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On the Rank Function of the 3-Dimensional Rigidity Matroid

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Abstract

It is an open problem to find a good characterization for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$. In this paper we give a brief survey of existing lower and upper bounds on the rank of the 3-dimensional rigidity matroid of a graph and introduce a new upper bound, which may lead to the desired good characterization.

1 Introduction

The theory of rigidity and flexibility of frameworks has a wide range of applications in applied geometry. It has been applied in statics [18], in molecular conformation problems [4], and in computer aided design [11]. More recent applications include localization problems of sensor networks [1] and formations of autonomous agents [6].

In this paper we focus on rigidity problems of 3-dimensional generic frameworks and consider one of the main unsolved problems in combinatorial rigidity: the characterization of rigid graphs in 3-space. First we give a brief survey of existing lower and upper bounds on the rank of the 3-dimensional rigidity matroid of a graph and then introduce a new upper bound, which may lead to the desired characterization.

A *framework* (G, p) in d-space is a simple graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$. The *rigidity matrix* of the framework is a matrix $R(G, p)$ of size $|E| \times d|V|$ with rows indexed by E and sets of d consecutive columns indexed by V. For each edge $e = v_iv_j \in E$, the entries in the row e and in the d columns v_i and v_j contain the d coordinates of $p(v_i) - p(v_j)$ and $p(v_j) - p(v_i)$, respectively. The remaining entries in row e are zeros. See [27, 29] for more details. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of rows of the rigidity matrix. A *framework* (G, p) is *generic* if the set of coordinates of the

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points $p(v)$, $v \in V$, is algebraically independent over the rationals. Any two generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the d-dimensional *rigidity matroid* $\mathcal{R}_d(G) = (E, r_d)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. The following fundamental result gives our first upper bound for $r_d(G)$.

Lemma 1.1. *[27, Lemma 11.1.3] Let (G, p) be a framework in $\mathbb{R}^d$. Then $\operatorname{rank} R(G, p) \leq S(n, d)$, where $n = |V(G)|$ and*

$$S(n, d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d+2 \\ \binom{n}{2} & \text{if } n \leq d+1. \end{cases}$$

We say that a graph $G = (V, E)$ is *rigid* in $\mathbb{R}^d$ if $r_d(G) = S(n, d)$. (This definition is motivated by the fact that if G is rigid and (G, p) is a generic framework on G, then every continuous deformation of (G, p) which preserves the edge lengths $||p(u) - p(v)||$ for all $uv \in E$, must preserve the distances $||p(w) - p(x)||$ for all $w, x \in V$, see [27].) We say that G is *M-independent*, *M-dependent* or an *M-circuit* in $\mathbb{R}^d$ if E is independent, dependent or a circuit, respectively, in $\mathcal{R}_d(G)$. For $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in $G[X]$, that is, in the subgraph induced by X in G. We use $E(X)$ or $i(X)$ when the graph G is clear from the context. A *cover* of G is a collection $\mathcal{X}$ of pairwise incomparable subsets of V, each of size at least two, such that $\cup_{X \in \mathcal{X}} E(X) = E$. Lemma 1.1 implies the following necessary condition for G to be M-independent.

Lemma 1.2. *If $G = (V, E)$ is M-independent in $\mathbb{R}^d$ then $i(X) \leq |X|d - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d + 2$.*

Note that, since G is simple, we automatically have $i(X) \leq S(|X|, d) = \binom{|X|}{2}$ when $|X| \leq d + 1$. Lemma 1.1 also gives the following stronger upper bound on $r_d(G)$.

Lemma 1.3. *If $G = (V, E)$ is a graph then*

$$r_d(G) \leq \min_{\mathcal{X}} \sum_{X \in \mathcal{X}} S(|X|, d)$$

where the minimum is taken over all covers $\mathcal{X}$ of G.

The converse of Lemma 1.2 also holds for $d = 1, 2$. The case $d = 1$ follows from the fact that the 1-dimensional rigidity matroid of G is the same as the cycle matroid of G, see [9, Theorem 2.1.1]. The case $d = 2$ is a result of Laman [16]. Similarly, the inequality given in Lemma 1.3 holds with equality when $d = 1, 2$, and leads to a good characterization of the rank function. The case $d = 2$ is a result of Lovász and Yemini [17]. Neither of these statements hold for $d \geq 3$. Indeed, it remains an open problem to find a good characterization for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$.

We shall give a brief survey of existing lower and upper bounds on the rank of the 3-dimensional rigidity matroid of a graph and introduce a new upper bound, which may lead to the desired good characterization.

2 Lower bounds on the rank function

We may certify a lower bound on $r_d(G)$ by providing a d-dimensional framework (G, p) whose rigidity matrix has sufficiently high rank. A lemma of Schwartz [20] implies that there is always such a matrix with small enough entries. This shows that d-dimensional rigidity is in NP. It also gives rise to a randomized polynomial time algorithm for computing $r_d(G)$.

