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Uniform partitioning to bases in a matroid

Zsolt Fekete and Jácint Szabó

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Zsolt Fekete⋆ and Jácint Szabó⋆⋆

Abstract

We say that a matroid $\mathcal{M}$ is *k-uniform* if – provided that it is the disjoint union of its two bases – for any given k-element subpartition $\mathcal{P}$ of its ground set, $\mathcal{M}$ can be partitioned into two disjoint bases B_1, B_2 such that $||B_1 \cap P| - |B_2 \cap P|| \leq 1$ for all $P \in \mathcal{P}$. The circuit matroid of an undirected graph G is called *k-star-uniform* if the above holds for all k-element subpartitions containing stars of independent vertices of G. In this paper we prove that the circuit matroids are 1-uniform and 3-star-uniform but not necessarily 2-uniform and 4-star-uniform.

Keywords: matroid, spanning tree

1 Introduction

Let $\mathcal{M}$ be a matroid on ground set E and let $l \geq 2$ an integer. We say the partition $E = \bigcup_{1 \leq i \leq l} B_i$ to bases B_i of $\mathcal{M}$ is *uniform to the subpartition* $\mathcal{P}$ *of* E if

$$||B_i \cap P| - |B_j \cap P|| \leq 1$$

for all $1 \leq i < j \leq l$ and $P \in \mathcal{P}$.

Let k be a positive integer. In this paper we consider matroid classes with the property that if a matroid in the class has a partition into l bases then given any k-element subpartition $\mathcal{P}$ of its ground set it has a partition into l bases which is uniform to $\mathcal{P}$. We will consider only matroid classes closed under deletion so observe that it is enough to consider matroids which are the disjoint union of their *two* bases, i.e. we assume that $l = 2$.

⋆Dept. of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary H-1117 and Communication Networks Laboratory, Pázmány Péter sétány 1/A, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). Research is supported by OTKA grant T 037547, by the Egerváry Research Group of the Hungarian Academy of Sciences and by European MCRTN Adonet, Contract Grant No. 504438. e-mail: `fezso@cs.elte.hu`.

⋆⋆Dept. of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). Research is supported by OTKA grant T 037547, by the Egerváry Research Group of the Hungarian Academy of Sciences and by European MCRTN Adonet, Contract Grant No. 504438. e-mail: `jacint@cs.elte.hu`.

Thus we say that a matroid $\mathcal{M}$ is *k-uniform* if, whenever $\mathcal{M}$ is the disjoint union of its two bases, for any k-element subpartition $\mathcal{P}$ of ground set E it has a partition into two bases uniform to $\mathcal{P}$.

In Section 2 we prove that weakly base orderable matroids are 1-uniform and strongly base orderable matroids are k-uniform for any k. (For definitions see Section 2.)

The rest of the paper is concerned with circuit matroids of undirected graphs. We are interested in graphs G where $E(G)$ is the disjoint union of two spanning trees of G. Such a graph is called a *2-tree*. We think of these trees as a coloring to red and blue such that both the red and the blue edges form a spanning tree of G. We call this a 2*-tree-coloring*. Note that a 2-tree may have double parallel edge pairs but no loops. The *star* of a vertex $v \in V(G)$, denoted by $\Delta(v)$, consists of the edges of G incident to v. We say that the circuit matroid $\mathcal{M}_G$ of a graph G is *k-star-uniform* if, provided that G is a 2-tree, for any k-element subpartition $\mathcal{P}$ of $E(G)$ consisting of stars of independent vertices, G has a 2-tree-coloring uniform to $\mathcal{P}$.

It is well known and we use it without proof that circuit matroids are 1-star-uniform. (It also follows from Theorem 3.1.) In Thm. 5.2 we prove that they even 3-star-uniform. However, they are not necessarily 4-star-uniform, shown by the counterexample of Section 4.

We need a some preliminaries on 2-trees. A characterization of 2-trees were given by Nash-Williams [1]. $i_G(X)$ denotes the number of edges of G induced by the vertex set $X \subseteq V(G)$. $d_G(X,Y)$ denotes the number of edges between X and Y for the disjoint vertex sets X, $Y \subseteq V(G)$.

Theorem 1.1. (Nash-Williams)*[1] The graph G is a 2-tree if and only if $|E(G)| = 2|V(G)| - 2$ and $i_G(X) \leq 2|X| - 2$ for all $\emptyset \neq X \subseteq V(G)$.*

Definition 1.2. $X \subseteq V(G)$ is *tight* if $i_G(X) = 2|X| - 2$.

By the supermodularity of i_G the next claim follows easily.

Claim 1.3. *If G is a 2-tree then the union of two intersecting tight sets is tight. Moreover, if X is tight and $u \notin X$ then $d_G(X,u) \leq 2$.*

An operation used throughout is the *split* at vertex $v \in V(G)$. We call a split *admissible* if it results in a 2-tree. The inverse operation of a split is called *unsplit*. Let $v \in V(G)$ with $\deg_G(v) \leq 4$.

- If $\deg_G(v) = 2$ then *splitting* v means simply to delete v from G. Clearly, $G - v$ is also a 2-tree and any 2-tree-coloring of $G - v$ can be extended to a 2-tree-coloring of G in two ways, by arbitrary coloring one of the edges of v to blue and the other one to red.

- If $\deg_G(v) = 3$ then let the edges incident to v be e_i joining v to u_i for $i = 1, 2, 3$. *Splitting the edge-pair* e_i, e_j $(i \neq j)$ means to delete v from G and to add the edge e joining u_i to u_j resulting in the graph H. We also say that we *split* v *to a* u_iu_j*-edge*. Note that splitting the pair e_i, e_j is admissible unless G has a tight set X such that $u_i, u_j \in X$ and $v \notin X$. So Claim 1.3 clearly implies that for at least two choices of the edge-pair e_i, e_j the graph H is a 2-tree. In this case any 2-tree-coloring of H can be extended to a 2-tree-coloring of G in the following way (the *unsplitting at* v). If the split edge e is, say, blue then delete e from H, add v, add the edges e_i, e_j colored blue and let the third edge incident to v be red.

- If $\deg_G(v) = 4$ then let the edges incident to v be e_i joining v to u_i for $1 \leq i \leq 4$. *Splitting the edge-pair* e_1, e_2 means to delete v from G and to add the edge e joining u_1 to u_2 and the edge f joining u_3 to u_4 resulting in the graph H. We also say that we *split* v *to a* u_1u_2*-edge and to a* u_3u_4*-edge*. It is easy to see that this split is admissible unless G has a tight set $v \notin X$ such that either $u_1, u_2 \in X$ or $u_3, u_4 \in X$. $\mathcal{M}_G$ is 1-star-uniform so for at least one from the three possible splits, the graph H is a 2-tree. In this case any 2-tree-coloring of H can be extended to a 2-tree-coloring of G in the following way (the *unsplitting at* v). First pinch e and f by vertex v. If e and f had different colors then we are done. Otherwise, say, both e and f were blue so we produced a circuit C in the blue tree. Now re-color an edge of C incident to v to red.

Thm. 1.1 implies that a 2-tree (with at least 2 edges) has either a vertex of degree 2 or two vertices of degree 3. Thus it is always possible to perform an admissible split.

In Section 3 we prove that circuit matroids are 1-uniform but not 2-uniform. Section 4 proves that circuit matroids are 2-star-uniform but not 4-star-uniform. Finally, in Section 5 we prove with a different method than that of Section 4, that circuit matroids are 3-star-uniform.

We mention here the following conjecture (see [2], Exercise 4.69).

Conjecture 1.4. *Let* $G = (E, V)$ *be an undirected graph. If* $|E| = 2|V| - 2$, $i(X) \leq 2|X| - 3$ *holds for every* $X \subset V$, $|X| \geq 2$ *and every vertex of* G *has degree at most 4 then* E *can be partitioned into two Hamiltonian paths.*

This conjecture gave some motivation for this paper. Indeed, observe that in this case a partition of E into two Hamiltonian paths can be regarded as a partition into two trees uniform to the set of all stars. So it would be interesting to investigate uniform partitions in 2-trees which satisfy properties like $i(X) \leq 2|X| - 3$ or connectivity requirements.

In this paper all graphs G are undirected. Let $\delta_G(X) = d_G(X, V(G) - X)$ for $X \subseteq V(G)$. We say that edge e *enters* $X \subseteq V(G)$ if exactly one end-vertex of e is contained in X, and e is an *xy-edge* if the two end-vertices of e are x and y.

2 Base orderable matroids

In this section we concern base orderable matroids.

Definition 2.1. We say that a matroid $\mathcal{M}$ is *weakly base orderable* if for every two bases B_1, B_2 there exists a bijection $f : B_1 \to B_2$ such that for every $v \in B_1$ the sets

$B_1 - v + f(v)$, $B_2 + v - f(v)$ are bases of $\mathcal{M}$. $\mathcal{M}$ is *strongly base orderable* if for every two bases B_1, B_2 there exists a bijection $f\colon B_1 \to B_2$ such that $B_1 - X + f(X)$ is a base of $\mathcal{M}$ for every $X \subseteq B_1$.

Note that strong base orderability implies the weak one, since the same f, and $X_1 = \{v\},\ X_2 = B_1 - \{v\}$ will do.

Theorem 2.2. *Weakly base orderable matroids are 1-uniform.*

Proof. Let $\mathcal{M}$ be a weakly base orderable matroid on ground set E which is the disjoint union of its two bases and let $P \subseteq E$. Choose disjoint bases B_1, B_2 minimizing

$$||B_1 \cap P| - |B_2 \cap P||$$

Suppose indirectly that, say, $|B_1 \cap P| - |B_2 \cap P| \geq 2$. This implies that there exists $v \in B_1 \cap P$ such that $f(v) \notin B_2 \cap P$ for the bijection f guaranteed by the definition. But then $||B_1 \cap P| - |B_2 \cap P|| > ||B_1' \cap P| - |B_2' \cap P||$ for the disjoint bases $B_1' = B_1 - v + f(v)$, $B_2' := B_2 + v - f(v)$, which is a contradiction. □

Theorem 2.3. *Strongly base orderable matroids are k-uniform for any k.*

Proof. Let $\mathcal{M}$ be a strongly base orderable matroid on ground set E which is the disjoint union of its two bases. Observe that it is enough to prove that $\mathcal{M}$ has two disjoint bases uniform to any *partition* $\mathcal{P} = \{P_1, \ldots, P_k\}$ of E.

Let B_1, B_2 be two disjoint bases of $\mathcal{M}$ and let $f\colon B_1 \to B_2$ be the bijection guaranteed by the definition. Let $D = (\mathcal{P}, \{e_v : v \in B_1\})$ be a directed graph where e_v joins $P_i \ni v$ to $P_j \ni f(v)$. We allow loops and parallel arcs, therefore D has exactly $|B_1| = |B_2|$ arcs. Take a new, almost Eulerian orientation of D, which means that $|\delta(P) - \varrho(P)| \leq 1$ for all $P \in \mathcal{P}$. Let $X = \{v \in B_1 : e_v$ is reoriented in the new orientation$\}$. Now $B_1' = B_1 - X + f(X)$ and $B_2' = B_1 - (B_1 - X) + f(B_1 - X)$ will do because for every $P \in \mathcal{P}$ we have $||B_1' \cap P| - |B_2' \cap P|| = |\delta(P) - \varrho(P)| \leq 1$. □

3 The circuit matroids are 1-uniform

In this section we prove that circuit matroids are 1-uniform. However, they are not necessarily 2-uniform, which is shown by the circuit matroid of K_4 and the subpartition of $E(K_4)$ consisting of two disjoint perfect matchings.

Theorem 3.1. *Circuit matroids are 1-uniform.*

Proof. Let G be a 2-tree and $P \subseteq E(G)$. We prove by induction on $E(G)$ that G has a 2-tree-coloring uniform to $\{P\}$. If $E(G) = \emptyset$ then the statement is trivially true. Recall that by Thm. 1.1, G has a vertex of degree at most 3.

Assume that G has a vertex v of degree 2. Let $\Delta(v) = \{e, f\}$. If, say, $e \in P$ and $f \notin P$ then by induction, $G - v$ has two disjoint spanning trees F_1 and F_2 uniform to $\{P - e\}$. Assume that, say, $|F_1 \cap (P - e)| \leq |F_2 \cap (P - e)|$. Now $F_1 + e$ and $F_2 + f$ are two disjoint spanning trees of G uniform to $\{P\}$. If $\{e, f\} \subseteq P$ or $\{e, f\} \cap P = \emptyset$ then apply the induction hypothesis for $G - v$ and $P - \{e, f\}$.

