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Source location with rigidity and tree packing requirements

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Abstract

We consider the following two problems. *(i)* Given a graph, add a minimum size clique to the graph such that the resulting graph is rigid in the plane. *(ii)* Given a graph, contract a minimum size vertex-set such that the resulting graph has two edge-disjoint spanning trees. We prove that these problems are polynomially solvable.

1 Introduction

We consider undirected graphs which may contain multiple edges but not loops. Let $G = (V, E)$ be a graph, $F \subseteq E$ and $X \subseteq V$. Let $F(X)$ denote the set of edges in F with both end-vertices in X and $i_F(X) := |F(X)|$. $e(X)$ is the number of edges with at least one endpoint in X, that is, $e(X) = |E| - i_E(V - X)$. Let $\mathcal{C}(G)$ denote the set of the components of G. If $(A, B; E)$ is a bipartite graph and $X \subseteq A$, then $\Gamma(X)$ denotes the set of the neighbors of X.

We will define two matroids $\mathcal{M}_{2,2}$ and $\mathcal{M}_{2,3}$, so let $l = 2$ or $l = 3$ in the rest of the paper. Suppose we are given a graph $G = (V, E)$. We define the following set system:

$$\mathcal{I}_{2,l} := \{F \subseteq E : i_F(X) \leq 2|X| - l \text{ for every } X \subseteq V, |X| \geq 2\}.$$

The following claim is well-known. We remark that the condition "$i_F(X) \leq k|X| - l$" defines a matroid for many other values of k and l (see the appendix of [14]).

Claim 1.1. *$\mathcal{I}_{2,l}$ forms the independent sets of a matroid on the underlying set E. In this matroid the rank of an $F \subseteq E$ is the following:*

$$\min_{\mathcal{X}} \sum_{X \in \mathcal{X}} (2|X| - l) + |F - F(\mathcal{X})|$$

Where the minimum is taken over set systems $\mathcal{X} = \{X_1, \ldots, X_t\}$ where $X_i \subseteq V$, $|X_i| \geq 2$ and $F(\mathcal{X}) := \cup_{X \in \mathcal{X}} F(X)$.

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This matroid will be denoted by $\mathcal{M}_{2,l}$ and let r be its rank function. We remark that if $l = 2$, then $\mathcal{X}$ can be chosen to be a sub-partition of V (this is a consequence of matroid union theorem, [13, Theorem 51.1 and Corollary 51.1c]), and if $l = 3$, then $\mathcal{X}$ can be chosen such that $F = F(\mathcal{X})$ [9].

If $Z \subseteq V$, then K_Z will denote (the edge-set of) a graph on Z which has $4 - l$ parallel edges between every two vertices of Z. We investigate the following problem.

> We are given a graph G and the task is to find a minimum cardinality Z such that the rank of $E + K_Z$ is $2|V| - l$.

If $l = 2$, then $r(E) = 2|V| - 2$ if and only if there exist 2 edge-disjoint spanning trees in G. (By Nash-Williams' theorem [11], see also [13, Corollary 51.1c.]) So in this case the problem is the following.

> We are given a graph G. Find a minimum cardinality Z so that $G + K_Z$ has 2 edge-disjoint spanning trees, or equivalently contract a minimum size Z so that G/Z has 2 edge-disjoint spanning trees.

This problem is a variation of the so called source location problem, where G/Z has to satisfy connectivity requirements. (See [1] and [6].)

If $l = 3$, then $r(E) = 2|V| - 3$ if and only if G is generically rigid in the plane (Laman's theorem [7], for basic concepts of rigidity see e.g. [5]). So in this case the problem is the following.

> Add a minimum size clique to a graph so as to make it generically rigid in the plane.

It can be shown that this problem is equivalent to finding a minimum size vertex set so that pinning down that vertex-set makes the graph generically infinitesimally rigid. The non-generic version of this problem was solved in [8]. (About pinning see also [12, Section 8.2 and 18.2].)

The main result is that these problems are polynomial time solvable (Theorem 2.2). In fact it is proved that a slight generalization of these problems are solvable, namely we can prescribe a vertex-set T that has to be contained in Z.

2 Main result

The main content of Lemma 2.1 is that we give an equivalent formulation of our problem in the case when E is independent in $\mathcal{M}_{2,l}$. Using Lemma 2.1 we are able to reduce our problem to a matching problem.

Lemma 2.1. *Let E be independent in $\mathcal{M}_{2,l}$ and $|E| < 2|V| - l$, and let $Z \subseteq V$.*
(i) If $|V - Z| \geq 2$, then

$$r(E + K_{V-Z}) = \min_{X \subseteq Z} 2|V - X| - l + e(X)$$

(ii) $r(E + K_{V-Z}) = 2|V| - l$ if and only if $e(X) \geq 2|X|$ holds for every $X \subseteq Z$.