It would be useful to also have combinatorial methods to verify lower bounds on $r_d(G)$. This could be accomplished by obtaining conditions which imply M-independence, since $r_d(G) \geq k$ if and only if G has an M-independent subgraph with k edges. Sufficient conditions for M-independence will also be relevant in the next section since our new upper bound for $r_3(G)$ requires us to be able to verify M-independence.

When $d = 2$, there are two different combinatorial characterizations of M-independence. We will describe these characterizations and give some partial extensions for the case when $d = 3$.

2.1 Laman type conditions

As mentioned in the Introduction, M-independence for $d = 2$ is characterized by the following result of Laman.

Theorem 2.1. *[16] A graph $G = (V, E)$ is M-independent in $\mathbb{R}^2$ if and only if $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 4$.*

The following example shows that the necessary condition for M-independence given in Lemma 1.2 is not sufficient when $d = 3$. We will refer to this condition henceforth as the *Laman condition*.

Example 1 Let G be the graph obtained from two disjoint K_5's by identifying an edge uv, and then deleting uv. Then $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $|X| \geq 5$ but, as we shall see later, $r_3(G) = 17 < |E(G)|$. Thus G is not M-independent in $\mathbb{R}^3$. (This example is often referred to in the literature as the 'double banana'.)

We next show that we may obtain a sufficient condition for M-independence in $\mathbb{R}^3$ by strengthening the Laman condition.

Theorem 2.2. *[12] Let $G = (V, E)$ be a graph. If*

$$i(X) \leq \frac{1}{2}(5|X| - 7) \tag{1}$$

for all $X \subseteq V$ with $|X| \geq 5$ then G is M-independent in $\mathbb{R}^3$.

We believe that the multiplicative constant, 5/2, in the upper bound on $i(X)$ given in Theorem 2.2 can be weakened to 3. The double banana shows that there exist graphs $G = (V, E)$ with $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $|X| \geq 5$, which are M-dependent in $\mathbb{R}^3$. We also have M-dependent examples satisfying $i(X) \leq 3|X| - 7$,

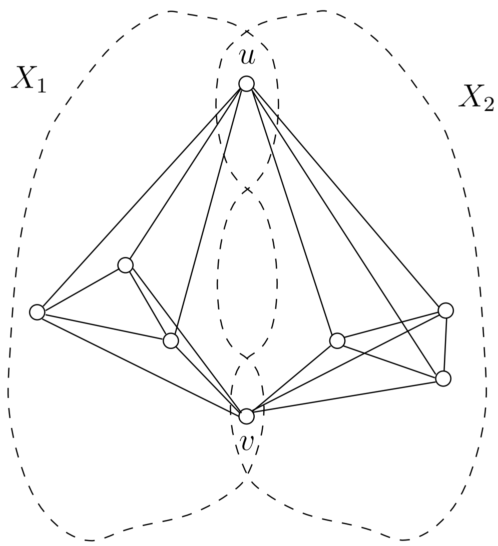

Figure 1: The 'double banana' graph of Example 1. A cover $\{X_1, X_2\}$ is indicated by the dashed curves.

but we know of no M-dependent graphs satisfying $i(X) \leq 3|X| - 8$ for all $X \subseteq V$, $|X| \geq 5$.

We close this subsection by describing two families of graphs for which the Laman condition is both necessary and sufficient for M-independence in $\mathbb{R}^3$.

Theorem 2.3. *[12] Let G be a connected graph of maximum degree at most five and minimum degree at most four. Then G is M-independent in $\mathbb{R}^3$ if and only if $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $|X| \geq 5$.*

We believe that Theorem 2.3 remains valid without the hypotheses that G is connected and has minimum degree at most four. Note that one can test, in polynomial time, whether an arbitrary graph $G = (V, E)$ satisfies $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $|X| \geq 5$. This can be done as follows. We may suppose that $|V| \geq 5$. For a path T of length two in G let $\bar{T}$ be a multigraph obtained from T by replacing one of its edges by two parallel edges. Thus $|E(\bar{T})| = 3$. It is not difficult to check that there is a set $X \subseteq V$ violating the Laman condition in G if and only if there exists a path T in G for which the edge set of $G_T = (V, E \cup E(\bar{T}))$ cannot be partitioned into 3 forests. There exist efficient algorithms for the forest partition problem, see [19, Chapter 51]. Thus we can test whether G is Laman by trying all possible paths T of length two in G.

This observation, and the fact that bases in a matroid can be constructed greedily, give rise to a polynomial algorithm for computing $r_3(G)$ when G satisfies the hypotheses of Theorem 2.3. Bereg [2] has given a linear time algorithm which determines $r_3(G)$ when G belongs to this family of graphs.

