits an odd cycle C.

If a maximum even factor N fits an odd cycle C, then by Lemma 3.11 $N' = N/V(C)$ is a maximum even factor in $D' = D/V(C)$. Thus $\{C\} \in L_{D'}$. By induction $L_{D'}$ is a verifying set in D', let X be its pre-image in D. We need to show that X is a verifying set and $X \subseteq L_D$, since by Claim 3.4 these imply $X = L_D$.

N is a maximum even factor fitting C, thus the nodes in $V(C)$ are easily seen to be in L_D. Consider a node $v \in X - \{C\}$, let M be a maximum even factor in D' with $\delta_M(v) = 0$. The construction in Claim 3.12 gives a maximum even factor $M'' := M' \cup M_{st}$ in D with $\delta_{M''}(v) = 0$, this proves $X \subseteq L_D$.

Claim 3.13 implies that X is a verifying set. □

Remark. In Theorem 4.1 the set L_D corresponds to the set "D", while $\Gamma^+_D(L_D)$ corresponds to the set "A" in the Gallai-Edmonds decomposition. The change of notation is to avoid ambiguity of the notation D for the digraph.

5 An algorithm for maximum even factors

The key to this algorithm is the observation that the set $L(D, M)$ defined in Section 3 can be constructed in linear time, as it reduces to alternating paths in an auxiliary bipartite graph. Other notions in the proof – like V^+, V^-, contractions of odd cycles – can be easily represented in linear time by an appropriate data structure for adjacency.

In order to construct a polynomial algorithm to find a maximum even factor and a verifying set in an odd-cycle-symmetric graph, we use the same approach as in the Section 3, the proof contains all essential components of the algorithm.

We describe a subroutine in Table 1 with input of an odd-cycle-symmetric graph $D = (V, A)$ and an even factor $M \subseteq A$. The subroutine returns either a larger even factor in D or a verifying set X for M. An algorithm to solve the maximum even factor problem is given by applying this subroutine repeatedly as follows. We begin with any even factor in D, for example $M = \emptyset$. We apply the subroutine to the input G, M – as long as we get a larger even factor we apply the subroutine with the larger even factor repeatedly. As an even factor has cardinality at most $|V|$, after at most $|V|$ applications of the subroutine we end up in a verifying set.

Proof of correctness and polynomiality. The subroutine has a REPEAT loop where in each loop we maintain a pair D_j, M_j with j increasing by 1 in each turn. D_{j+1} is

Table 1: The subroutine of the algorithm.

INPUT: An odd-cycle-symmetric graph $D = (V, A)$ and an even factor $M \subseteq A$.
OUTPUT: An even factor in D larger than M or a verifying set in D for M.

$j \leftarrow 0$, $D_0 \leftarrow D$, $M_0 \leftarrow M$

REPEAT UNTIL "Larger even factor" or "Verifying set".

Determine $L_j = L(D_j, M_j)$, $W_j(x)$ for each $x \in L_j$.

IF there is $ab = e \in A(D_j)$ with $a \in L_j$ and $b \in V(D_j) - V^-(M_j)$. (Case I)

Put any $ab = e \in A(D_j)$ with $a \in L_j$ and $b \in V(D_j) - V^-(M_j)$.

$2q \leftarrow$ length of $W_j(a)$.

$W_{j,i} \leftarrow$ starting segments of $W_j(a)$ of length $2i$ for $0 \leq 2i \leq 2q$.

$M_{j,i} := M_j \Delta W_{j,i}$.

IF some $M_{j,i}$ is not an even factor. (Subcase Ib)

$i \leftarrow$ the smallest for which $M_{j,i}$ is not an even factor.

$C_j \leftarrow$ the unique odd cycle in $M_{j,i}$.

$D_{j+1} \leftarrow D_j/V(C_j)$, $M_{j+1} \leftarrow M_{j,i-1}/V(C_j)$, $j \leftarrow j+1$.

ELSE IF $M_{j,q} + ab$ is not an even factor. (Subcase Ia)

$C_j \leftarrow$ the unique odd cycle in $M_{j,q} + ab$.