Proof. *(i)* Let $E' := E + K_{V-Z}$. For a set system $\mathcal{X} = \{X_1, \dots, X_t\}$, $X_i \subseteq V$, $|X_i| \geq 2$ let $v(\mathcal{X}) := \sum_{X \in \mathcal{X}}(2|X| - l) + |E' - E'(\mathcal{X})|$. By Claim 1.1, $r(E') = \min_{\mathcal{X}} v(\mathcal{X})$. It is enough to prove that the minimum is attained on a one element set system $\mathcal{X} = \{X\}$ where $Z - V \subseteq X \subseteq V$ because $v(\{X\}) = 2|X| - l + e(V - X)$, thus by complementing X we get *(i)*.

Let the set system $\mathcal{X}$ lexicographically minimize the vector $(v(\mathcal{X}), |E' - E'(\mathcal{X})|, |\mathcal{X}|)$. Then the following holds.

$$|X \cap Y| \leq 1 \text{ for } X \neq Y \in \mathcal{X} \tag{1}$$

$$\nexists X, Y, Z \in \mathcal{X} : \ |X \cap Y| = |X \cap Z| = |Y \cap Z| = 1 \text{ and } X \cap Y \cap Z = \emptyset \tag{2}$$

$$\text{for } u \neq v \in V - Z \text{ there exists a unique } X_{uv} \in \mathcal{X} : u, v \in X_{uv} \tag{3}$$

(1) is true because $2|X| - l + 2|Y| - l = 2|X \cup Y| - l + 2|X \cap Y| - l > 2|X \cup Y| - l$ if $|X \cap Y| \geq 2$, so $\mathcal{X} - X - Y + X \cup Y$ would contradict the minimality of $v(\mathcal{X})$. We prove (2) indirectly, suppose there exists such a configuration then $2|X| - l + 2|Y| - l + 2|Z| - l = 2(|X \cup Y \cup Z| + 3) - 3l \geq 2|X \cup Y \cup Z| - l$. Thus by replacing X, Y, Z with $X \cup Y \cup Z$ we lexicographically decrease $(v(\mathcal{X}), |E' - E'(\mathcal{X})|, |\mathcal{X}|)$, a contradiction. To prove (3) suppose that $u \neq v \in V - Z$ and there does not exist an X containing u, v. Then adding set $\{u, v\}$ lexicographically decrease $(v(\mathcal{X}), |E' - E'(\mathcal{X})|, |\mathcal{X}|)$. The unicity of such a set follows by (1).

We prove that if $u, v, w \in V - Z$, then $X_{uv} = X_{vw}$. Indirectly suppose not. Then by (1) $X_{uv} \cap X_{vw} = \{v\}$, but then $X_{uv} \cap X_{uw} = \{u\}$ and $X_{vw} \cap X_{uw} = \{w\}$. This contradicts (2).

This implies that $X_{uv} = X_{u'v'}$ for every $u, v, u', v' \in V - Z$, that is, there exists a unique $X \in \mathcal{X}$ containing $V - Z$, and $|X \cap Y| \leq 1$ for every $Y \in \mathcal{X} - X$. It is easy to see that $v(\mathcal{X}) \geq v(\{X\})$ because $i_{E'}(X') = i_E(X') \leq 2|X'| - l$ for every $X' \in \mathcal{X} - X$. Therefore the one element set system $\{X\}$ satisfies that $r(E') = v(\{X\})$.

(ii) If $|V - Z| \geq 2$, then *(i)* implies *(ii)*. If $|V - Z| \leq 1$, then $r(E + K_{V-Z}) = r(E) = |E| < 2|V| - l$ and $e(Z) = |E| < 2|V| - l \leq 2|Z|$. □

We call a set $Z \subseteq V$ a *good subset* if $e(X) \geq 2|X|$ holds for every $X \subseteq Z$. By Lemma 2.1 (ii) if E is independent in $\mathcal{M}_{2,l}$, then the problem of finding a minimum cardinality subset Y such that $r(E + K_Y) = 2|V| - l$ is equivalent to finding a maximum size good set Z.