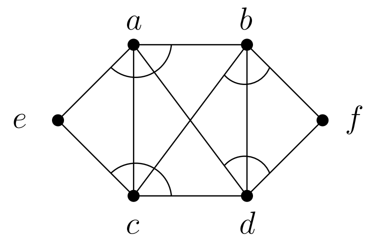

Fig. 1. The graph H

Assume now that G has a vertex v of degree 3. Let $\Delta(v) = \{e_1, e_2, e_3\}$ such that e_i joins v to $u_i \in V(G)$ for $i = 1, 2, 3$. Recall that $G - v + u_iu_j$ is a 2-tree for at least two choices of $1 \le i < j \le 3$.

If $|\Delta(v) \cap P| = 0$ then apply induction to any admissible split at v and P. Now the unsplitting at v gives a 2-tree-coloring of G uniform to $\{P\}$.

If $|\Delta(v) \cap P| = 1$ then we can assume that, say, $G - v + u_1u_2$ is a 2-tree and $e_1 \in P$. Apply induction to $G - v + u_1u_2$ and $P - e_1 + u_1u_2$. Then the unsplitting at v gives a 2-tree-coloring of G uniform to $\{P\}$.

If $|\Delta(v) \cap P| = 2$ then we can assume that, say, $G - v + u_1u_2$ is a 2-tree and $e_1, e_3 \in P$. Apply induction to $G - v + u_1u_2$ and $P - \{e_1, e_3\}$. As before, the unsplitting at v gives a 2-tree-coloring of G uniform to $\{P\}$.

Finally, if $|\Delta(v) \cap P| = 3$ then simply apply induction to an admissible split at v to some edge u_iu_j and $P - \Delta(v) + u_iu_j$. □

4 The circuit matroids are 2-star-uniform

In the next section we prove that the circuit matroids are 3-star-uniform. Since this proof relies on the fact that they are 2-star-uniform, first we prove this in Theorem 4.1 in this section. Although the solution method for the case $k = 3$ would give a shorter proof for Theorem 4.1, here we include a different proof, not to repeat ourselves.

We cannot state a stronger result, which is shown by the non 4-star-uniform circuit matroid below. Consider the graph H in **Fig. 1** and the subpartition $\mathcal{P} = \{\{af, ab\}, \{ef, ed\}, \{be, bc\}, \{da, dc\}\}$. One can check that H has no 2-tree-coloring uniform to $\mathcal{P}$. Observe that each set of $\mathcal{P}$ consists of two adjacent edges. Now pinch each edge pair $\{e_i^1, e_i^2\} \in \mathcal{P}$ by a new vertex v_i for $1 \le i \le 4$ resulting in the 2-tree G. Note that the new vertices are independent in G. Assume that G has two disjoint spanning trees uniform to $\{\Delta(v_i) : 1 \le i \le 4\}$. Observe that each vertex v_i is incident to exactly one parallel edge-pair. Contracting this edge-pair for each v_i would give a 2-tree-coloring of H, which is impossible.

Theorem 4.1. *The circuit matroids are 2-star-uniform.*

Proof. Let G be a 2-tree and u, v be non-adjacent vertices of G. We will prove that there exists a partition of E into two spanning trees which is uniform to $\{\Delta(u), \Delta(v)\}$. Let $E(G)$ be the disjoint union of the spanning trees $F_b, F_r \subseteq E(G)$, called the *blue* and the *red* trees. In **Figs. 2–5** thick edges are blue and thin are red. For vertices $x \neq y$, the *blue branch of* x *wrt.* y denotes the component of $F_b - y$ containing x. For $s = u, v$, the blue edge e joining s and x is said to be *free (at s)* if the second vertex of the red $s - x$ path is not contained in the blue branch of x wrt. s. In this case x is said to be a *free blue neighbor of* s and the blue branch of x wrt. s is called *free* as well. Finally, we say that the blue edge e_b can be *exchanged* to the red edge e_r if both $F_b - e_b + e_r$ and $F_r - e_r + e_b$ are spanning trees of G.

If edge e joining x and y has multiplicity 1 (e.g. a free edge) then we use the notation 'xy' for e. If $y \neq u$ then denote the blue branch of y wrt. u by B_y, and the red $u - y$ path by Q_y. Note that for $s = u, v$ a free edge sx has the property that it can be exchanged only to edges not incident to s.

Let $\alpha = \deg_{F_b}(u) - \deg_{F_r}(u)$ and $\beta = \deg_{F_b}(v) - \deg_{F_r}(v)$. We prove the theorem by induction on $|\alpha| + |\beta| + \mathrm{sgn}(\alpha\beta)$. We are done if the value of this expression is at most 1. In the proof we pose some assumptions on the trees F_b and F_r with the property that if any assumption fails then it is possible to apply induction. Each assumption is valid from its declaration till the end of the proof.

Suppose that our 2-tree-coloring is not uniform because, say, $\alpha \geq 2$. Denote by α' (resp. β') the number of blue edges which are free at u (resp. v). Clearly $\alpha' \geq \alpha \geq 2$ and $\beta' \geq \beta$. If $\beta < 0$ then we can exchange any free blue edge ux to any edge of Q_x which enters B_x. In this way α decreased by 2 and β is increased by either 0 or 2, hence we can apply induction.

So we assume that

Assumption 4.2. $\beta \geq 0$.

Let ux be a free blue edge. If Q_x has an edge e which is not incident to v and which enters B_x then we can exchange ux to e, keeping β and decreasing α by 2. This must be the case when Q_x enters B_x more than once, because the number of edges of Q_x entering B_x is odd. Thus we may assume 4.3 since otherwise we can apply induction.

Assumption 4.3. For each free blue neighbor x of u the red $u - x$ path Q_x enters B_x exactly once, and this only entering edge of Q_x is incident to v.

Proposition 4.4. *$\beta' \geq \beta + \alpha' - 1$ and equality implies that there are exactly α' red edges joining v to the blue branch of u wrt. v, and that this branch is not free at v.*

Proof. Assumption 4.3 implies that the following considerations hold for Q_x whenever x is a free blue neighbor of u. First, Q_v is a subpath of Q_x.

In case v is not contained in a free blue branch wrt. u (**Fig. 2**) then Q_v does not intersect any free blue branches wrt. u, and the path $Q_x - Q_v$ starts from v, with its first edge it enters B_x and remains in B_x.

In case v is contained in some free branch B_z (**Fig. 3**) then $Q_v - v$ does not intersect any free blue branches wrt. u. For $x \neq z$, the path $Q_x - Q_v$ starts from v, with its first edge it enters B_x and remains in B_x. Finally, the path $Q_z - Q_v$ remains in B_z.

In both cases G has at least α' red edges joining v to the blue branch of u wrt. v. This clearly implies the statement. □

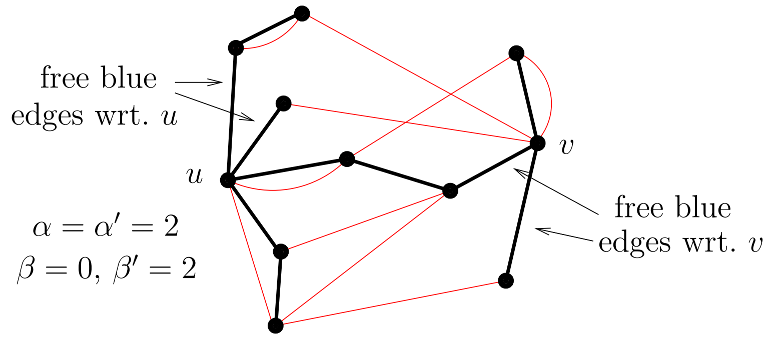

Fig. 2.

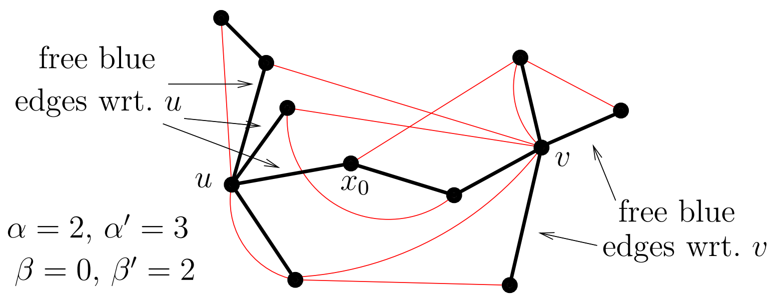

Fig. 3.

Suppose that $\beta > 0$. What we try to do now is to decrease β instead of α by applying the above considerations to v instead of u. If Assumption 4.3 fails with the $u \leftrightarrow v$ substitution then we can exchange a pair of edges decreasing β by 2 and keeping α. So we can apply induction because the value of the expression $|\alpha|+|\beta|+\text{sgn}(\alpha\beta)$ decreased even when $\beta = 1$. On the other hand, observe that the proof of Proposition 4.4 did not use that $\alpha \geq 2$. So if Assumption 4.3 holds with the $u \leftrightarrow v$ substitution then Proposition 4.4 yields that $\alpha' \geq \alpha + \beta' - 1$. But this contradicts to $\beta' \geq \beta + \alpha' - 1$ since $\alpha \geq 2$ and $\beta \geq 1$. Thus we pose the next assumption.

Assumption 4.5. $\beta = 0$.

Let the number of free blue branches wrt. v which do not contain u be β''. Proposition 4.4 implies that independently of the freeness of the blue branch of u wrt. v, $\beta'' \geq \alpha' - 1$ holds.

Suppose that vx is a free blue edge with the property that the blue branch of x wrt. v does not contain u, and there exists a red edge e not incident to u which can be exchanged to vx, see **Fig. 4**. Let, moreover, uy be a free blue edge wrt. u such that B_y does not contain v (there are at least $\alpha' - 1 \geq 1$ such edges). By Assumption 4.3 there exists only one red edge vz which can be exchanged to uy. Now exchange vx to e setting $\beta = -2$ and keeping α. Denote the new blue branch of y wrt. u by B'_y and the new red $u-y$ path by Q'_y. Clearly B_y is a subgraph of B'_y. We state that after this exchange uy can still be exchanged to vz, decreasing α by 2 and setting back

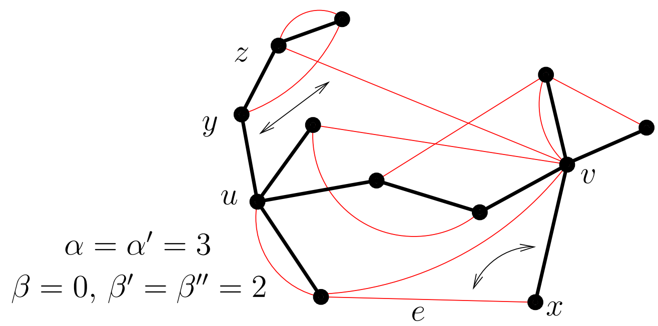

Fig. 4.

$\beta = 0$. First observe that the red edge vz enters B'_y. Second, the $z - y$ subpath of Q_y contains only edges induced by B_y so the $v - y$ subpath of Q_y does not contain e. So it is easy to see that the $v - y$ subpath of Q_y is a subpath of Q'_y and hence vz can be exchanged to uy.

Thus we pose the following assumption since otherwise we could apply induction.

Assumption 4.6. For each free blue neighbor x of v whose blue branch wrt. v does not contain u, the red $v - x$ path enters the blue branch of x wrt. v exactly once, and this only entering edge is incident to u.

Proposition 4.7. *If Assumptions* 4.3, 4.5 *and* 4.6 *hold then*

1. *the number of red edges joining u to B_v is β'',*
2. *B_v is not free at u,*
3. *$\alpha = 2$,*
4. *$\beta'' = \alpha' - 1$.*

Proof. Just like in the proof of Proposition 4.4, Assumption 4.6 implies that G has at least β'' red edges joining u to B_v. This gives that $\alpha' \geq \alpha + \beta'' - 1$ where equality holds only if there are exactly β'' red edges joining u to B_v, and B_v is not free at u. Now $\alpha' \geq \alpha + \beta'' - 1 \geq 2 + (\alpha' - 1) - 1 = \alpha'$ so equality holds throughout. The first equality gives *1.* and *2.*, while the second *3.* and *4.* □

We have two possibilities. Denote the second vertex of the *blue* $v - u$ path by w and denote the red $v - w$ path by R_w. Define B_u to be the blue branch of u wrt. v.