It follows from Euler's formula that a planar graph $G = (V, E)$ on at least three vertices has at most $3|V| - 6$ edges, with equality if and only if G is a plane triangulation. Thus planar graphs satisfy the Laman condition. Gluck showed that they are, indeed, M-independent in $\mathbb{R}^3$.

Theorem 2.4. *[8] Every planar graph is M-independent in $\mathbb{R}^3$ and every plane triangulation is rigid in $\mathbb{R}^3$.*

An important family of graphs for which it is conjectured that the Laman condition is both necessary and sufficient for M-independence in $\mathbb{R}^3$ are squares of graphs (also called molecular graphs). The *square* H^2 of graph H is the graph obtained from H by adding a new edge uv for each pair $u, v \in V(H)$ for which $uv \notin E(H)$ but $uw, vw \in E(H)$ for some $w \in V(H)$.

Conjecture 2.5. *[15] Let H be a graph and $G = H^2$. Then G is M-independent in $\mathbb{R}^3$ if and only if $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $|X| \geq 5$.*

For lower (and upper) bounds on the rank of molecular graphs see Franzblau [7].

2.2 Henneberg sequences

The second combinatorial certificate for M-independence in $\mathbb{R}^2$ uses Henneberg sequences [10]. To describe these sequences we need some terminology. Let $G = (V, E)$ be a graph and d be a fixed integer. A $(0, d)$-*extension* of G is a graph obtained by choosing a set S of at most d vertices of G, adding a new vertex v and edges from v to all vertices in S. For $1 \leq j \leq d - 1$, a (j, d)-*extension* of G is a graph obtained by choosing a set X of $d+j$ vertices of G such that $i(X) \geq j$, deleting j edges between the vertices of X, and then adding a new vertex v and edges from v to all vertices in X. It is known, see [9, Theorem 5.3.1], that if H is M-independent in $\mathbb{R}^d$ and $v \in V$ with $d(v) = d+j$ for some $0 \leq j \leq d-1$ then H is a (j, d)-extension of an M-independent graph G. It is also known, see [27, Lemma 11.1.1, Theorem 11.1.7], that $(0, d)$- and $(1, d)$-extensions preserve M-independence in $\mathbb{R}^d$. Thus graphs which can be obtained from K_2 by a sequence of $(0, d)$- and $(1, d)$-extensions are M-independent in $\mathbb{R}^d$.

Since a graph $G = (V, E)$ which is M-independent in $\mathbb{R}^2$ must contain a vertex of degree at most three by Lemma 1.2, we may deduce:

Theorem 2.6. *[10, 24] A graph $G = (V, E)$ is M-independent in $\mathbb{R}^2$ if and only if it can be obtained from K_2 by a sequence of $(0, 2)$- and $(1, 2)$-extensions.*

Theorem 2.6 implies that we may certify that a graph G is M-independent in $\mathbb{R}^2$ by exhibiting a sequence of graphs $K_2 = G_0, G_1, \ldots, G_m = G$ such that G_i is obtained from G_{i-1} by a $(0, 2)$- or $(1, 2)$-extension for $1 \leq i \leq m$. We call such a sequence a 2-dimensional *Henneberg sequence*.

Lemma 1.2 implies that every graph which is M-independent in $\mathbb{R}^3$ must contain a vertex of degree at most five. Thus, an analogous result to Theorem 2.6 in $\mathbb{R}^3$ would follow if we could show that $(2, 3)$-extensions preserve M-independence in $\mathbb{R}^3$. Unfortunately, this is not true in general. It may be true, however, if we put more restrictions on the $(2, 3)$-extension.

Conjecture 2.7. *[24] Let $G = (V, E)$ be a graph which is M-independent in $\mathbb{R}^3$, and $X \subseteq V$ with with $|X| = 5$. Let H be obtained from G by adding a new vertex v adjacent to each vertex of X. Then H is M-independent in $\mathbb{R}^3$ if either*
(a) $G \cup \{e, f\}$ is M-independent in $\mathbb{R}^3$ for edges $e = x_1x_2$ and $f = x_3x_4$ where x_1, x_2, x_3, x_4 are distinct vertices in X and $e, f \notin E$, or
(b) $G \cup \{e, f\}$ and $G \cup \{e', f'\}$ are both M-independent in $\mathbb{R}^3$ for two pairs of distinct edges $e = x_1x_2$, $f = x_2x_3$, and $e' = x'_1x'_2$, $f' = x'_2x'_3$, where $x_1, x_2, x_3, x'_1, x'_2, x'_3$ are vertices in X, $x_2 \neq x'_2$, and $e, f, e'f' \notin E$.

If true, Conjecture 2.7 may be a useful tool for proving results on rigidity in $\mathbb{R}^3$. It is not clear, however, that it would provide a polynomial length certificate for M-independence in $\mathbb{R}^3$. The problem is that to verify condition (b) in the conjecture we have to consider two graphs. This could give rise to an exponential number of graphs in the certificate.