Theorem 2.2. *We are given a graph $G = (V, E)$ and a set $T \subseteq V$. Then there exists an $O\left((|E| + |V|)^{3/2}\right)$ running time algorithm to find a maximum cardinality good set Z not intersecting T, and there exists an $O(|V|^2)$ running time algorithm to find a minimum cardinality set Z containing T so that $r(E + K_Z) = 2|V| - l$.*

Proof. First we state that in the second problem we can assume that E is independent in $\mathcal{M}_{2,l}$ because if E' is a base of E in matroid $\mathcal{M}_{2,l}$, then $r(E + K_Z) = r(E' + K_Z)$ holds for all $Z \subseteq V$. Finding a base of an edge-set in $\mathcal{M}_{2,l}$ can be done in $O(|V|^2)$ time (if $l = 2$, then see [13, Section 51.5a], if $l = 3$, then see e.g. [3] and [2] for other references).

By Lemma 2.1 (ii) the two optimization problems are equivalent. We prove that finding a maximum size good set can be reduced to finding a maximum matching.

Let us define the following bipartite graph B. Let $\tilde{V}$ denote a set which contains two nodes v', v'' for each $v \in V - T$. For a set $X \subseteq V - T$ let $\tilde{X}$ will denote the set $\{v' : v \in X\} \cup \{v'' : v \in X\}$. The two color classes of B will be $\tilde{V}$ and E. The edge-set is $F := \{(ev') : v \in V - T \text{ is an endpoint of } e\} \cup \{(ev'') : v \in V - T \text{ is an endpoint of } e\}$. By definition it follows that $Z \subseteq V - T$ is good if and only if $|\Gamma(\tilde{X})| \geq |\tilde{X}|$ for each $X \subseteq Z$, and it is easy to see that this is equivalent to the following: $|\Gamma(X)| \geq |X|$ for each $X \subseteq \tilde{Z}$. By Hall's theorem (see e.g. [13, Theorem 16.6]) Z is good if and only if there is a matching $M \subseteq F$ covering $\tilde{Z}$.

We construct graph $G' = (V', E')$ such that $V' := \tilde{V} \cup E$ and $E' := F \cup \{v'v'' : v \in V - T\}$. Let M be a maximum matching in G' and Z a maximum cardinality good set in G. We claim that $|M| = |V - T| + |Z|$. If we have a good set Z' and $M_{Z'} \subseteq F$ is a matching covering Z', then $M_{Z'} \cup \{v'v'' : v \in V - T - Z'\}$ is a matching of cardinality $2|Z'| + |V - T| - |Z'| = |V - T| + |Z'|$. If M' is a maximal matching, then $Z := \{v \in V - T : v' \text{ and } v'' \text{ is covered by } M' \cap F\}$ is good set of cardinality $|M'| - |V - T|$.

We can apply an algorithm finding a maximum matching in G' to find a maximum size good set in G. This matching can be found in $O(\sqrt{|V'|}|E'|)$ time (for literature see [13, Section 24.4a]). By the definition of G': $|V'| \leq 2|V| + |E|$ and $|E'| \leq 4|E| + |V|$, hence $O(\sqrt{|V'|}|E'|) = O\left((|E| + |V|)^{3/2}\right)$. Using the fact that $|E| \leq 2|V|$ if E is independent in $\mathcal{M}_{2,l}$ we get an $O(|V|^{3/2})$ running time algorithm for independent E. So the total running time of the algorithm to find a minimum cardinality Z set containing T so that $r(E + K_Z) = 2|X| - l$ is $O(|V|^2)$. □

We use the above reduction and the Berge-Tutte formula (see e.g. [13, Theorem 24.1]) to deduce a minimax theorem on the maximum size of a good set.

Theorem 2.3. *If we are given a graph $G = (V, E)$ and a set $T \subseteq V$, then the following equality holds:*

$$\max_{Z \text{ is good, } Z \cap T = \emptyset} |Z| = \min_{Y : T \subseteq Y \subseteq V} \sum_{C \in \mathcal{C}(G-Y)} \left\lfloor \frac{e(C)}{2} \right\rfloor + |Y - T| \tag{4}$$

Proof. max ≤ min: If $Z \subseteq V - T$ is good and $T \subseteq Y \subseteq V$, then $|Z| \leq \sum_{C \in \mathcal{C}(G-Y)} |Z \cap C| + |Y - T| \leq \sum_{C \in \mathcal{C}(G-Y)} \left\lfloor \frac{e(C)}{2} \right\rfloor + |Y - T|$.

max ≥ min: Using the notations and results of the previous proof, it is enough to prove that the size of a maximum matching of G' is at least

$$\min_{Y : T \subseteq Y \subseteq V} \left(\sum_{C \in \mathcal{C}(G-Y)} \left\lfloor \frac{e(C)}{2} \right\rfloor + |Y| \right) + |V| - 2|T|.$$