Case 1. *The blue edge vw is free at v*, see **Fig. 5**. Let y be a free blue neighbor of u. $v \notin V(B_y)$ by Proposition 4.7, *2*. Let vz be the only red edge which can be exchanged to uy by Assumption 4.3. Exchange uy to vz setting $\alpha = 0$ by Proposition 4.7, *3* and $\beta = 2$. After this exchange denote the new blue branch of u wrt. v by B'_u. We state that B'_u is free at v. The freeness of vw at v has the following consequences. First, $vz \notin R_w$ hence the new red $v - w$ path is still R_w. Second, the first edge of R_w does not enter B_u thus it does not enter $B'_u = B_u - B_y$. So after the exchange the blue branch of u is free at v and $\alpha = 0$, $\beta = 2$. Hence after the exchange Proposition 4.7, *2*

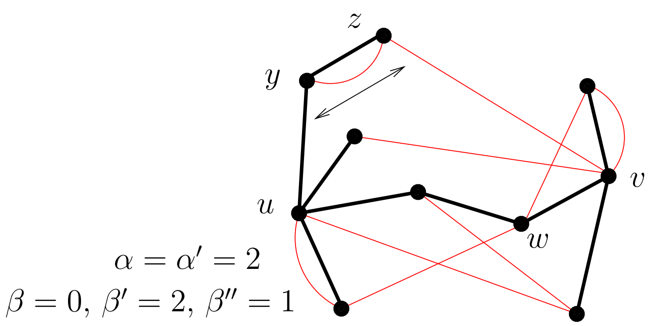

Fig. 5.

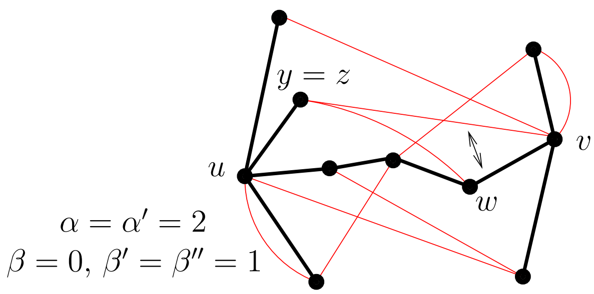

Fig. 6.

fails with the $u \leftrightarrow v$ substitution so at least one of Assumptions 4.3, 4.5 and 4.6 fail with the $u \leftrightarrow v$ substitution. Thus it is possible to apply induction.

Case 2. *The blue edge vw is not free at v*, see **Fig. 6**. Now $\beta' = \beta'' = \alpha' - 1$ by Proposition 4.7, *4*. Hence Proposition 4.4 implies that there are exactly α' red edges joining v to B_u. These α' edges enter the α' free blue branches wrt. u by Assumption 4.3. w is not free at v hence the second vertex z of R_w is contained in the branch of some free blue neighbor y of u. Exchange vw to vz keeping α and β. After this exchange v and y are contained in the same blue branch wrt. u, denoted by B'_v. We state that B'_v is free at u. Denote the new red $u - y$ path by Q'_y. Assumption 4.3 implies that Q_v is a subpath of Q_y and that Q_v does not contain vz. Hence Q_v is still a subpath of Q'_y. Note that the first edge of Q_v does not enter B_y by the freeness of y. Moreover, Q_v contains no edge joining u to $G - B_u$ by Assumption 4.6 and Proposition 4.7, *1*. So after the exchange B_v is free and $\alpha = 2$, $\beta = 0$. Hence after the exchange Proposition 4.7, *2* fails so at least one of Assumptions 4.3, 4.5 and 4.6 fail. That is, it is possible to apply induction. □

5 The circuit matroids are 3-star-uniform

In this section we prove that circuit matroids are 3-star-uniform. Actually, we use a different formulation of the problem in terms of substars.

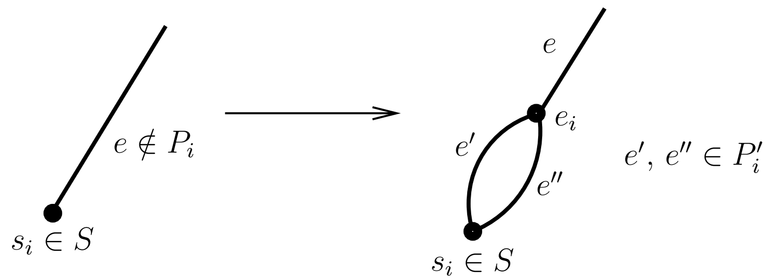

Fig. 7. Reducing the substar case to the star case

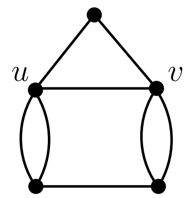

Fig. 8.

Definition 5.1. A subset of a star of a vertex is called a *substar*.

In Thm. 5.2 we prove that each 2-tree G has a 2-tree-coloring uniform to a subpartition $\{P_i\colon P_i \text{ is a substar of } s_i \in S\}$ when $|S| = 3$. This clearly implies that circuit matroids are 3-star-uniform with the choice $P_i = \Delta(s_i)$. S need not be stable, the only restriction is that the P_i are disjoint. In fact, this formulation is equivalent to the original by the following reduction.

For each edge $e \notin P_i$ which is incident to $s_i \in S$ do the following, see **Fig. 7**. Subdivide e by a new vertex e_i and double the edge s_ie_i. Add these two parallel s_ie_i-edges to P_i resulting in the new subpartition $\mathcal{P}'$. Note that $\mathcal{P}'$ consists of stars of pairwise non-adjacent vertices. Now a 2-tree-coloring uniform to $\mathcal{P}'$ gives a 2-tree-coloring uniform to $\mathcal{P}$ by contracting each new vertex e_i to s_i.

We return for a while to the case $|\mathcal{P}| = 2$. Recall that Thm. 4.1 states that a 2-tree has a 2-tree-coloring uniform to $\{\Delta(u), \Delta(v)\}$ whenever u and v are non-adjacent. The reduction of **Fig. 7** implies that this is also true in some cases when u and v are adjacent. E.g. if $e \neq f$ are uv-edges then define $P(u) = \Delta(u) - \{e, f\}$ and $P(v) = \Delta(v) - \{e, f\}$. If the uv-edge has multiplicity 1 and say, u has odd degree then let $P(u) = \Delta(u) - uv$ and $P(v) = \Delta(v)$. Now there exists a 2-tree-coloring uniform to $\{P(u), P(v)\}$ by the reduction of **Fig. 7** and by Theorem 4.1, which is automatically uniform to $\{\Delta(u), \Delta(v)\}$. If uv has multiplicity 1 and the degrees of u and v are even then it is possible that no 2-tree-coloring exists which is uniform to $\{\Delta(u), \Delta(v)\}$, see **Fig. 8**.

Theorem 5.2. *Let G be a 2-tree, $S = \{s_1, s_2, s_3\} \subseteq V(G)$ and P_i be a substar of s_i for $i = 1, 2, 3$. If the substars P_i are pairwise disjoint then G has 2-tree-coloring uniform to $\mathcal{P} = \{P_1, P_2, P_3\}$.*

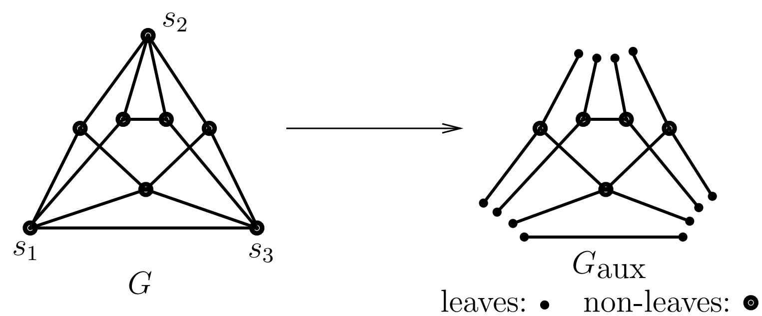

Fig. 9.

Theorem 5.2 remains true if $\mathcal{P}$ consists of three disjoint substars without the restriction that the substars must belong to three distinct vertices. Indeed, if $\mathcal{P}$ has two substars P_1 and P_2 belonging to the same vertex v, then replace v by two vertices v_1 and v_2 joined by a parallel edge-pair, and detach the incident edges of v in such a way that P_i will be a substar of v_i for $i = 1, 2$.

Proof of Theorem 5.2

The proof proceeds as follows. First we show some properties which a minimal counterexample must have and then we explore the possible connected components of the auxiliary graph G_{aux} (definition below). Finally, using the description of the components of G_{aux}, we prove that no counterexample exists.

Definition 5.3. Replace each vertex $s \in S$ by $\deg_G(s)$ vertices of degree 1, called *leaves*, see **Fig. 9**. The resulting graph is G_{aux}. For a connected component C of G_{aux} let $V_2(C)$ denote the set of the non-leaf vertices of C, i.e. $V_2(C) = \{v \in V(C) : \deg_C(v) \geq 2\}$.

Definition 5.4. The edges of $\bigcup_i P_i$ are called *significant*. For a vertex $v \notin S$ we denote by $s\text{-}\deg_G(v)$ the number of significant edges incident to v.

Let the pair $(G, \mathcal{P})$ be a counterexample to the theorem minimizing $|V(G)|$. We may assume that $|P_i|$ is even for $i = 1, 2, 3$. Otherwise, delete one edge from each P_i of odd size, giving a new subpartition $\mathcal{P}'$. Since each 2-tree-coloring of G which is uniform to $\mathcal{P}'$ is also uniform to $\mathcal{P}$, we get that $(G, \mathcal{P}')$ is also a counterexample to the theorem. Thus we assume that

$(G, \mathcal{P})$ is a minimal counterexample to the theorem, and $|P_i|$ is even for $i = 1, 2, 3$.

Proposition 5.5. $\deg_G(s) \geq 4$ *for each* $s \in S$.

Proof. If $\deg_G(s_1) = 2$ then a 2-tree-coloring uniform to $\{P_2, P_3\}$ guaranteed by Theorem 4.1 and by the reduction of **Fig. 7** is uniform to P_1 as well, contradicting that $(G, \mathcal{P})$ is a counterexample to the theorem. Similarly, if $\deg_G(s_1) = 3$ then a

2-tree-coloring uniform to $\{P_2, P_3\}$ is uniform to P_1 as well, except possibly when $|P_1| = 2$. So assume that $P_1 = \{e_1, e_2\}$ and $\Delta(s_1) = \{e_1, e_2, e_3\}$, where e_i joins s_1 to v_i for $i = 1, 2, 3$. Assume that, say, splitting the edge-pair e_1, e_3 to the v_1v_3-edge f results in a 2-tree H. If $e_3 \in P_i$ for $i = 2$ or 3 then let $P_i^H = P_i - e_3 + f$, otherwise let $P_i^H = P_i$. Theorem 4.1 with the reduction of **Fig. 7** gives a 2-tree-coloring of H uniform to $\{P_2^H, P_3^H\}$. Now the unsplitting at v results in a 2-tree-coloring of G uniform to $\{P_2, P_3\}$ such that also e_1, e_2 have different colors, a contradiction. □

Proposition 5.6. *G has at most one vertex of degree 2. If v is such a vertex then $s\text{-}\deg_G(v) = 2$ and the edges incident to v are not parallel.*

Proof. Let $v \in V(G)$ be a vertex with $\deg_G(v) = 2$. Let $P_i' = P_i - \Delta(v)$ for $i = 1, 2, 3$. Suppose first that $s\text{-}\deg_G(v) \leq 1$. By the minimality of G, the graph $G - v$ has a 2-tree-coloring uniform to $\mathcal{P}' = \{P_1', P_2', P_3'\}$, which can trivially be extended to a 2-tree-coloring of G uniform to $\mathcal{P}$. We could also extend this uniform 2-tree-coloring of $G - v$ if $s\text{-}\deg_G(v) = 2$ and the edges of v are parallel. So $\deg_G(v) = 2$ implies that $s\text{-}\deg_G(v) = 2$ and that the edges incident to v are not parallel. Suppose that v_1 and v_2 are two such vertices with neighbors s_1, s_3 and s_2, s_3, resp. Let H be the following 2-tree: add to G a vertex v of degree 2 with neighbors s_1 and s_2 and delete v_1 and v_2. Let $P_1^H = P_1 - v_1s_1 + vs_1$, $P_2^H = P_2 - v_2s_2 + vs_2$ and $P_3^H = P_3 - v_1s_3 - v_2s_3$. By the minimality of G, the graph H has a 2-tree-coloring uniform to $\{P_1^H, P_2^H, P_3^H\}$, and this coloring can easily be extended to a 2-tree-coloring of G uniform to $\mathcal{P}$, a contradiction. □

Corollary 5.7. *There exists at most one component C of G_{aux} such that $V_2(C)$ contains a vertex v with $\deg_G(v) = 2$. Such a component is called the* null-component, *and it has the property that $V_2(C) = \{v\}$.*

Proposition 5.8. $\deg_G(v) = 3$ *implies* $s\text{-}\deg_G(v) \geq 2$.