$D_{j+1} \leftarrow D_j/V(C_j)$, $M_{j+1} \leftarrow M_{j,q}/V(C_j)$, $j \leftarrow j+1$.

ELSE (here $M_{j,q} + ab$ is a *larger even factor*)

OUTPUT: EXPAND($M_{j,q} + ab$) is larger than M.

ELSE (Case II)

OUTPUT: M is maximum and the pre-image of L_j is a *verifying set*.

constructed by contracting a specially chosen odd cycle C_j in D_j, thus by Claims 3.6, 3.8, 3.9 each D_j will be odd-cycle-symmetric. In each loop we define $M_{j+1} := N_j/C_j$ for some N_j which fits C_j and $|N_j| = |M_j|$, thus the deficiency of even factor M_j stays the same for each j, that is $\text{def} := \text{def}_{D_j} M_j = \text{def}_D M$.

The sketch of a REPEAT loop:

1. we find an even factor N_j in D_j fitting an odd cycle C_j in form of $N_j = M_{j,q} + ab$, with $\text{def}_D N_j = \text{def}$; or

2. we find an even factor in D_j larger than M_j – then we expand it to a larger even factor than M in D; or

3. we find out that $L(D_j, M_j)$ is a verifying set in D_j for M_j – then we conclude that the pre-image of $L(D_j, M_j)$ in $D = D_0$ is a verifying set in D for M.

To clarify the notation in the algorithm: definitions for M-alternating walk, special M-alternating walk, starting segment, $L(\cdot, \cdot)$, V^+, V^-, K^+ apply from Sections 2, 3. In line 3 $W_j(x)$ is a special even M_j-alternating walk with last node x which is obtained in the process of determining $L(D_j, M_j)$.

In line 3 we need to determine $L(D_j, M_j)$ for some odd-cycle-symmetric graph D_j and an even factor M_j – without loss of generality, let us show how to determine $L(D, M)$. M-alternating walks reduce to alternating walks in an auxiliary bipartite graphs as formulated below, these are easier to work with. Let D' be the bipartite graph with node set $V' \cup V''$ and arc set $A' = \{u'v'' : uv \in A\}$, let us use $e' = u'v''$ if $e = uv \in A$. Then $M' := \{e' : e \in M\}$ is a matching in D' and $K' := \{v' : v \in K^+\}$ is the set of nodes in V' exposed by M'. In fact matchings are exactly the images of path-cycle-factors.

An M'-*alternating walk* in D' is a walk on edges $f_0, f_1, f_2, \ldots$ for which $f_i \in M'$ whenever i is odd, and the first node in the walk is in K'. An M'-alternating walk is an M'-*alternating path* if it is a path (i.e. it uses no edge or node twice). Let R be the set of nodes in V' which can be reached by an M'-alternating path with first node in K'. We make the following observations:

1. M'-alternating walks $W' = (v'_0, e'_0, v'_1, e'_1, \ldots, v'_{n-1}, e'_{n-1}, v'_n)$ in D' are exactly the images of M-alternating walks $W = (v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n)$ in D.

2. In particular, the images of special alternating walks are exactly the M'-alternating paths.

3. $L(D, M) = \{v : v' \in R\}$.

Alternating paths in a bipartite graph, as well as the set R can be found in linear time $O(|V'| + |A'|)$, so $L(D, M)$ can be determined in linear time $O(|V| + |A|)$. We use a data-structure to encode an alternating forest on $V' \cup V''$ by a pointer $\phi : V' \cup V'' \to V' \cup V''$ so that for any $v_0 \in R$ the sequence defined by $v_{i+1} := \phi(v_i)$ will be the node-sequence of an even alternating path ending in v_0. This provides a special even M-alternating walk $W(x)$ for all nodes $x \in L(D, M)$.

In case of the "**IF**" in line 4, the operations in lines 5-8 yield a sequence $M_{j,i}$ as a counterpart of the sequence in Case I of Section 3. By Claim 3.18, $M_{j,i}$ is a path-cycle-factor; of course it has the same deficiency as M_j. Here $M_{j,0} = M_j$ is an even factor, and $M_{j,i+1} = M_{j,i} + ab - cb$ for some arcs $ab, cb \in A(D_j)$.