We close this section by describing another operation which preserves M-independence in $\mathbb{R}^3$. Let $G = (V, E)$ be a graph, $v \in V$, and $vu_1, vu_2, ..., vu_k \in E$. The *vertex splitting* operation *on two edges* at v deletes the edges $vu_j, vu_{j+1}, ..., vu_k$ for some $j \geq 3$ and adds a new vertex v' and new edges $v'v, v'u_1, v'u_2$,$v'u_j$, $v'u_{j+1}, ..., v'u_k$. The *vertex splitting* operation *on three edges* at v deletes the edges $vu_j, vu_{j+1}, ..., vu_k$ for some $j \geq 4$ and adds a new vertex v' and new edges $v'u_1, v'u_2$,$v'u_3, v'u_j$, $v'u_{j+1}, ..., v'u_k$.

Theorem 2.8. *[25] Let G be a graph which is M-independent in $\mathbb{R}^3$. Let H be obtained from G by a vertex splitting operation (on two or three edges) at some vertex v. Then G' is M-independent in $\mathbb{R}^3$.*

One application of Theorem 2.8 is a quick proof for Theorem 2.4, see [26].

3 Upper bounds on the rank function

Let G be a graph. We saw in the last section that we can verify a lower bound for $r_d(G)$ by determining the rank of the rigidity matrix of a suitable framework (G, p). To show that our lower bound is (near) optimal, we need to be able to obtain a good upper bound on $r_d(G)$. Our first such upper bound is given by Lemma 1.3. As noted in the Introduction, Lovász and Yemini showed that this upper bound is tight when $d = 2$.

Theorem 3.1. *[17] If $G = (V, E)$ is a graph then*

$$r_2(G) = \min_{\mathcal{X}} \sum_{X \in \mathcal{X}} (2|X| - 3)$$

where the minimum is taken over all covers $\mathcal{X}$ of G.

The double banana graph of Example 1 shows that Lemma 1.3 need not be tight when $d = 3$.

Henceforth we will be concerned entirely with the 3-dimensional rigidity matroid of a graph. To simplify terminology we will suppress reference to the dimension and say for example that G is rigid to mean G is rigid in $\mathbb{R}^3$. We denote the function $S(n,3)$ given in Lemma 1.1 by $f(n)$. Thus $f(2) = 1$ and $f(n) = 3n - 6$ for $n \geq 3$.

We can use Theorem 2.3 to show that Lemma 1.3 becomes tight when $d = 3$, if we restrict the degrees of the vertices.

Theorem 3.2. *[12] Let G be a connected graph of maximum degree at most five and minimum degree at most four. Then $r_3(G) = \min_{\mathcal{X}} \sum_{X\in\mathcal{X}} f(|X|)$ where the minimum is taken over all covers $\mathcal{X}$ of G.*

We can obtain upper bounds on $r_3(G)$ for general graphs, which are stronger than that given by Lemma 1.3, by considering special kinds of covers.

3.1 2-thin covers

Let G be a graph. A cover $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ of G is *t-thin* if $|X_i \cap X_j| \leq t$ for all $1 \leq i < j \leq m$. It is known that Theorems 3.1 and 3.2 remain true if we add the condition that the minimum is taken over all 1-thin covers $\mathcal{X}$. In addition, a cover $\mathcal{X}$ of G which minimizes $\sum_{x\in\mathcal{X}} f(|X|)$ and has as few elements as possible is 1-thin. (If $X_1, X_2 \in \mathcal{X}$ and $|X_1 \cap X_2| \geq 2$ then we can replace $\mathcal{X}$ by $\mathcal{X}' = (\mathcal{X} - \{X_1, X_2\}) \cup \{X_1 \cup X_2\}$.) Thus the upper bound on $r_3(G)$ given by Lemma 1.3 remains the same if we add the condition that the minimum is taken over all 1-thin covers. We shall see, in Corollary 3.6 below, that 2-thin covers can be used to improve this upper bound.

Let $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ be a 2-thin cover of G. Let $H(\mathcal{X})$ be the set of all pairs of vertices uv such that $X_i \cap X_j = \{u, v\}$ for some $1 \leq i < j \leq m$. For each $uv \in H(\mathcal{X})$ let $d(uv)$ be the number of sets X_i in $\mathcal{X}$ such that $\{u, v\} \subseteq X_i$ and put

$$val(\mathcal{X}) = \sum_{X\in\mathcal{X}} f(|X|) - \sum_{uv\in H(\mathcal{X})} (d(uv) - 1).$$

In 1983, Dress, Drieding and Haegi [5, equation (39)], [23, Conjecture 3] conjectured that

$$r_3(G) = \min_{\mathcal{X}} \{val(\mathcal{X})\}, \tag{2}$$

where the minimum is taken over all 2-thin covers $\mathcal{X}$ of G. This conjecture would have provided a good characterization for $r_3(G)$. It was recently disproved in [13] by showing that $\min_{\mathcal{X}}\{val(\mathcal{X})\}$ can be negative and hence will not give an upper bound for $r_3(G)$ in general.

