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Merging hyperedges to meet edge-connectivity requirements

Tamás Király

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Tamás Király*

Abstract

We give a short proof of a slight generalization of a theorem of Szigeti [1] on hypergraph connectivity augmentation.

Given a finite ground set V, a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *skew supermodular* if at least one of the following two inequalities holds for every $X, Y \subseteq V$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{1}$$
$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{2}$$

A set function is *symmetric* if $p(X) = p(V - X)$ for every $X \subseteq V$. In [1], Szigeti proved the following result, which is fundamental for solving local edge-connectivity augmentation problems in hypergraphs.

Theorem 1 ([1]). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew supermodular set function, and $m : V \to \mathbb{Z}_+$ a degree prescription. There exists a hypergraph H s.t. $d_H(v) = m(v)$ for every $v \in V$ and $d_H(X) \geq p(X)$ for every $X \subseteq V$ if and only if*

$$\sum_{v \in X} m(v) \geq p(X) \quad \textit{for every } X \subseteq V. \tag{3}$$

In this note we give a short proof of a slight generalization of this theorem. Let $H = (V, \mathcal{E})$ be a hypergraph. By *merging* two disjoint hyperedges of H we mean the operation of replacing them in H by their union. "Merging some hyperedges of H" means repeating this operation a few times. Let us define the set function

$$b_H(X) := |\{e \in \mathcal{E} : \ e \cap X \neq \emptyset\}|.$$

It is easy to see that b_H is fully submodular, and

$$b_H(X) + b_H(Y) \geq b_H(X - Y) + b_H(Y - X) + |\{e \in \mathcal{E} : \ \emptyset \neq e \cap Y \subseteq X \cap Y\}|.$$

*MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117. e-mail: tkiraly@cs.elte.hu. Supported by the Hungarian National Foundation for Scientific Research, OTKA T037547, and by European MCRTN Adonet, Contract Grant No. 504438.

Theorem 2. *Let $H = (V, \mathcal{E})$ be a hypergraph, and let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew supermodular set function for which*

$$b_H(X) \geq p(X) \quad \text{for every } X \subseteq V. \tag{4}$$

Then by merging some hyperedges of H we can obtain a hypergraph $H_* = (V, \mathcal{E}_*)$ such that

$$d_{H_*}(X) \geq p(X) \quad \text{for every } X \subseteq V. \tag{5}$$

Proof. We prove the theorem by induction on the number of hyperedges of H (it is clearly true if $\mathcal{E} = \emptyset$). A set $X \subseteq V$ is called *tight* if $b_H(X) = p(X)$. By the properties of b_H and p, if X and Y are tight, then either $X \cap Y$ and $X \cup Y$ are tight, or $X - Y$ and $Y - X$ are tight. Furthermore, if X and Y are tight and there is a hyperedge e such that $\emptyset \neq e \cap Y \subseteq X \cap Y$, then $X \cap Y$ and $X \cup Y$ are tight.

Let e_0 be an arbitrary hyperedge of H. If there is no tight set X such that $e_0 \subseteq X$, then let $H' := H - e_0$ and

$$p'(X) := \begin{cases} p(X) - 1 & \text{if } e_0 \cap X \neq \emptyset \text{ and } e_0 \cap (V - X) \neq \emptyset, \\ p(X) & \text{otherwise.} \end{cases}$$

The set function p' is symmetric and skew supermodular, and $b_{H'}(X) \geq p'(X)$ for every $X \subseteq V$, so by induction there is a hypergraph H'_*, obtained by merging some hyperedges of H', such that $d_{H'_*}(X) \geq p'(X)$ for every $X \subseteq V$. It follows that $H_* := H'_* + e_0$ satisfies (5). We can thus assume that there is a tight set X_0 such that $e_0 \subseteq X_0$.

Let $\mathcal{Y}$ be the family of tight sets Y for which $e_0 \cap Y \neq \emptyset$. If $Y \in \mathcal{Y}$, then $\emptyset \neq e_0 \cap Y \subseteq X_0 \cap Y$, thus $X_0 \cap Y$ and $X_0 \cup Y$ are also in $\mathcal{Y}$. If Y_1, and Y_2 are maximal sets in $\mathcal{Y}$, then $e_0 \subseteq Y_1 \cap Y_2$ by the previous observation, thus $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ are also in $\mathcal{Y}$, which means that $\mathcal{Y}$ has a unique maximal set Y_0.

Suppose that there is no hyperedge $e \in \mathcal{E}$ such that $e \cap Y_0 = \emptyset$. Then $p(V - Y_0) = p(Y_0) = b_H(Y_0) > b_H(V - Y_0)$ since $e_0 \subseteq Y_0$, contradicting (4). Thus there is a hyperedge $e_1 \in \mathcal{E}$ such that $e_1 \cap Y_0 = \emptyset$. Consider the hypergraph $H' := (V, \mathcal{E} - \{e_0, e_1\} + (e_0 \cup e_1))$, i.e. the hypergraph obtained by merging e_0 and e_1. If $b_{H'}(X) < p(X)$ for some $X \subseteq V$, then $e_0 \cap X \neq \emptyset$, $e_1 \cap X \neq \emptyset$, and X is tight. But then $X \in \mathcal{Y}$, a contradiction because $e_1 \cap Y = \emptyset$ for every $Y \in \mathcal{Y}$.

We proved that H' satisfies the conditions of the theorem, so by induction there is a hypergraph H_*, obtained by merging some hyperedges of H' (hence obtained by merging some hyperedges of H), that satisfies (5). □

Theorem 1 corresponds to the case when H consists of hyperedges of size 1, and $m(v)$ is the multiplicity of $\{v\}$ in H.

References

[1] Z. Szigeti, *Hypergraph connectivity augmentation*, in: Connectivity Augmentation of Networks: Structures and Algorithms, Mathematical Programming (ed. A. Frank), Ser. B, Vol. 84, No. 3 (1999), 519–527.