When the "**IF**" in lines 9-12 takes effect – corresponding to Subcase Ib – we have $i \geq 1$ and $M_{j,i-1}$ is an even factor. There are some arcs $ab, cb \in A(D_j)$ for which $M_{j,i} = M_{j,i-1} + ab - cb$, hence there are at most one odd cycle in $M_{j,i}$ (this justifies line 11). This unique odd cycle C_j contains ab and it is easy to see that $M_{j,i-1}$ fits the odd cycle C_j. A similar argument shows that if the "**ELSE IF**" part in lines 13-15 takes effect – corresponding to Subcase Ib – then $M_{j,q}$ fits an odd cycle C_j.

If the subroutine gets to line 16, then it exhibits a larger even factor in D. To do this, we need the procedure EXPAND(N) which starts with an even factor $N = N_j := M_{j,q} + ab$ in a contracted graph D_j and outputs an even factor in $D = D_0$ with deficit $\text{def}_{D_j} N_j < \text{def}$. Procedure EXPAND is done by running "expand one odd cycle" for $D_{j'}, N_{j'}$ repeatedly for j' decreasing from j to 1. The procedure "expand one odd cycle" expands an even factor in some D'_j to construct an even factor in $D_{j'-1}$ with the same deficiency. "expand one odd cycle" can be done in linear time $O(|V(C_{j'-1})|)$ by adding an even path on one segment of $\overleftrightarrow{C_{j'}}$ and some two-arc cycles in $\overleftrightarrow{C_{j'}}$.

An odd cycle is contracted in each round of the REPEAT loop in the lines 12 or 15. In the last loop – when we get to line 18 – we have Case II from the Section 3, thus L_j is a verifying set for M_j. Let $L_j^{(i)}$ denote the set of nodes in D_i corresponding to the nodes in L_j, so in line 19 the pre-image of L_j is equal to $L_j^{(0)}$.

Claim 5.1. *Using the notation of the subroutine, if for some* $1 \leq j' \leq j$ *the set* $L_j^{(j')}$ *is a verifying set for* $M_{j'}$*, then* $L_j^{(j'-1)}$ *is a verifying set for* $M_{j'-1}$.

Proof. Claim 3.13 applies for $D = D_{j'-1}$, $N' = M_{j'}$, $D' = D_{j'}$, $C = C_{j'}$ and $N = M_{j',m}$ for some $1 \leq m \leq q$, with $X' = L_j^{(j')}$ and $X = L_j^{(j'-1)}$. □

This Claim completes the proof of correctness of the algorithm, we determine the running time expressed in terms of $n = |V|, m = |A|$. Let us also declare that graphs, path-cycle-factors are given by adjacency structures in the algorithm.

We have at most n augmentations, in one augmentation we have at most n contractions. Suppose, when we get C_j we construct D_{j+1} by building its whole graph structure with the necessary pointers from and to D_j, this takes $O(n + m)$ time. As already discussed before, determining $L_j, W_j(x)$ is done in $O(n + m)$ time. Finding ab in line 4 needs $O(m)$ time, then W_i is constructed in $O(n)$ time. For some i, j we determine $M_{j,i}$ in $O(n)$ time since a special walk has length at most $2n$. It takes $O(n)$ time to check whether $M_{j,i}$ is an even factor or to exhibit C_j in lines 11 or 14. In total we need $O(n^2)$ time to complete lines 9-15 for each $M_{j,i}, 0 \leq i \leq q$ and for $M_{j,q} + ab$. In an augmentation step we need one EXPAND subroutine, which is done by applying "expand one odd cycle" $j \leq n$ times, thus EXPAND needs $O(n^2)$ time. This adds up to $O(n \cdot n \cdot (n + m + n^2)) = O(n^4)$ time if we consider each augmentation and each contraction.

ORIGIN is needed once in the end, needing $O(n^2)$ time, thus the total running time of the maximum even factor algorithm is $O(n^4)$.

□□

Remark. The algorithm described in this section does not necessarily return the canonical verifying set L_D from Theorem 4.1, but only minor refinements are needed to find L_D. Consider the situation when the algorithm finds out that L_j is a verifying set.