Proof. Suppose that $s\text{-}\deg_G(v) \leq 1$ and let the edges incident to v be e_1, e_2, e_3 such that $e_2, e_3 \notin P_i$ for $i = 1, 2, 3$. We may assume that splitting the edge-pair e_1, e_2 to edge e results in a 2-tree H. For $i = 1, 2, 3$, if $e_1 \in P_i$ then let $P_i^H = P_i - e_1 + e$, otherwise let $P_i^H = P_i$. Now H has a 2-tree-coloring uniform to $\{P_1^H, P_2^H, P_3^H\}$ by the minimality of G. This coloring gives a 2-tree-coloring of G uniform to $\mathcal{P}$, a contradiction. □

Theorem 5.9. *Only the following type of sets can be tight in G:*

1. *a singleton,*
2. *$V(G)$,*
3. *$V(G) - v$ where* $\deg_G(v) = 2$,
4. *$\{s_i, s_j\}$ such that $s_i, s_j \in S$ and $E(G)$ contains a parallel s_is_j-edge-pair.*

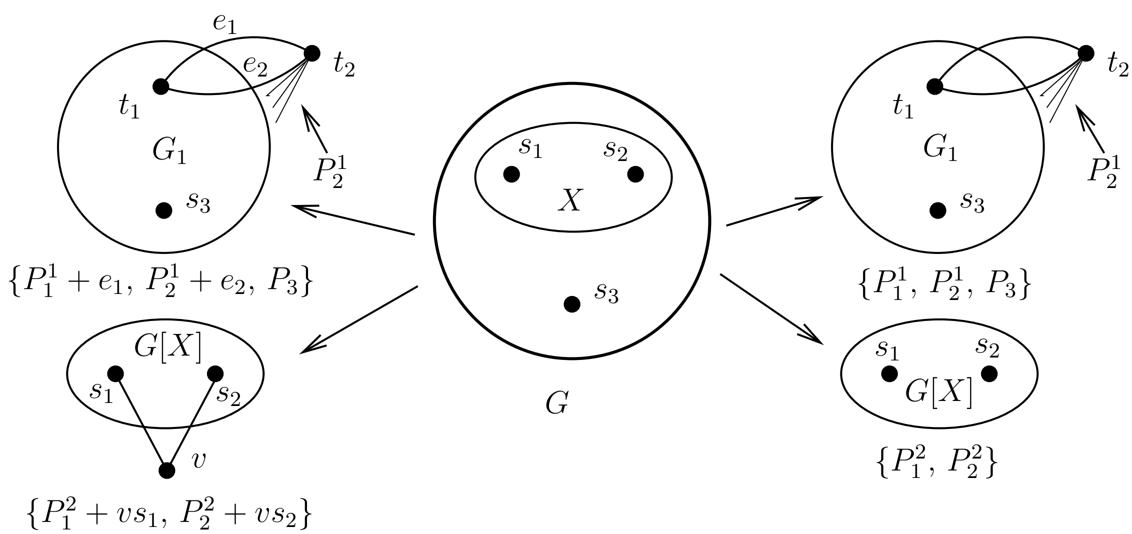

Fig. 10.

Proof. The graph we get when contracting $X \subseteq V(G)$ to one vertex and deleting the loops created is denoted by G/X. Suppose that $X \subseteq V(G)$ is a tight set of G not listed in the theorem. Observe that by Theorem 1.1 both G/X and $G[X]$ are 2-trees. We have four cases depending on the size of $X \cap S$.

- $X \cap S = \emptyset$. X is not a singleton so by the minimality of G, the graph G/X has a 2-tree-coloring uniform to $\mathcal{P}$. Extending this by an arbitrary 2-tree-coloring of $G[X]$ gives a 2-tree-coloring of G uniform to $\mathcal{P}$, a contradiction.

- $|X \cap S| = 1$. Let $s_1 \in X \cap S$. X is not a singleton so by the minimality of G, the graph G/X has a 2-tree-coloring uniform to $\{P_1 \cap E(G/X),\ P_2,\ P_3\}$. Moreover, $G[X]$ has a 2-tree-coloring uniform to $P_1 \cap E(G[X])$. By possibly reversely coloring the edges of $G[X]$, these 2-tree-colorings give a 2-tree-coloring of G uniform to $\mathcal{P}$, a contradiction.

- $|X \cap S| = 2$, see **Fig. 10**. Observe that $|X| \geq 3$. Let $s_1, s_2 \in X \cap S$ and let $P_i^1 = E(G/X) \cap P_i$ and $P_i^2 = E(G[X]) \cap P_i$ for $i = 1, 2$. Denote the vertex of G/X to which X was contracted by t_1. Let G_1 be the following graph: add to G/X a new vertex t_2, join it by two parallel edges e_1, e_2 to t_1 and re-join the edges of P_2^1 to t_2 instead of t_1. Moreover, let $G_2 = G[X]$. By the minimality of G, the graph G_1 has a 2-tree-coloring uniform to $\{P_1^1,\ P_2^1,\ P_3\}$ and G_2 has a 2-tree-coloring uniform to $\{P_1^2,\ P_2^2\}$. If not all $|P_i^1|$, $|P_i^2|$ are odd for $i = 1, 2$ then by possibly reversely coloring the edges of G_2, these 2-tree-colorings of G_1 and G_2 give a 2-tree-coloring of G uniform to $\mathcal{P}$, a contradiction. If $|P_i^1|$, $|P_i^2|$ are odd for $i = 1, 2$ then let G_2' be the graph we get when adding a new vertex v to G_2 and joining it to s_1 and s_2. Note that $|V(G_2')| < |V(G)|$ since $\deg_G(s_3) \geq 4$ by Proposition 5.5. Now G_1 has a 2-tree-coloring uniform to $\{P_1^1 + e_1,\ P_2^1 + e_2,\ P_3\}$, and G_2' has a 2-tree-coloring uniform to $\{P_1^2 + vs_1,\ P_2^2 + vs_2\}$. By a possible reverse coloring these 2-tree-colorings give a 2-tree-coloring of G uniform to $\mathcal{P}$, a contradiction.

- $|X \cap S| = 3$. Assume that X is a maximal tight set which is not of the form $V(G)$ or $V(G) - v$ for $\deg_G(v) = 2$. Now there exists a component C of G_{aux} different from the null-component such that $Y = V_2(C) - X \neq \emptyset$. By Corollary 5.7, $\deg_G(y) \geq 3$ holds for all $y \in Y$. Suppose that $y \in Y$ is a vertex with $\deg_G(y) = 3$. Now $s\text{-}\deg_G(y) \geq 2$ holds by Proposition 5.8, hence $d_G(y, X) = 2$ by Claim 1.3. But then $Y - y \neq \emptyset$ so the tight set $X + y$ would contradict to the maximality of X. So even $\deg_G(y) \geq 4$ holds for all $y \in Y$, implying $2i_G(Y) + \delta_G(Y) \geq 4|Y|$. But now

$$i_G(X \cup Y) = i_G(X) + (2i_G(Y) + \delta_G(Y)) - i_G(Y) \geq$$

$$\geq (2|X| - 2) + 4|Y| - (2|Y| - 2) = 2|X \cup Y|,$$

contradicting to Theorem 1.1.

□

Proposition 5.10. *G contains no parallel edges except possibly induced by S.*

Corollary 5.10 implies that if e is an edge joining x to y such that $\{x, y\} \not\subseteq S$ then it has multiplicity 1 hence we may use the notation 'xy' for e.

Corollary 5.11. *If $\deg_G(v) = 3$ or 4 then all three splits at v give 2-trees, except a split to a parallel $s_i s_j$-edge-pair with $s_i, s_j \in S$.*

Proposition 5.12. *If $v \in V(G) - S$ with $\deg_G(v) = 4$ then $s\text{-}\deg_G(v) = 2$ or 3.*

Proof. Corollary 5.10 implies that the edges incident to v go to 4 distinct vertices u_1, u_2, u_3, u_4. This already excludes $s\text{-}\deg_G(v) = 4$. Suppose that $s\text{-}\deg_G(v) \leq 1$. We know that at least one split at v gives a 2-tree, say, splitting v to the u_1u_2-edge e and to the u_3u_4-edge f results in a 2-tree H. If $s\text{-}\deg_G(v) = 1$ and, say, $u_1 = s_1 \in S$ and $vs_1 \in P_1$ then let $P_1' = P_1 - vs_1 + e$ and $P_i' = P_i$ for $i = 2, 3$. If $s\text{-}\deg_G(v) = 0$ then let $P_i' = P_i$ for $i = 1, 2, 3$. By the minimality of G, the graph H has a 2-tree coloring uniform to $\{P_1', P_2', P_3'\}$. Now pinch the edges e and f by the vertex v and if e and f had the same color then re-color an edge vu_i different from vs_1. This gives a 2-tree-coloring of G uniform to $\mathcal{P}$. □

The components of G_{aux}

Our next step in proving Theorem 5.2 is to describe the possible connected components of G_{aux}. There are altogether 24 of them.

Definition 5.13. For a component C of G_{aux} let $b = b(C) = |E(C)| - 2|V_2(C)|$. We denote the number of vertices $v \in V(C)$ with $\deg_C(v) = k$ by $d_k = d_k(C)$, especially, the number of leaves of C by $\delta = \delta(C) = d_1$. Moreover, $V_4 = V_4(C) = \{v \in V(C) \colon \deg_G(v) \geq 4\}$.

Observe that $b(C) \geq 0$ because $|E(C)| = |E(G)| - i_G(V(G) - V_2(C)) \geq 2|V(G)| - 2 - (2|V(G) - V_2(C)| - 2) = 2|V_2(C)|$. E.g. the null-component C has $b(C) = 0$. Besides,

$$\sum\{b(C) : C \text{ is a component of } G_{\text{aux}}\} = |E(G)| - 2|V(G) - S| = 4.$$

So $0 \leq b(C) \leq 4$ holds for each component C of G_{aux}, and the b values of the components can sum up to 4 in five ways (not taking into consideration the components with $b = 0$): 4, 3+1, 2+2, 2+1+1, 1+1+1+1.

Proposition 5.14. $\sum\{\delta(C) : C$ *is a component of* $G_{\text{aux}}\} \geq 12$.

Proof. $\sum_C \delta(C) = \sum_{i=1}^{3} \deg_G(s_i) \geq 12$ by Proposition 5.5. □

Next we list some properties of these components.

Proposition 5.15. *For each component C of G_{aux} the following properties hold.*

$$b \geq 1 \;\Rightarrow\; d_2 = 0 \tag{1}$$

$$\sum_{k \geq 2} (4 - k) d_k = \delta - 2b \tag{2}$$

$$d_2 = 0 \;\Rightarrow\; \delta - 2b \leq d_3 \leq \left\lfloor \frac{\delta}{2} \right\rfloor \tag{3}$$

$$d_2 = 0\,,\;\; \delta - 2b \geq 3 \;\Rightarrow\; \delta - 2b + 1 \leq d_3 \tag{4}$$

$$b \geq 2 \;\Rightarrow\; d_3 \leq \delta - b - 2 \text{ and equality implies } |V_4| = 1 \tag{5}$$

$$b \geq 1 \;\Rightarrow\; 2 \leq \delta \leq 4b \tag{6}$$

Proof. (1) If $\deg_C(v) = 2$ for $v \in V(C)$ then trivially $\deg_G(v) = 2$ so Corollary 5.7 implies that C is the null-component, which has $b(C) = 0$.

(2) $\sum_{k \geq 2} (4-k) d_k = \sum\{4 - \deg_C(v) : v \in V_2(C)\} = 4|V_2(C)| - 2|E(C)| + \delta = \delta - 2b$.

(3) The lower bound is implied by (2) while the upper by Proposition 5.8.