At a conference on rigidity held in Montreal in 1987, Dress conjectured that equality is obtained in (2) for the special 2-thin cover defined as follows. For $u, v \in V$, the edge uv is an *implied edge* of G if $uv \notin E$ and $r_3(G + uv) = r_3(G)$. The closure $\hat{G}$ of G is the graph obtained by adding all the implied edges to G. A *rigid cluster* of G is a set of vertices which induce a maximal complete subgraph of $\hat{G}$. It is not difficult to see that any two rigid clusters of G intersect in at most two vertices. Thus the set of rigid clusters of G is a 2-thin cover of G.

Conjecture 3.3. *(see [3],[9, Conjecture 5.6.1], and [21, Conjecture 2.3]) Let $G = (V, E)$ be a graph and $\mathcal{C}$ be the set of rigid clusters of G. Then*

$$r_3(G) = val(\mathcal{C}). \tag{3}$$

This conjecture is still open. Note, however, that even if Conjecture 3.3 was shown to be true, it would not provide a good characterization for the rank function.

We have verified Conjecture 3.3 for a special family of graphs of low degree.

Theorem 3.4. *[13] Let G be a 3-edge-connected graph of maximum degree at most five and minimum degree at most four. Let $\mathcal{C}$ be the set of rigid clusters of G. Then $r_3(G) = val(\mathcal{C})$.*

It is conceivable that Conjecture 3.3 is true because of the special intersection properties of rigid clusters. If so, then it may be possible to resurrect the first conjecture of Dress et al. by only considering 2-thin covers whose intersection properties reflect those of rigid clusters.

3.2 Independent 2-thin covers

We say that a 2-thin cover $\mathcal{X}$ of a graph $G = (V, E)$ is *independent* if the graph $(V, H(\mathcal{X}))$ is M-independent. The following lemma will imply that independent 2-thin covers can be used to give an upper bound on $r_3(G)$.

Lemma 3.5. *Let $G = (V, E)$ be a graph, and $\mathcal{X}$ be an independent 2-thin cover of G. Let $G_1 = G \cup H(\mathcal{X})$. Then*

$$r_3(G) \leq \sum_{X_i \in \mathcal{X}} r_3(G_1[X_i]) - \sum_{uv \in H(\mathcal{X})} (d(uv) - 1).$$

Proof: We may suppose that $H = H(\mathcal{X}) \subseteq E$ and hence $G = G_1$. For each $X_i \in \mathcal{X}$ let S_i be the set of edges of H which join two vertices of X_i. Since $\mathcal{X}$ is independent, (X_i, S_i) is an M-independent subgraph of $G[X_i]$ and hence S_i can be extended to a basis B_i for the rigidity matroid of $G[X_i]$. Let $S = \cup_{X_i \in \mathcal{X}} B_i$. Then S spans E since, if $e \in E$ then $e \in E(X_i)$ for some $X_i \in \mathcal{X}$ and hence e is spanned by $B_i \subseteq S$. Thus $r_3(G) \leq |S|$. Furthermore, $|B_i| = r_3(G[X_i])$ for all $X_i \in \mathcal{X}$. Since S covers each $uv \in S - H$ exactly once and covers each $uv \in H$ exactly $d(uv)$ times, we have

$$|S| = \sum_{X_i \in \mathcal{X}} |B_i| - \sum_{uv \in H} (d(uv) - 1) \leq \sum_{X_i \in \mathcal{X}} r_3(G[X_i]) - \sum_{uv \in H(\mathcal{X})} (d(uv) - 1),$$

as claimed. •

Corollary 3.6. *[13, Lemma 3.4] Let G be a graph, and $\mathcal{X}$ be an independent 2-thin cover of G. Then $r_3(G) \leq val(\mathcal{X})$.*

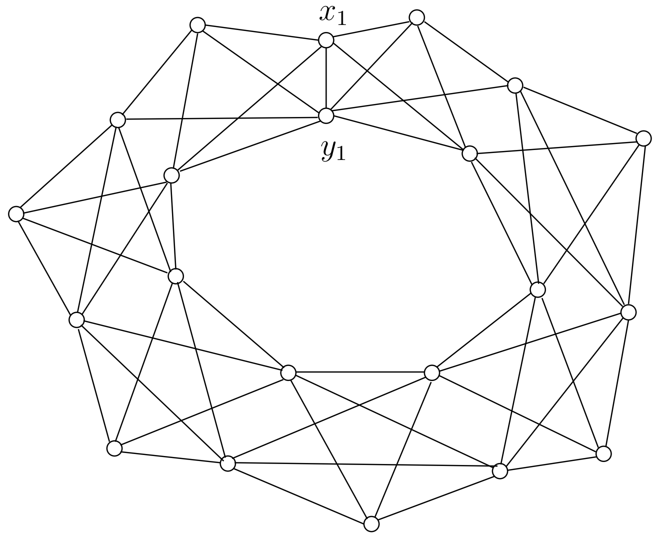

Figure 2: The graph of Example 2.