To see that the pre-image of L_j is exactly L_D we need $L_j = L_{D_j}$. That is, we need to find for all $x \in L_j$ an even factor $M(x)$ in D_j with $\delta_{M(x)}(x) = 0$. To this end, let $W_j(x)$ be a special even M_j-alternating walk with last node x, let $M(x) := M_j \Delta W_j(x)$. If for all nodes $x \in L_j$ $M(x)$ is an even factor, then we are done. Otherwise, we apply lines 9-12 to $W_j(x)$, we go on after the new contraction.

6 Conclusions and examples

1.) Let us try to distinguish concepts which are basically the same as for the matching case, and concepts which needed to be modified – compared to Edmonds' matching algorithm. The following lemma is the analogue of Lemma 3.11 for maximum matchings.

Lemma 6.1. *Suppose N is a matching in an undirected graph G and C is an N-alternating odd cycle adjacent to an exposed node; define $G' = G/V(C)$ and $N' = N/V(C)$.*

1. *If N is maximum, then N' is maximum.*
2. *If Z' is a barrier (in the Berge-Tutte formula) for N', then the pre-image Z is a barrier for N.*

In fact the definition of "N fitting C" was motivated by the nice correspondence of lemmas 6.1 and 3.11. An odd cycle C with the above properties can be used in an inductive or algorithmic way – both for matchings and for even factors. The key difference between the concept of this paper and well-known concepts for matchings is the way how we find this odd cycle.

2.) Notice that M-alternating walks are easier to handle than alternating paths: the concatenation of two M-alternating walks is an M-alternating walk, too. The analogue statement neither holds for M-alternating paths for matchings, nor for special walks for even factors.

For this reason, the proof and the algorithm uses M-alternating walks, let us sketch how this works for the special case of matchings. The algorithm first tries to find an alternating walk W joining two exposed nodes, consider a shortest one. If W is an alternating path, then the actual matching can be augmented. Otherwise, let W' be a shortest segment of W which is not a path. One can easily see that W' is an M-alternating blossom, i.e. $W' = P \cup C$ where P is an M-alternating even path from an exposed node p to q, and C is an M-alternating odd cycle for which

$V(P) \cap V(C) = \{q\}$ (see Figure 3). We apply Lemma 6.1 for $M\Delta P$, C, we are done by induction.

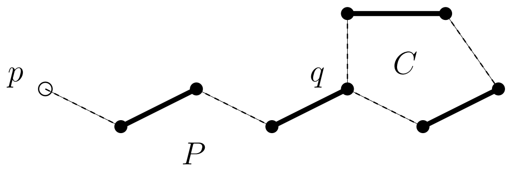

Figure 3: An M-alternating blossom.

If there is no M-alternating walk W joining two exposed nodes, then the set $L := L(\overleftrightarrow{G}, \overleftrightarrow{M})$ is an independent set, and $\Gamma_{\overleftrightarrow{G}}(L)$ will be a verifying set in the Berge-Tutte formula.

3.) Consider the reduction of the maximum matching problem in a graph G to the maximum even factor problem in $\overleftrightarrow{G}$. A maximum matching in G has exactly $\nu(\overleftrightarrow{G})/2$ edges, since

(a) for a matching M in G we can give the even factor $\overleftrightarrow{M}$ having two-arc cycles in place of the edges of M, and

(b) if M is an even factor in $\overleftrightarrow{G}$, then an even cycle in M can be replaced by a matching with half as many edges, a directed path of length l can be replaced by a matching with $\lceil l/2 \rceil$ edges.

The proof of Theorem 3.1 can be specialized to the instance $\overleftrightarrow{G}$ to prove the Berge-Tutte formula, but we cannot avoid directed paths to occur. Hence, we in fact solve the "maximum 2-matching without odd cycles" problem.

A 2-matching is a node-disjoint family of edges with weight 2, and paths or cycles with weight 1 on each edge. In Figures 4, 5 only the supporting edge sets of 2-matchings are indicated with bold lines. Alternating walks, the symmetric difference should be understood by the reduction to $\overleftrightarrow{G}$, an alternating walk is indicated by a dotted line on the left hand side, while the symmetric difference is indicated on the right hand side.