(4) Let $C' = C[V_2(C)]$. Now $\delta - 2b \leq d_3$ holds by (3) so suppose that $\delta - 2b = d_3$. This implies that $\deg_C(v) \leq 4$ holds for all vertices $v \in V_2$ by (2). Moreover, $\deg_C(v) \geq 3$ for $v \in V_2$ since $d_2 = 0$. Let $v \in V_2$. Now $\deg_{C'}(v) \leq 1$ if $\deg_C(v) = 3$ by Proposition 5.8 and $\deg_{C'}(v) \leq 2$ if $\deg_C(v) = 4$ by Proposition 5.12. Thus the highest degree of C' is at most 2. So C' is a path or a circuit because it is connected. So C' has at most 2 vertices of degree one hence $\delta - 2b = d_3 \leq 2$, a contradiction.

(5) Note that $d_2 = 0$ by (1). Let $v \in V_2$ be a vertex with $\deg_G(v) = 3$. If $s\text{-}\deg_G(v) = 3$ then v and its three leaves would form a component C of G_{aux} with $b(C) = |E(C)| - 2|V_2(C)| = 1$. Thus $s\text{-}\deg_G(v) = 2$ holds by Proposition 5.8. Denote by e the non-significant edge incident to v. If e joins v to a vertex $w \in V(C)$ with $\deg_G(w) = 3$ then the vertices v, w together with the 4 incident leaves would form a component of G_{aux} with $b = 1$. Hence e joins v to V_4 implying that $V_4 \neq \emptyset$ and $\delta_G(V_4) = \delta - d_3$. Moreover,

$$\delta - 2b = \sum_{k \geq 2} (4-k)d_k = d_3 + \sum \{4 - \deg_C(v) : v \in V_4(C)\} =$$

$$= d_3 + 4|V_4| - 2i_G(V_4) - \delta_G(V_4).$$

So

$$\delta - b = d_3 + 2|V_4| - i_G(V_4) \geq d_3 + 2,$$

where the last inequality is due to Theorem 1.1. Theorem 5.9 implies that equality holds only if V_4 is a singleton.

(6) The lower bound is due to the 2-edge connectivity of G. The upper is implied by the inequalities $\delta - 2b \leq d_3$ (by Properties (1) and (3)) and $2d_3 \leq \delta$ (by Proposition 5.8).

□

Now we are ready to describe the connected components of G_{aux}. These components are depicted in **Figs. 11–14** and in the rest of the proof of Theorem 5.2 we refer to them using the notations ***(a)*** - ***(x)*** of these figures. The leaves of the components are not shown in the figures at all, only their incident edges. The notations (1) – (6) refer to the statements of Proposition 5.15. Without even mentioning, we frequently use Corollary 5.10, Propositions 5.8, 5.12 and statements (1) – (6) of Proposition 5.15.

Components with $b = 0$

Assume that $d_2 = 0$. (3) yields that $d_3 \geq \delta - 2b = \delta \geq 2$. For all vertices $v \in V(C)$ with $\deg_G(v) = 3$ we know that $s\text{-}\deg_G(v) \geq 2$ by Proposition 5.8. Thus $\delta \geq 2d_3$, a contradiction. So $d_2 \geq 1$. Now Proposition 5.7 implies that C is the null-component (see **Fig. 11 (*a*)**) and that G_{aux} contains no other components with $b = 0$.

Components with $b = 1$

$\delta = 2$: Now $d_3 \leq 1$ by (3). If $v \in V(C)$ with $\deg_C(v) = 3$ then two edges would join v to S in G and the third edge incident to v would be a cut edge of G. So $d_3 = 0$. (2) implies that $\deg_C(v) = 4$ for all $v \in V_2(C)$. $d_4 \leq 1$ by Proposition 5.12 but $d_4 = 1$ is impossible. So $d_4 = 0$ and we get the edge-graph shown in **Fig. 11 (*b*)**. Such a component comes from an edge induced by S.

$\delta = 3$: Now $d_3 = 1$ by (3). (2) implies that $\deg_C(v) = 4$ for all $v \in V_4(C)$ but $d_4 = 0$ by Proposition 5.12. So $|V_2| = 1$ and we get **Fig. 11 (*c*)**. Observe that by

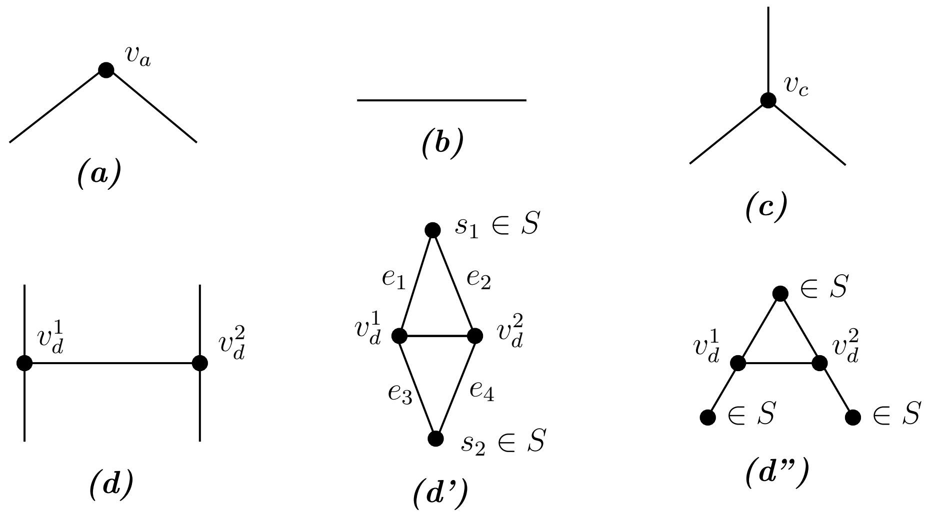

Fig. 11. The components of G_{aux}, I.
***(a)* has $b = 0$, the others $b = 1$**

Corollary 5.10, such a component comes from a subgraph of G where the three edges are incident to three distinct vertices of S.

$\delta = 4$: (3) implies that $d_3 = 2$. $\deg_C(v) \leq 4$ for $v \in V(C)$ by (2) and $d_4 = 0$ by Proposition 5.12. Hence this component is as shown in **Fig. 11 *(d)***. There is a strong restriction on the position of this component in G. First, there are no parallel edges in G by Corollary 5.10. Second, assume that C comes from a subgraph depicted in **Fig. 11 *(d')***. Proposition 5.8 implies that $e_1, e_2 \in P_1$ and $e_3, e_4 \in P_2$. Now replace this subgraph by an edge e joining s_1 to s_2 (i.e. with component ***(b)***) resulting in the 2-tree H. Let $P_1^H = P_1 - \{e_1, e_2\}$, $P_2^H = P_2 - \{e_3, e_4\}$ and $P_3^H = P_3$. By the minimality of G, H has a 2-tree-coloring uniform to $\{P_1^H, P_2^H, P_3^H\}$, which gives a 2-tree-coloring of G uniform to $\mathcal{P}$, a contradiction. So only the subgraph of **Fig. 11 *(d")*** remained.

Components with $b = 2$

(3) and (5) give that $d_3 = \delta - 4$, moreover, $|V_4| = 1$ by (5). Let $V_4 = \{w\}$. Now $\deg_C(w) = 4$ by (2). Finally, $d_3 = \delta - 4$ gives that $\delta \geq 4$.

$\delta = 4$: $d_3 = 0$ so $\deg_C(w) = 4$ contradicts to Corollary 5.10.

$\delta = 5$: Now $d_3 = 1$, giving the component of **Fig. 12 *(e)***.

$\delta = 6$: Now $d_3 = 2$, giving a component shown in **Fig. 12 *(f)***.

$\delta = 7$: Contradicts to (4).

$\delta = 8$: Contradicts to (4).

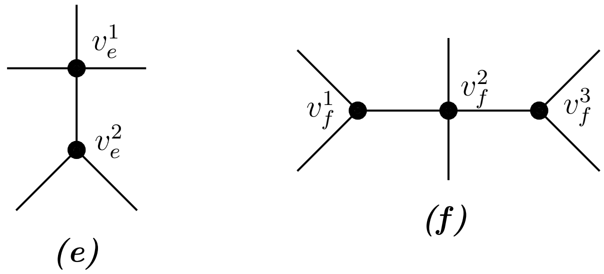

Fig. 12. The components of G_{aux}, II.
Components with $b = 2$

Components with $b = 3$

Such a component is accompanied in G_{aux} with one component of $b = 1$ for which $\delta \le 4$ by (6), and possibly with the null-component, where $\delta = 2$. So $\delta \ge 6$ holds by Proposition 5.14. (3) and (5) give that $\delta - 6 \le d_3 \le \delta - 5$. If $d_3 = \delta - 6$ then $\deg_C(v) = 4$ for all $v \in V_4$ by (2), and if $d_3 = \delta - 5$ then $V_4 = \{w\}$ by (5) and $\deg_C(w) = 5$ by (2).

$\delta = 6$: If $d_3 = 0$ then the vertices of V_4 are adjacent to altogether 6 leaves so $d_4 = 2$ or 3 by Proposition 5.12. The first case gives **Fig. 13 *(g)*** and the second **Fig. 13 *(h)***. Finally, if $d_3 = 1$ then $\deg_C(w) = 5$ would contradict to Corollary 5.10.

$\delta = 7$: If $d_3 = 1$ then the vertices of V_4 are adjacent to altogether 5 leaves so $|V_4| = 2$ by Proposition 5.12, and one vertex of V_4 is adjacent to 3 leaves and the other one to 2 leaves, see **Fig. 13 *(i)***. If $d_3 = 2$ then we get **Fig. 13 *(j)***.

$\delta = 8$: If $d_3 = 2$ then $|V_4| = 2$ by Proposition 5.12, and both vertices of V_4 are adjacent to 2 leaves, see **Fig. 13 *(k)***. In the case $d_3 = 3$ we get **Fig. 13 *(l)***.

$\delta = 9$: $d_3 = 3$ is excluded by (4) so $d_3 = 4$. Now we get **Fig. 13 *(m)***.

$\delta = 10$: $d_3 = 4$ is excluded by (4) so $d_3 = 5$ yielding **Fig. 13 *(n)***.

$\delta = 11$: $d_3 = 5$ by (3) but this is excluded by (4).

$\delta = 12$: $d_3 = 6$ by (3) but this is excluded by (4).

Components with $b = 4$

Such a component can be accompanied in G_{aux} only with the null-component. The null-component has $\delta = 2$ so now $\delta \ge 10$ by Proposition 5.14. Moreover, $\delta \le 16$ by (6). (3) and (5) give that $\delta - 8 \le d_3 \le \delta - 6$. If $d_3 = \delta - 8$ then $\deg_C(v) = 4$ for all $v \in V_4$ by (2). If $d_3 = \delta - 7$ then $\deg_C(w) = 5$ for a specified vertex $w \in V_4$ and $\deg_C(v) = 4$ for $v \in V_4 - w$ by (2). Finally, if $d_3 = \delta - 6$ then $V_4 = \{w\}$ by (5) and $\deg_C(w) = 6$ by (2).

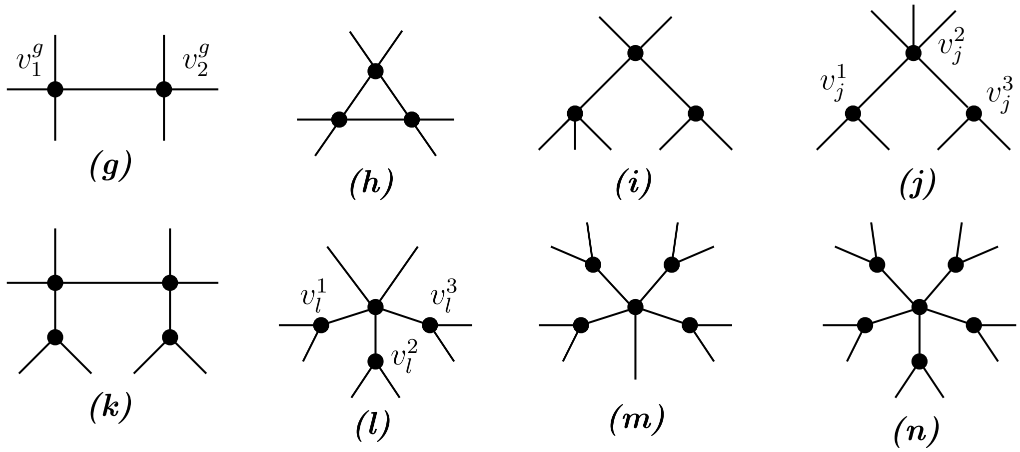

Fig. 13. The components of G_{aux}, III. Components with $b = 3$

$\delta = 10$: Let first $d_3 = 2$. The vertices of V_4 are adjacent to altogether 6 leaves so Proposition 5.12 yields that $d_4 = 2$ or 3. Now $d_4 = 2$ would give a disconnected graph and $d_4 = 3$ gives **Fig. 14 (*o*)**. If $d_3 = 3$ then $d_4 \leq 2$ by Proposition 5.12. Now $d_4 = 2$ gives **Fig. 14 (*p*)**, $d_4 = 1$ gives **Fig. 14 (*q*)** and **(*r*)**, while $d_4 = 0$ is impossible. Finally, $d_3 = 4$ gives **Fig. 14 (*s*)**.