Proof: Let $G_1 = G \cup H(\mathcal{X})$. Since $r_3(G_1[X_i]) \leq f(|X_i|)$ we may apply Lemma 3.5 to deduce that $r_3(G) \leq r_3(G_1) \leq val(\mathcal{X})$. •

We can use Corollary 3.6 to determine the rank of the double banana graph G of Example 1. Taking $\mathcal{X} = \{X_1, X_2\}$ to be the independent cover of G consisting of the vertex sets of the two K_5's we have $H(\mathcal{X}) = \{uv\}$ and $r_3(G) \leq f(5) + f(5) - 1 = 9 + 9 - 1 = 17$. See Figure 1. On the other hand, for all edges e we can use a Henneberg sequence of 0- and 1-extensions to deduce that $G - e$ is M-independent. Thus $r_3(G) = |E(G)| - 1 = 17$. It follows that G is a 2-connected non-rigid M-circuit. (Non-rigid M-circuits do not exist in 2-dimensions. Their existence in 3-dimensions makes the problem of characterizing independence in the 3-dimensional rigidity matroid significantly more difficult.)

The following example due to Tay [22] is perhaps more interesting.

Example 2 Let $G = G_1 \cup G_2 \cup \ldots \cup G_7$, where $V(G_i) \cap V(G_{i+1}) = \{x_i, y_i\}$, $E(G_i) \cap E(G_{i+1}) = \emptyset$, $G_i = K_5 - \{x_{i-1}y_{i-1}, x_iy_i\}$ and subscripts are read modulo seven. Let $H = G + x_1y_1$ and let $\mathcal{X}$ be the independent 2-thin cover of H obtained by taking the vertex sets $V(G_i)$, $1 \leq i \leq 7$. Then Corollary 3.6 gives $r_3(H) \leq val(\mathcal{X}) = 7 \times 9 - 7 = 56 = |E(H)| - 1$. It can be seen that $H - e$ is M-independent for all $e \in E(H)$. Thus H is an M-circuit. Since $r_3(H) = 56 = 3|V(H)| - 7$, H is an example of a 4-connected non-rigid M-circuit.

We have recently shown in [14] that the 'Molecular Conjecture', due to Tay and Whiteley, see [23, 27, 28], would imply that Corollary 3.6 is tight for squares of graphs (and that it would also imply Conjecture 2.5). This family is important since it is used to model the rigidity properties of molecules. Biologists and physicists have developed heuristic algorithms for computing the 3-dimensional rank of the square of a graph, see for example [15]. If the Molecular Conjecture is true, then Corollary 3.6

could be used as part of a certificate of correctness for these algorithms.

The next example, again due to Tay [22], shows that Corollary 3.6 is not tight for all graphs.

Example 3 Let $G_0 = (V_0, E_0)$ be a complete graph on five vertices with $V_0 = \{v_i : 1 \le i \le 5\}$. For $1 \le i < j \le 5$ let $G_{i,j} = (V_{i,j}, E_{i,j})$ be a complete graph on five vertices with $V_{i,j} \cap V_0 = \{v_i, v_j\}$ and $E_{i,j} \cap E_0 = \{v_i v_j\}$ for $1 \le i < j \le 5$. Let

$$G = \left(G_0 \cup (\bigcup_{1 \le i < j \le 5} G_{i,j}) \right) - E_0.$$

We will see later that $r_3(G) \le |E(G)| - 1 = 89$. On the other hand, $\min_{\mathcal{X}} val(\mathcal{X})$ over all independent 2-thin covers $\mathcal{X}$ of G is 90. (Note that the rigid cluster cover of G, $\mathcal{C} = \{V_0\} \cup \{V_{i,j} : 1 \le i < j \le 5\}$ has value 89, but we cannot use it and Corollary 3.6 to deduce that $r_3(G) \le 89$ because $\mathcal{C}$ is not an independent cover.)

3.3 Iterated 2-thin covers

We may strengthen Corollary 3.6 by applying Lemma 3.5 iteratively.

An *iterated 2-thin cover* $\mathcal{T} = (T : G_0, G_1, \ldots, G_m)$ of a graph $G = (V, E)$ *of depth* m is a rooted tree T whose nodes are subsets of V, and a sequence of graphs $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_m$, with the following properties.

(i) The root of T is V and $G_0 = G$.

(ii) Each leaf of T is at distance m from the root.