We explain Subcase Ib by the example shown in Figure 4. q, b are the exposed nodes, the dotted line is a special even M-alternating walk W from q to a, the edge ab has the role of Case I. The algorithm constructs the sequence of M_i's step by step with i increasing. When the algorithm gets to $i = 4$ an odd cycle C occurs which will be contracted.

In Subcase Ib, $M_l + ab$ is not an even factor, so it has at least one odd cycle. In general, $M_l + ab$ may have several odd cycles in this case; for example in Figure 4 we have $l = 12$ and the 2-matching $M_{12} + ab$ has three odd cycles. The algorithm does not choose one of them as C for the contraction, Figure 5 (using the same notation) shows an example where the cycle C does not appear in $M_l + ab$.

4.) A maximum 2-matching can be found in polynomial time as it reduces to bipartite matching. It could be an easy treatment of non-bipartite matching if the following

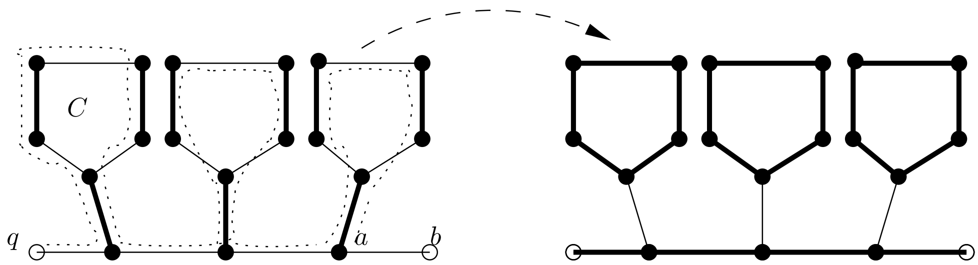

Figure 4:

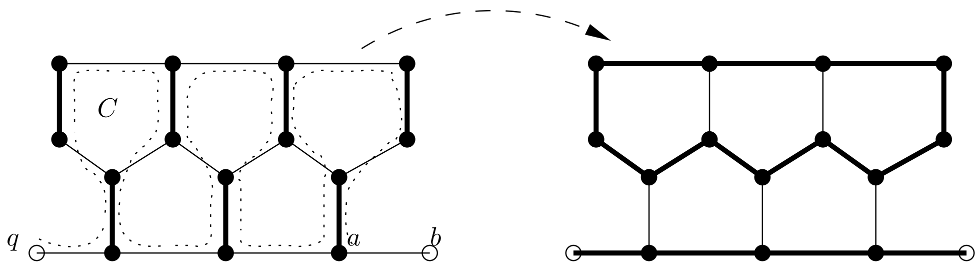

Figure 5:

algorithm would output a maximum matching:

> "Find a maximum 2-matching in an undirected graph G. If it has no odd cycles, then a maximum matching is given by decomposing its even cycles into a matching. Suppose it has some odd cycles, let C be one of them. Find a large matching M' in $G' = G/C$ recursively, expand M' to a matching M of size $|M'| + |C| - 1$. Output M."

Unluckily, this does not work out if we choose C arbitrarily. No rule has been found for the choice of C which guarantees that M will be maximum.

The subroutine in Table 1 walks around this uncertainty instead of solving it. By "walking around it" we mean the part when the sequence of M_i's is constructed in Subcase Ib, there $M_l + ab$ corresponds to a larger 2-matching. We do not concentrate on the odd cycles at hand in $M_l + ab$, we rather find another 2-matching which has exactly one fitting odd cycle.

5.) In the algorithm we use the auxiliary bipartite graph D' to determine $L(D, M)$, there an alternating forest can be constructed to encode alternating paths reaching nodes in R. In Edmonds' matching algorithm this structure is maintained after the contraction of an odd cycle C, in fact C is chosen so that it is a fundamental cycle for the alternating forest. However, in the present algorithm, the contraction of an odd cycle can completely destroy the alternating forest in the auxiliary graph. We have found no way to avoid this, and we regard this as one of the reasons why it turned out to be difficult to find a combinatorial algorithm. We deal with this by not even trying to maintain any structure of alternating walks after a contraction.

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