$\delta = 11$: $d_3 = 3$ is excluded by (4). If $d_3 = 4$ then the vertices of V_4 are adjacent to altogether 3 leaves so $d_4 \leq 1$ by Proposition 5.12. Now $d_4 = 1$ gives **Fig. 14 (*t*)** and **Fig. 14 (*u*)**, while $d_4 = 0$ is impossible. $d_3 = 5$ gives **Fig. 14 (*v*)**.

$\delta = 12$: $d_3 = 4$ is excluded by (4). If $d_3 = 5$ then $d_4 \leq 1$ by Proposition 5.12. Now $d_4 = 1$ gives **Fig. 14 (*w*)**, while $d_4 = 0$ is impossible. Finally, $d_3 = 6$ gives **Fig. 14 (*x*)**.

$\delta = 13$: $d_3 = 6$ by (4) and by the upper bound of (3). $d_4 = 0$ by Proposition 5.12, so this graph would be disconnected.

$\delta = 14$: $d_3 = 7$ by (3) and (4). $d_4 = 0$ by Proposition 5.12, so this graph would be disconnected.

$\delta = 15$: Impossible by (3) and (4).

$\delta = 16$: Impossible by (3) and (4).

Reductions to smaller graphs

Using the above description of the components we enumerate all possibilities for G_{aux}. For two cases of G_{aux} we cannot do else than directly giving a 2-tree-coloring of G uniform to $\mathcal{P}$, see **Figs. 19–20**. However, in all the other cases we prove that we can apply admissible splits to G to reduce the problem to a smaller 2-tree H with

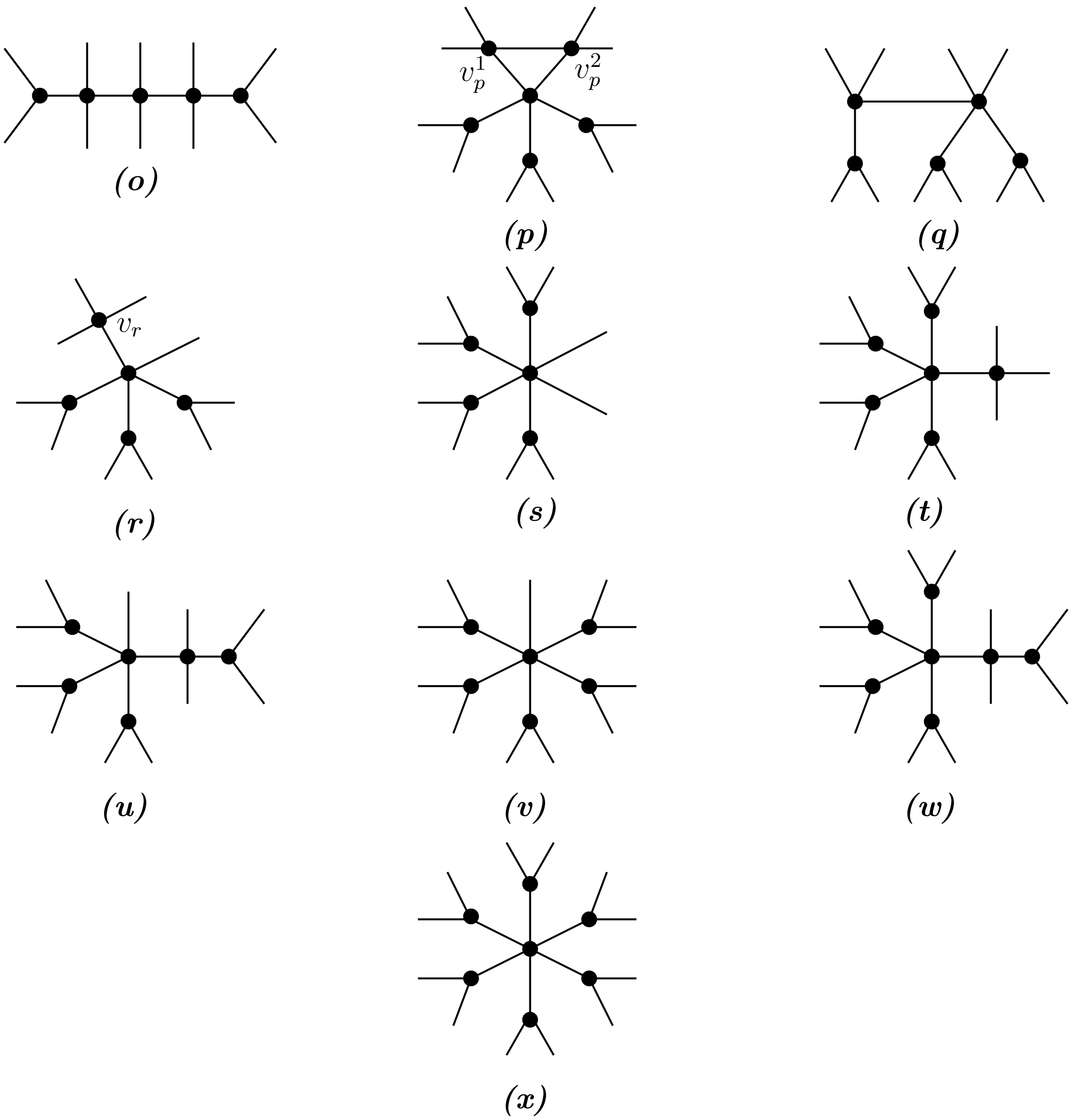

Fig. 14. The components of $G_{\mathbf{aux}}$, IV.
Components with $b = 4$

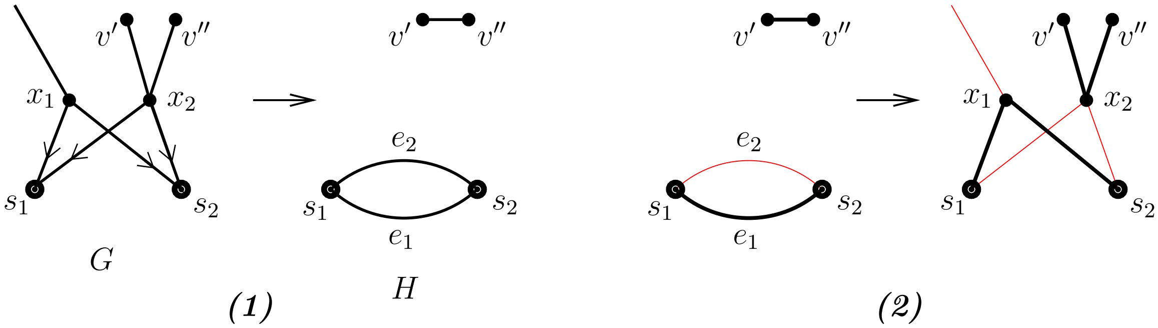

Fig. 15. Reduction 1.

subpartition $\mathcal{P}^H = \{P_1^H, P_2^H, P_3^H\}$. We use Reductions 1 – 3 below. These reductions all have the property that if H is really a 2-tree then a 2-tree-coloring of H uniform to $\mathcal{P}^H$ can be extended to a 2-tree-coloring of G uniform to $\mathcal{P}$. So our only task will be to prove that H is indeed a 2-tree. Since proving that H is a 2-tree will be always easy, we will not consider this issue, we only show a general scheme after Reduction 2 and an example in **Fig. 18**.

We will apply the following reductions. We use Corollary 5.10 and Propositions 5.8, 5.12 without mentioning. In **Figs. 15–20** the vertices of S are shown as big dots and *each edge $vs_i \in P_i$ is indicated by an arrow showing from v to s_i.*

Reduction 1. (**Fig. 15**.) Let $x_1, x_2 \in V(G) - S$ be two vertices such that $\deg_G(x_i) \in \{3, 4\}$, $s\text{-}\deg_G(x_1) = 2$, $s\text{-}\deg_G(x_2) \le 3$ and $x_i s_1 \in P_1$, $x_i s_2 \in P_2$ for $i = 1, 2$. We pose the restriction that if x_1 and x_2 are adjacent in G then $\deg_G(x_2) = 4$ must hold. Now first split x_1 to the s_1s_2-edge e_1 resulting in the graph G_2. Then in G_2 split x_2 to the s_1s_2-edge e_2 (note that if x_1 and x_2 are adjacent in G and $\deg_G(x_1) = 3$ then $\deg_{G_2}(x_2) = 3$ holds.) The second splitting results in the graph H, see **Fig. 15 (*1*)**. Let $P_1^H = P_1 - x_1s_1 - x_2s_1$ and $P_2^H = P_2 - x_1s_2 - x_2s_2$. If $s\text{-}\deg_G(x_2) = 2$ then let $P_3^H = P_3$. If $s\text{-}\deg_G(x_2) = 3$ then let $P_3^H = P_3 - x_2s_3$ in case $\deg_{G_2}(x_2) = 3$, and let $P_3^H = P_3 - x_2s_3 + e$ in case $\deg_{G_2}(x_2) = 4$ and x_2 was split to the edges e_2 and e. If H is a 2-tree then it has a 2-tree-coloring uniform to $\mathcal{P}^H$ by the minimality of G. In this coloring e_1 and e_2 have different colors. By possibly exchanging the colors of e_1 and e_2 we can achieve that at the unsplitting at x_2

- we can keep the uniformity to P_3^H in case $s\text{-}\deg_G(x_2) = \deg_{G_2}(x_2) = 3$, and
- we do not need to re-color any edges in case $\deg_{G_2}(x_2) = 4$.

Next unsplit at x_1 yielding a 2-tree-coloring of G uniform to $\mathcal{P}$, see **Fig. 15 (*2*)**. Note that if $\deg_G(x_1) = 4$ and both split edges of x_1 had the same color before the unsplitting at x_1 then it is possible to re-color an edge incident to x_1 keeping uniformity.

Reduction 2. (**Fig. 16**.) Let $x_1, x_2 \in V(G) - S$ be two vertices such that $\deg_G(x_i) \in \{3, 4\}$, $s\text{-}\deg_G(x_i) = 2$ for $i = 1, 2$ and $x_2s_1 \in P_1$, $x_1s_2 \in P_2$ and

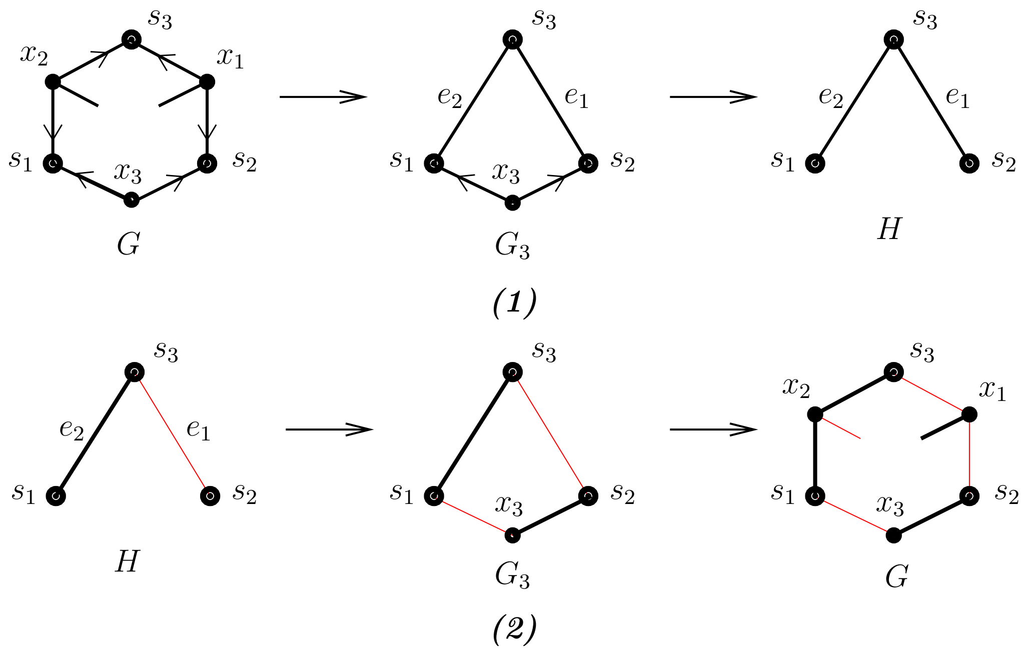

Fig. 16. Reduction 2.

x_1s_3, $x_2s_3 \in P_3$. We pose the restriction that if x_1 and x_2 are adjacent in G then $\deg_G(x_2) = 4$ must hold. Now first split x_1 to the s_2s_3-edge e_1 resulting in the graph G_2. Then in G_2 split x_2 to the s_1s_3-edge e_2 resulting in the graph G_3. Finally, let $\deg_{G_3}(x_3) = 2$ for some $x_3 \in V(G) - S$ such that the neighbors of x_3 in G_2 are s_1 and s_2 and $x_3s_1 \in P_1$, $x_3s_2 \in P_2$. Now delete x_3 from G_3 resulting in the graph H, see **Fig. 16 (*1*)**. Let $P_1^H = P_1 - x_2s_1 - x_3s_1$, $P_2^H = P_2 - x_1s_2 - x_3s_2$ and $P_3^H = P_3 - x_1s_3 - x_2s_3$. Assume that H is a 2-tree and that it has a 2-tree-coloring uniform to $\mathcal{P}^H$ such that e_1 *and* e_2 *have different colors.* First unsplit at x_3 such that x_3s_1 has the color of e_1 and x_3s_2 has the color of e_2. Next unsplitting at x_2 and then at x_1 gives a 2-tree-coloring of G uniform to $\mathcal{P}$, see **Fig. 16 (*2*)**.