(iii) For $1 \le i \le m$ the set of nodes of T at level i, $\mathcal{X}_i$, is a 2-thin cover of G_{i-1} and $G_i = G_{i-1} \cup H(\mathcal{X}_i)$.

(iv) For each node W at level i of T, $0 \le i \le m-1$, the set of children of W is an independent 2-thin cover $\mathcal{X}^W$ of $G_i[W]$.

Note that the graphs $G_0, G_1, \ldots, G_m$ are uniquely determined by the graph G and the labelled tree T.

Let $\mathcal{T} = (T : G_0, G_1, \ldots, G_m)$ be an iterated 2-thin cover of a graph $G = (V, E)$. For $1 \le j \le m$, let

$$\mathcal{F}_j = \{\mathcal{X}^W : W \in \mathcal{X}_{j-1}\}$$

be the family of covers corresponding to the sets of children of each node at level $j-1$ in T. For $\mathcal{X} \in \mathcal{F}_j$, let

$$\gamma(\mathcal{X}) = \sum_{(u,v) \in H(\mathcal{X})} (d(u, v) - 1).$$

Put

$$\gamma_j = \sum_{\mathcal{X} \in \mathcal{F}_j} \gamma(\mathcal{X}),$$

and

$$\gamma(\mathcal{T}) = \sum_{j=0}^{m} \gamma_j.$$

Let
$$val(\mathcal{T}) = \sum_{X \in \mathcal{X}_m} f(|X|) - \gamma(\mathcal{T}).$$

Lemma 3.7. *Let G be a graph. Then $r_3(G) \leq \min\{val(\mathcal{T})\}$ where the minimum is taken over all iterated 2-thin covers $\mathcal{T}$ of G.*

Proof: Let $\mathcal{T} = (T : G_0, G_1, \ldots, G_m)$ be an iterated 2-thin cover of G. We prove the lemma by induction on m. If $m = 0$ then $val(\mathcal{T}) = f(|V(G)|) \geq r_3(G)$ by Lemma 1.2. So suppose $m \geq 1$. Since $\mathcal{X}_1$ is an independent 2-thin cover of G_0, Lemma 3.5 implies that
$$r_3(G) \leq r_3(G_1) \leq \sum_{W \in \mathcal{X}_1} r_3(G_1[W]) - \gamma(\mathcal{X}_1). \tag{4}$$

For each $W \in \mathcal{X}_1$ let T^W be the subtree of T rooted at W, and let $\mathcal{X}_i^W$ be the set of nodes of T^W at level j, $0 \leq j \leq m-1$. Let $\mathcal{T}^W$ be the iterated 2-thin cover of $G_1[W]$ determined by T^W. Then $\mathcal{T}^W$ has depth $m-1$. By induction, we have
$$r_3(G_1[W]) \leq val(\mathcal{T}^W).$$

Furthermore $\gamma(\mathcal{T}) = \gamma(\mathcal{X}_1) + \sum_{W \in \mathcal{X}_1} \gamma(\mathcal{T}^W)$ and
$$\sum_{X \in \mathcal{X}_m} f(|X|) = \sum_{W \in \mathcal{X}_1} \sum_{Y \in \mathcal{X}_{m-1}^W} f(|Y|).$$

This implies that the lemma holds for G. •

We illustrate this lemma by considering the graph $G = (V, E)$ of Example 3. Let $\mathcal{T}$ be the iterated 2-thin cover of depth 2 defined as follows. Put $\mathcal{X}_0 = \{V\}$ and $G_0 = G$. Let $\mathcal{X}_1 = \{V_{i,j} \ : \ 1 \leq i < j \leq 5,\ (i,j) \neq (1,2)\} \cup \{W\}$ where $W = V_0 \cup V_{1,2}$ and $G_1 = G_0 \cup \{\{v_i v_j \ : \ 1 \leq i < j \leq 5,\ (i,j) \neq (1,2)\}$. Let $\mathcal{X}^W = \{V_0, V_{1,2}\}$ and $\mathcal{X}^{V_{i,j}} = \{V_{i,j}\}$ for the remaining nodes on level one. Then
$$\sum_{X \in \mathcal{X}_2} f(|X|) - \gamma(\mathcal{T}) = 99 - 10 = 89.$$

Hence $r_3(G) \leq 89$.

We know of no examples for which strict inequality holds in Lemma 3.7. This leads us to:

Conjecture 3.8. *Let G be a graph. Then*
$$r_3(G) = \min\{val(\mathcal{T})\}$$
where the minimum is taken over all iterated 2-thin covers $\mathcal{T}$ of G.

It can be seen that Conjecture 3.8 is equivalent to the following partial converse of Lemma 3.5.

Conjecture 3.9. *Let $G=(V,E)$ be a graph. Suppose G is not rigid in $\mathbb{R}^3$. Then there exists an independent 2-thin cover $\mathcal{X}$ of G such that $|\mathcal{X}|\geq 2$ and*

$$r_3(G)=\sum_{X_i\in\mathcal{X}} r_3(G_1[X_i])-\sum_{uv\in H(\mathcal{X})}(d(uv)-1),$$

where $G_1=G\cup H(\mathcal{X})$.