Let $w(X) := \sum_{C \in \mathcal{C}(G'-X)} \left\lfloor \frac{|C|}{2} \right\rfloor + |X|$ $(X \subseteq V')$. The Berge-Tutte formula implies that the cardinality of a maximum matching is equal to $\min_{X \subseteq V'} w(X)$. Suppose that $v' \in C \in \mathcal{C}(G' - X)$ and $v'' \in X$ where $v \in V - T$. If $|C|$ is even, then $w(X + v') \leq w(X)$. If $|C|$ is odd, then $w(X - v'') \leq w(X)$. Thus we can assume that

$X \cap \tilde{V}$ is the union of v', v'' pairs. If an edge $e \in X \cap E$, then $w(X-e) \leq w(X)$. Thus there exists an optimal X such that $X = \tilde{U}$ for some $U \subseteq V - T$. Now the equality $w(X) = \sum_{C \in \mathcal{C}(G'-X)} \left\lfloor \frac{|C|}{2} \right\rfloor + |X| = |V - T - U| + \sum_{C \in \mathcal{C}(G-U \cup T)} \left\lfloor \frac{e(C)}{2} \right\rfloor + 2|U|$ implies that in (4) equality holds for $Y = U \cup T$. □

We present a short proof for the NP-completeness of the source location problem concerning 3 edge-disjoint trees and the 3 dimensional pinning problem. (*(ii)* was already proved in [10].)

Theorem 2.4. *Let $G = (V, E)$ be a graph.*
(i) The problem of finding a minimum size $Z \subseteq V$ such that G/Z has 3 edge-disjoint spanning trees is NP-complete.
(ii) The problem of finding a minimum size $Z \subseteq V$ such that $G + K_Z$ is rigid in 3 dimensions is NP-complete.

Proof. (i) This problem is clearly in NP. The problem of finding a maximum stable set in a 3-regular graph is NP-complete, see [4]. It is easy to see that if G is 3-regular, then G/Z has 3 edge-disjoint spanning trees if and only if $V - Z$ is stable.
(ii) This problem is in NP because 3 dimensional rigidity is known to be in NP. It is easy to see that if G is 3-regular, then $G + K_Z$ is rigid in 3 dimensions if and only if $V - Z$ is stable. □

We mention that the problem of finding a minimum cardinality set Z so that $r(E + K_Z) = 2|V| - l$ is equivalent to a special matroid parity problem. If $r_0(Z) := r(E + K_{V-Z}) - r(K_{V-Z})$ $(Z \subseteq V)$, then r_0 is a 2-polymatroid function on V because the submodularity can be deduced from the submodularity of r and $r_0(V - Z) = r(E + K_Z) - r(K_Z) \leq r(K_V) - r(K_Z) \leq 2|V - Z|$. By investigating the latter inequality it can be checked that if $r(E) < 2|V| - l$, then $r_0(V - Z) = 2|V - Z|$ holds if and only if $r(E + K_Z) = 2|V| - l$. Hence the question of finding a minimum cardinality Z so that $r(E + K_Z) = 2|V| - l$ is equivalent to finding a maximum cardinality W so that $r_0(W) = 2|W|$. Now we prove that a simple minimax formula for this matroid parity problem is a consequence of Theorem 2.3.

Theorem 2.5. *If we are given a graph $G = (V, E)$, then*

$$\max_{W \subseteq V-T: r_0(W)=2|W|} |W| = \min_{\mathcal{F} \text{ is a partition of } V-T} \sum_{X \in \mathcal{F}} \left\lfloor \frac{r_o(X)}{2} \right\rfloor .$$

Proof. The min $\geq$ max direction is true for every 2-polymatroid function. To prove min $\leq$ max we observe that it is enough to prove the formula if E is independent in matroid $\mathcal{M}_{2,l}$ (we can choose a base). We show a sub-partition where the left hand side is greater or equal than the right hand side. Let $\mathcal{F}'$ be a sub-partition of $V - T$ formed by the components of $G - Y$ where Y is an optimal set in (4) and let $\mathcal{F} := \mathcal{F}' \cup \{\{y\} : y \in Y - T\}$. The following inequality proves the theorem.

$$\sum_{X \in \mathcal{F}} \left\lfloor \frac{r_o(X)}{2} \right\rfloor \leq \sum_{X \in \mathcal{F}'} \left\lfloor \frac{e(X)}{2} \right\rfloor + |Y - T| = \max_{Z \text{ is good}, Z \cap T = \emptyset} |Z| =$$

$$= \max_{W \subseteq V-T : r_0(W)=2|W|} |W|$$

□

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