These reductions are of no use unless H is a 2-tree. To show that H is really a 2-tree it is enough to show sequential splits described in page 2 which reduce H to a 2-tree with vertex set S. Observe that a graph with vertex set S and with 4 edges is always a 2-tree unless it has a loop or an edge with multiplicity at least 3. Every time we apply Reductions 1 and 2 it will be an easy task to show such sequential splits. For an example see one case below (**Fig. 18**). Recall that when using Reduction 2 one also has to check whether H has a 2-tree-coloring uniform to $\mathcal{P}^H$ such that the split edges e_1 and e_2 have different colors. We will leave this to the reader when applying Reduction 2.

Unlike in Reductions 1 and 2, in the next Reduction there is no need to check if H is a 2-tree.

Reduction 3. Let x_1, $x_2 \in V(G) - S$ be two non-adjacent vertices such that $\deg_G(x_i) = 3$ and $x_is_1 \in P_1$, $x_is_2 \in P_2$ hold for $i = 1, 2$. Assume also that the

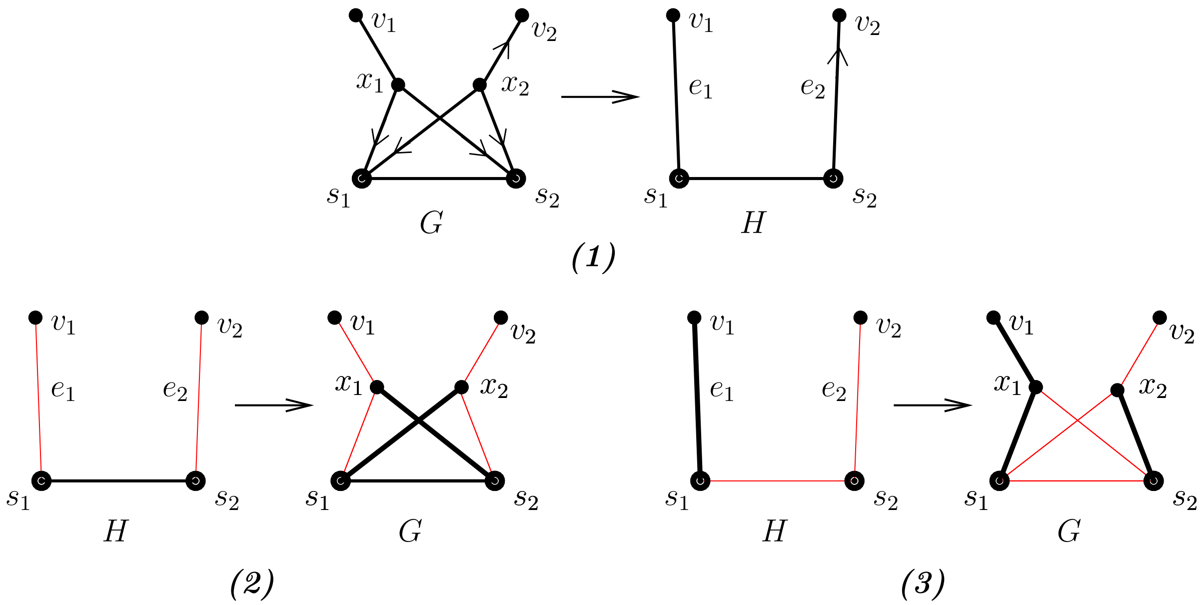

Fig. 17. Reduction 3.

edge s_1s_2 has multiplicity 1 in G. Let the neighbor of x_i distinct from s_1, s_2 be v_i for $i = 1, 2$. Splitting the vertices x_i to the s_iv_i-edge e_i for $i = 1, 2$ results in a graph H (see **Fig. 17 (*1*)**). If s_iv_i had multiplicity 2 then G_{aux} would have 3 components of type **(*b*)**, so x_1, x_2 would belong to a component **(*d*)** which is impossible. Thus if H is not a 2-tree then by Theorem 1.1, there exists a vertex set $W \subseteq V(H)$ such that $i_H(W) \geq 2|W| - 1$. Corollary 5.11 implies that s_i, $v_i \in W$ for $i = 1, 2$. But then $i_G(W \cup \{x_1, x_2\}) \geq 2|W \cup \{x_1, x_2\}| - 1$, a contradiction. So H is *always a 2-tree*. Let $P_1^H = P_1 - x_1s_1 - x_2s_1$, $P_2^H = P_2 - x_1s_2 - x_2s_2$ and $P_3^H = (P_3 \setminus \{x_1s_3, x_2s_3\}) \cup \{e_i : x_is_3 \in P_3\}$. H has a 2-tree-coloring uniform to $\mathcal{P}^H$ by the minimality of G. If e_1 and e_2 have the same colors in this coloring then simply unsplit x_1 and x_2, see **Fig. 17 (*2*)**. If e_1 and e_2 have different colors then use the extension of **Fig. 17 (*3*)**. In both cases we get a 2-tree-coloring of G uniform to $\mathcal{P}$.

Now we enumerate the possibilities for G_{aux}, according to how the b values of the components can sum up to 4. Recall that G_{aux} contains at most one component with $b = 0$, the null-component **(*a*)**. We use the notations of **Figs. 11–14**, i.e. we refer to the components of G_{aux} as **(*a*)** - **(*x*)** and to specified vertices of these components as v_a, v_d^1, v_d^2 etc. (see **Fig. 11**).

4(+0)

Denote the component of G_{aux} with $b = 4$ by C_4. C_4 has at least 4 vertices x with $\deg_G(x) \in \{3, 4\}$ and $s\text{-}\deg_G(x) = 2$ by Propositions 5.8 and 5.12 (except if $C_4 =$ **(*r*)** in the case $s\text{-}\deg_G(v_r) = 3$). In any case we can choose two vertices x_1, $x_2 \in V_2(C)$ such that, say, $x_is_1 \in P_1$ and $x_is_2 \in P_2$ for $i = 1, 2$. If x_1 and x_2 are adjacent in G then make sure that $\deg_G(x_2) = 4$ holds. Now apply Reduction 1 to x_1, x_2 resulting in the graph H. We have to prove that H is a 2-tree. $\deg_G(s_3) \geq 4$ by Proposition 5.5 which clearly implies that $\deg_H(s_3) \geq 3$, so it is straightforward to show a sequence of splits in H which gives a 2-tree with vertex set S. We illustrate this in the case $C_4 = (\boldsymbol{o})$, see **Fig. 18**. If $C_4 = (\boldsymbol{o})$ then G_{aux} also contains the null-component $(\boldsymbol{a})$ by Proposition 5.14. For instance, assume that H is the graph shown in **Fig. 18** ***(1)***. Now split v_1 to s_1s_3 resulting in the graph H_1. Then split v_2 to s_1s_3 resulting in the graph H_2. Finally delete from H_2 the vertices v_3 and v_a resulting in H_3, see **Fig. 18** ***(4)***. Since H_3 is trivially a 2-tree, we get that H is a 2-tree as well so we are done. You only have to be careful if $C_4 = (\boldsymbol{p})$ where it is forbidden to choose x_1, x_2 to be v_p^1 and v_p^2 because H would contain a loop.

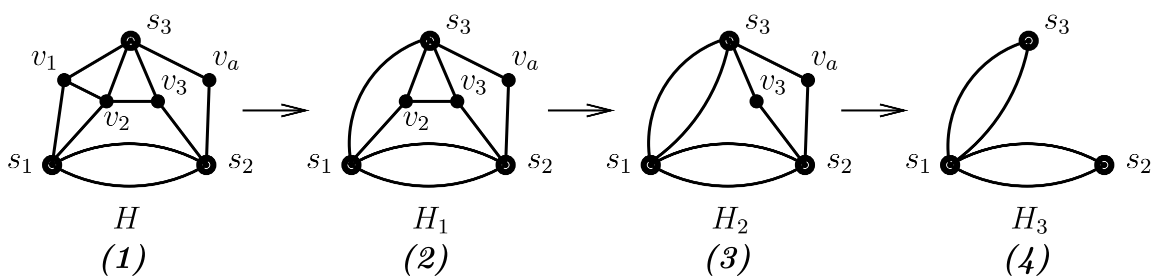

Fig. 18. Proving that H is a 2-tree

3+1(+0)

Let these components be denoted by C_3, C_1 (and C_0), resp.

- $C_3 = (\boldsymbol{g})$. Proposition 5.14 gives that $C_1 = (\boldsymbol{d})$ and also the null-component $C_0 = (\boldsymbol{a})$ is present. Independently of the value of $s\text{-}\deg_G(v_i^g)$ for $i = 1, 2$, we can apply Reduction 1.

- $C_3 = (\boldsymbol{h})$. Proposition 5.14 gives that $C_1 = (\boldsymbol{d})$ and also the null-component $C_0 = (\boldsymbol{a})$ is present. Proposition 5.5 yields that $V_2(C_3)$ is adjacent to each $s_i \in S$. So we can apply Reduction 1 by appropriately choosing $x_1 \in V_2(C_3)$ and $x_2 \in V_2(C_1)$.

- $C_3 = (\boldsymbol{i})$. If $C_1 = (\boldsymbol{c})$ or $C_1 = (\boldsymbol{d})$ then we can apply Reduction 1. $C_1 = (\boldsymbol{b})$ is excluded by Proposition 5.5.

- $C_3 = (\boldsymbol{j})$. $C_1 \neq (\boldsymbol{b})$ by Proposition 5.5 and if $C_1 = (\boldsymbol{d})$ then we are done by Reduction 1. If $C_1 = (\boldsymbol{c})$ then also the null-component $C_0 = (\boldsymbol{a})$ is present. The only possibility when we cannot apply Reduction 1 is when $s\text{-}\deg_G(v_c) = 2$ and the two significant edges incident to v_j^1, v_j^3 and v_c go to pairwise distinct pairs of vertices in S. So we can apply Reduction 2 by appropriately choosing x_1, $x_2 \in \{v_j^1, v_j^3, v_c\}$ and $x_3 = v_a$.

- $C_3 = (\boldsymbol{k})$. If $C_1 = (\boldsymbol{c})$ or $(\boldsymbol{d})$ then we can apply Reduction 1. Assume $C_1 = (\boldsymbol{b})$, i.e. an edge s_1s_2. Proposition 5.5 implies that $\deg_G(s_3) \geq 4$ so at least three vertices of $V_2(C_3)$ are adjacent to s_3. Thus Reduction 1 can be applied.

- $C_3 = \boldsymbol{(l)}$. If $C_1 = \boldsymbol{(c)}$ or $C_1 = \boldsymbol{(d)}$ then Reduction 1 can be applied. Assume that $C_1 = \boldsymbol{(b)}$, i.e. an edge s_1s_2. Proposition 5.14 implies that G_{aux} contains the null-component $\boldsymbol{(a)}$ as well. Now the only case when we cannot apply Reduction 1 or 3 is when the two significant edges incident to v_l^i go to pairwise distinct pairs of vertices in S for $i = 1, 2, 3$. So we can apply Reduction 2 by appropriately choosing $x_1,\ x_2 \in \{v_l^1,\ v_l^2,\ v_l^3\}$ and $x_3 = v_a$.
- $C_3 = \boldsymbol{(m)}$ or $C_3 = \boldsymbol{(n)}$. If $C_1 = \boldsymbol{(c)}$ or $\boldsymbol{(d)}$ then we can apply Reduction 1. If $C_1 = \boldsymbol{(b)}$ then Reduction 1 or 3 can be applied.