To see that Conjectures 3.8 and 3.9 are equivalent we proceed as follows.

(a) We first assume that Conjecture 3.8 is true. Let $\mathcal{T}=(G:G_1,G_2,\dots,G_m)$ be an iterated 2-thin cover of G such that $r_3(G)=val(\mathcal{T})$. Then equality must hold for $\mathcal{T}$ throughout the proof of Lemma 3.7. In particular, equality must hold in (4). Thus $\mathcal{X}_1$ is an independent 2-thin cover of G with

$$r_3(G)=\sum_{W\in\mathcal{X}_1} r_3(G_1[W])-\sum_{uv\in H(\mathcal{X}_1)}(d(uv)-1).$$

Hence Conjecture 3.9 holds for G.

(b) We next assume that Conjecture 3.9 is true. Suppose that Conjecture 3.8 is false and let $G=(V,E)$ be a counterexample with $|V|$ as small as possible. Let $\mathcal{X}_1$ be an independent 2-thin cover of G with

$$r_3(G)=\sum_{X_i\in\mathcal{X}_1} r_3(G_1[X_i])-\sum_{uv\in H(\mathcal{X}_1)}(d(uv)-1).$$

Since the sets in $\mathcal{X}_1$ are incomparable and $|\mathcal{X}_1|\geq 2$, we have $|X_i|<|V|$ for all $X_i\in\mathcal{X}_1$. Thus we may apply Conjecture 3.8 to $G_1[X_i]$ to obtain an iterated 2-thin cover $\mathcal{T}_i$ of $G_1[X_i]$ with $r_3(G_1[X_i])=val(\mathcal{T}_i)$ for each $X_i\in\mathcal{X}_1$. Let $\mathcal{T}$ be the iterated 2-thin cover of G with rooted tree T, which has V at its root, $\mathcal{X}_1$ at its first level, and in which the subtree of T rooted at X_i is the rooted tree T_i of $\mathcal{T}_i$ for each $X_i\in\mathcal{X}_1$. Then $r_3(G)=val(\mathcal{T})$. Hence Conjecture 3.8 holds for G.

We close this section by showing that Conjecture 3.8, or equivalently, Conjecture 3.9, would give a Co-NP characterization for the rank function of $\mathcal{R}_3(G)$.

Lemma 3.10. *Let G be a graph on $n\geq 3$ vertices and let $\mathcal{T}=(T:G_0,G_1,\dots,G_m)$ be an iterated 2-thin cover of G. Then:*
(a) the number of subsets at level i in T containing a fixed pair of vertices is at most $n-2$;
(b) there exists an iterated 2-thin cover $\mathcal{T}'$ of G of depth at most $n-2$ on at most $(n-2)(n-2)\binom{n}{2}$ nodes such that $val(\mathcal{T}')\leq val(\mathcal{T})$.

Proof: Part (a) follows from the fact that the subsets at any given level of T are a 2-thin cover of a graph on n vertices.

To prove (b), choose an iterated 2-thin cover $\mathcal{T}'=(T':G'_0,G'_1,\dots,G'_p)$ of G such that $val(\mathcal{T}')\leq val(\mathcal{T})$, p is as small as possible, and subject to this condition, the number of nodes of T' is as large as possible. Suppose there exists a node W_i on level

i in T' such that W_i has exactly one child W_{i+1} on level $i+1$ and then at least two children on level $i+2$. Construct T'' from T' by contracting the edge W_iW_{i+1} and adding a new leaf to each leaf in the subtree rooted at W_{i+1}. Let $\mathcal{T}''$ be the iterated 2-thin cover of G determined by T''. Then $val(\mathcal{T}'') = val(\mathcal{T}')$, $depth(\mathcal{T}'') = depth(\mathcal{T}')$, and $\mathcal{T}''$ has more nodes than $\mathcal{T}'$, a contradiction. If each node on level $m-1$ of T' has exactly one child then we may construct an iterated 2-thin cover of G, $\mathcal{T}''$, with $val(\mathcal{T}'') = val(\mathcal{T}')$ and $depth(\mathcal{T}'') = p-1$ by deleting the leaves of T'. Hence $\mathcal{X}'_{m-1}$ has a node W with at least two children. Each node on the path in T' from the root to W has at least two children. Hence each node is a proper subset of its parent. It follows that $p \leq n-2$. The upper bound on the number of nodes now follows from (a). •

Lemma 3.10(b) implies that Conjecture 3.8 would give a good characterization for $r_3(G)$. This follows from the polynomial upper bound on the size of the tree in the iterated 2-thin cover of G and the fact that the M-independence of a graph can be certified in polynomial time.

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