2+2(+0)

In all cases Reduction 1 can be applied.

2+1+1(+0)

Denote the component with $b = 2$ by C_2. We list the cases according to the two components with $b = 1$.

- $\boldsymbol{(b)} + \boldsymbol{(b)}$. Proposition 5.5 implies that G_{aux} contains the null-component $\boldsymbol{(a)}$, $C_2 = \boldsymbol{(f)}$ and $\deg_G(s_i) = 4$ for $i = 1, 2, 3$. Assume that the two edges of the components $\boldsymbol{(b)}$ are parallel, say, s_1s_2-edges. Then each vertex of $V_2(C_2)$ is adjacent to s_3 because $\deg_G(s_3) = 4$. Thus we can apply Reduction 1. So assume that the two edges of the components $\boldsymbol{(b)}$ are, say, s_1s_2 and s_1s_3. If the two significant edges incident to v_f^i go to pairwise distinct pairs of vertices in S for $i = 1, 2, 3$ then we can apply Reduction 2 by choosing $x_i = v_f^i$ for $i = 1, 2, 3$. The fact that $\deg_G(s) = 4$ for $s \in S$ implies that otherwise for at least two indices i the two significant edges incident to v_f^i go to s_2 and s_3. So we can apply Reduction 1.
- $\boldsymbol{(b)} + \boldsymbol{(c)}$. Assume that the edge of the component $\boldsymbol{(b)}$ joins s_1 to s_2. Proposition 5.5 implies that G_{aux} contains the null-component $\boldsymbol{(a)}$, too.
  - $C_2 = \boldsymbol{(e)}$. Proposition 5.5 implies that $\deg_G(s_i) = 4$ for $i = 1, 2, 3$. So there is only one choice for G up to isomorphism, namely, say, v_a is adjacent to s_2 and s_3 and v_e^2 is adjacent to s_1 and s_3. If $s\text{-}\deg_G(v_c) = 3$ or $s\text{-}\deg_G(v_e^1) = 3$ then we can apply Reduction 1 with $x_1 = v_e^2$ and $x_2 = v_c$ or v_e^1 resp. So assume that $s\text{-}\deg_G(v_c) = s\text{-}\deg_G(v_e^1) = 2$. For $v = v_c$ or v_e^1, if the two significant edges incident to v go to s_1 and s_2, then we can apply Reduction 2 with $x_1 = v_e^2$, $x_2 = v$, $x_3 = v_a$. Otherwise $v_cs_3,\ v_e^1s_3 \in P_3$ so we can apply Reduction 1.
  - $C_2 = \boldsymbol{(f)}$. If the two significant edges incident to v_f^i go to pairwise distinct pairs of vertices in S for $i = 1, 2, 3$ then we can apply Reduction 2 with $x_i = v_f^i$. Otherwise v_f^i and v_f^j are adjacent to the same pair of vertices in S for some $1 \leq i < j \leq 3$. This pair cannot be s_1, s_2 since $\deg_G(s_3) \leq 3$ would hold. Thus we can apply Reduction 1.

- ***(b)*** + ***(d)***. If $C_2 =$ ***(f)*** then G contains 4 vertices v with $\deg_G(v) = 3$ and $s\text{-}\deg_G(v) = 2$ so we can apply Reduction 1 or 3. So assume that $C_2 =$ ***(e)*** and that component ***(b)*** joins s_1 to s_2. We can apply Reduction 1 or 3 unless the two significant edges incident to v_d^1, v_d^2 and v_e^2 go to pairwise distinct pairs of vertices in S. In this case Reduction 1 can be applied unless $s\text{-}\deg_G(v_e^1) = 2$ and the two significant edges incident to v_e^1 go to s_1 and s_2. $|P_i|$ is even for $i = 1, 2, 3$ so also v_a is adjacent to s_1 and s_2. But then $\deg_G(s_3) \leq 3$ would hold, which is impossible.

- ***(c)*** + ***(c)***, ***(c)*** + ***(d)*** and ***(d)*** + ***(d)***. Apply Reduction 1.

1+1+1+1(+0)

We list all possible cases.

- ***(b)*** + ***(b)*** + ***(b)*** + ***(b)*** and ***(b)*** + ***(b)*** + ***(b)*** + ***(c)*** are impossible by Proposition 5.14.

- ***(b)*** + ***(b)*** + ***(b)*** + ***(d)***. G_{aux} contains also the null-component by Proposition 5.14. Here we cannot apply any reductions. Assume that v_d^1 is adjacent to s_1, s_3 and v_d^2 is adjacent to s_2, s_3. There are two cases on the position of the null-component up to isomorphism.

  - First, let v_a be adjacent to s_1 and s_2, see **Fig. 19 *(1)***. Denote $P^1 = \{v_d^1 s_1,\ v_a s_1\}$, $P^2 = \{v_d^2 s_2,\ v_a s_2\}$ and $P^3 = \{v_d^1 s_3,\ v_d^2 s_3\}$. $|P_i|$ is even for $i = 1, 2, 3$ so there are 3 possibilities for $\mathcal{P}$ up to isomorphism: $\{P^1, P^2, P^3\}$, $\{P^1 + s_3 s_1 + s_2 s_1, P^2, P^3\}$ and $\{P^1, P^2, P^3 + s_1 s_3 + s_2 s_3\}$. The 2-tree-coloring of the graph in **Fig. 19 *(1)*** is uniform to all these 3 cases of $\mathcal{P}$.

  - Let v_a be adjacent to s_1 and s_3, see **Fig. 19 *(2)***. Now the evenness of $|P_i|$ implies that with an $s_1 s_2$-edge e it holds that $\mathcal{P} = \{\{v_a s_1,\ v_d^1 s_1\},\ \{e,\ v_d^2 s_2\},\ \{v_a s_3,\ v_d^1 s_3,\ v_d^2 s_3,\ s_2 s_3\}\}$. **Fig. 19 *(2)*** shows a 2-tree-coloring uniform to $\mathcal{P}$.

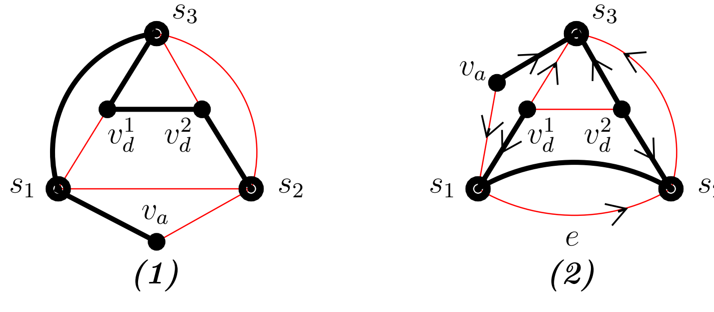

Fig. 19.

- $(\boldsymbol{b}) + (\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{c})$. G_{aux} contains also the null-component by Proposition 5.14. Denote the two vertices v_c of the two components $(\boldsymbol{c})$ by v_c' and v_c''. Proposition 5.5 implies that, say, the two edges of the components $(\boldsymbol{b})$ are s_1s_3- and s_2s_3-edges and v_a is adjacent to s_1 and s_2. If $s\text{-}\deg_G(v_c') = 3$ or $s\text{-}\deg_G(v_c'') = 3$ then we can apply Reduction 1 or 3 so assume otherwise. If the two significant edges incident to v_c' and v_c'' go to the same pair of vertices in S then apply Reduction 1 or 3. Otherwise there are two cases up to isomorphism. First, if the significant edges incident to v_c' go to s_2, s_3 and the significant edges incident to v_c'' go to s_1, s_3 then apply Reduction 2 with $x_1 = v_c'$, $x_2 = v_c''$, $x_3 = v_a$. Second, if the significant edges incident to v_c' go to s_1, s_2 and the significant edges incident to v_c'' go to s_2, s_3 then the evenness of $|P_i|$ implies that there is only one choice for $\mathcal{P}$. A 2-tree-coloring of G uniform to $\mathcal{P}$ is shown in **Fig. 20**.

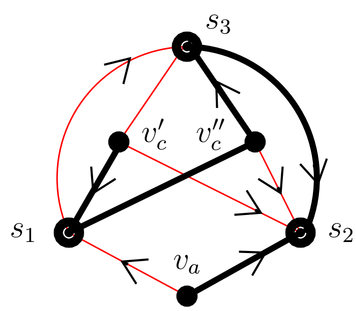

Fig. 20.

- $(\boldsymbol{b}) + (\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{d})$. If $s\text{-}\deg_G(v_c) = 3$ then apply Reduction 1 or 3. Assume that $s\text{-}\deg_G(v_c) = 2$. If the edges of the components $(\boldsymbol{b})$ are not parallel then we can apply Reduction 1 or 3 unless the two significant edges incident to v_c, v_d^1 and v_d^2 go to pairwise distinct pairs of vertices in S. In this latter case Reduction 2 can be applied with $x_1 = v_c$, $x_2 = v_d^1$ and $x_3 = v_d^2$. Now assume that the edges of the components $(\boldsymbol{b})$ are parallel s_1s_2-edges. $\deg_G(s_3) \geq 4$ so v_a is adjacent to s_3 and, say, v_d^i is adjacent to s_i for $i = 1, 2$. $|P_3|$ is even thus $v_cs_3 \in P_3$. So we can apply Reduction 1 with $x_1 = v_c$ and $x_2 = v_d^i$ for $i = 1$ or 2.

- $(\boldsymbol{b}) + (\boldsymbol{b}) + (\boldsymbol{d}) + (\boldsymbol{d})$. If the edges of the components $(\boldsymbol{b})$ are not parallel then apply Reduction 1 or 3. Assume that the edges of the components $(\boldsymbol{b})$ are parallel s_1s_2-edges. $\deg_G(s_3) \geq 4$ so at least three vertices of type v_d^i $(i = 1, 2)$ are adjacent to s_3, so it is possible to apply Reduction 1.

- $(\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{c}) + (\boldsymbol{c})$. Denote the vertices v_c of the components $(\boldsymbol{c})$ by v_c^i for $i = 1, 2, 3$. If $s\text{-}\deg_G(v_c^i) = 3$ for at least one index i then apply Reduction 1 or 3. Otherwise we can apply Reduction 1 or 3 unless the two significant edges incident to v_c^i go to pairwise distinct pairs of vertices in S for $i = 1, 2, 3$. Suppose that the edge of the component $(\boldsymbol{b})$ is an s_1s_2-edge. $\deg_G(s_3) \geq 4$ so G_{aux} contains also the null-component $(\boldsymbol{a})$ and v_a is adjacent to s_3. But then $|P_3| = 3$ would hold, which is impossible.

- **(*b*)** + **(*c*)** + **(*c*)** + **(*d*)** and **(*b*)** + **(*c*)** + **(*d*)** + **(*d*)** and **(*b*)** + **(*d*)** + **(*d*)** + **(*d*)**. Apply Reduction 1 or 3.

- **(*c*)** + **(*c*)** + **(*c*)** + **(*c*)**. Denote the vertices v_c of the components **(*c*)** by v_c^i for $1 \leq i \leq 4$. If $s\text{-}\deg_G(v_c^i) = 2$ for some $1 \leq i \leq 4$ then apply Reduction 1. Assume that $s\text{-}\deg_G(v_c^i) = 3$ for all i. Now independently of the existence of the null-component, split v_c^1 and v_c^2 to s_1s_2-edges and split v_c^3 and v_c^4 to s_2s_3-edges. The resulting graph is H with $V(H) = S$. Now take any 2-tree-coloring of H and unsplit the vertices v_c^i giving a 2-tree-coloring of G uniform to $\mathcal{P} = \{\Delta(s_1), \Delta(s_2), \Delta(s_3)\}$.

- **(*c*)** + **(*c*)** + **(*c*)** + **(*d*)** and **(*c*)** + **(*c*)** + **(*d*)** + **(*d*)** and **(*c*)** + **(*d*)** + **(*d*)** + **(*d*)** and **(*d*)** + **(*d*)** + **(*d*)** + **(*d*)**. Apply Reduction 1.

End of proof of Theorem 5.2. □

References

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