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Matroid matching with Dilworth truncation

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Abstract

Let $H = (V, E)$ be a hypergraph and let $k \geq 1$ and $l \geq 0$ be fixed integers. Let' $\mathcal{M}$ be the matroid with ground-set E s.t. a set $F \subseteq E$ is independent if and only if each $X \subseteq V$ with $k|X| - l \geq 0$ spans at most $k|X| - l$ hyperedges of F. We prove that if H is dense enough, then $\mathcal{M}$ satisfies the double circuit property, thus Lovász' min-max formula on the maximum matroid matching holds for $\mathcal{M}$. Our result implies the Berge-Tutte formula on the maximum matching of graphs ($k = 1$, $l = 0$), generalizes Lovász' graphic matroid (cycle matroid) matching formula to hypergraphs ($k = l = 1$) and gives a min-max formula for the maximum matroid matching in the 2-dimensional rigidity matroid ($k = 2$, $l = 3$).

1 Introduction

The theory of matroid matching is known to involve a range of combinatorial optimization problems concerning parity. One of its numerous equivalent definitions is as follows. Let $\mathcal{M}$ be a matroid with ground-set E, with rank-function $r_{\mathcal{M}}$, with span-function $\mathrm{sp}_{\mathcal{M}}$ and let $A \subseteq \binom{E}{2}$ be a set of (not necessarily disjoint) pairs of E. For short, if $F \subseteq E$ and $M \subseteq A$, then $r_{\mathcal{M}}(F \cup M)$ stands for $r_{\mathcal{M}}(F \cup \bigcup M)$ and $\mathrm{sp}(F \cup M)$ stands for $\mathrm{sp}(F \cup \bigcup M)$. A set of pairs $M \subseteq A$ is said to be a *matroid matching* of A w.r.t. $\mathcal{M}$ if $r_{\mathcal{M}}(M) = 2|M|$. The *matroid matching problem* is to compute a matroid matching of maximum size, the size of which is denoted by $\nu_{\mathcal{M}}(A)$.

Jensen and Korte [6] and Lovász [10] proved that the computation of $\nu_{\mathcal{M}}(A)$ needs exponential time if $\mathcal{M}$ is given by independence oracle. On the other hand, matroid matching relates the two important fields of combinatorial optimization involving submodularity and parity. This phenomenon shows its particular importance.

Starting from the matching problem of graphs and the matroid intersection problem, good characterization of the maximum size of a matroid matching [9] and also a polynomial algorithm [10] was obtained by Lovász for matroids represented over the

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field of reals. Important applications where the notion of matroid matching is less apparent are the maximum forest problem of 3-regular hypergraphs, the minimum number of vertices of a graph to pin to obtain a 2-dimensional rigid graph, and the maximum genus of graphs. By a construction of Schrijver [12], also Mader's maximum number of vertex-disjoint $\mathcal{A}$-paths can be computed by this algorithm. However, Lovász' min-max formula is given in a geometric language, it cannot be translated to a combinatorial one. This gap is partially filled by Lovász' structure theorem for 2-polymatroids [8] which enables us to derive combinatorial min-max formulae for some of the above problems.

Dress and Lovász [3] pointed out that the tractability of the above solvable cases is due to the *double circuit property* of the above matroids. A set $U \subseteq E$ is said to be a *double circuit* of $\mathcal{M}$ if $r_{\mathcal{M}}(U) = |U| - 2$ and $r_{\mathcal{M}}(U - \{e\}) = |U - \{e\}| - 1 = |U| - 2$ for every $e \in U$. A double circuit U has a rather simple structure, it has a partition $U_1 \dot\cup U_2 \dot\cup \dots \dot\cup U_d$ (called *principal partition*) s.t. $C_i = U - U_i$ $(i = 1, 2, \dots, d)$ are all of its circuits. The double circuit is said to be *non-trivial* if its principal partition has at least three classes. The crucial situation where most of the solution approaches to the matroid matching problem can get stuck is the existence of a non-trivial double circuit of a certain distinguished size. In this case, good characterization is known only for matroids having some special structural property. Lovász proved that in the case of full linear matroids (linear matroids with ground-set the full linear space), the modularity of lattice of flats (subspaces) is sufficient for this. However, there are cases where modularity does not hold but the maximum matching has a good characterization.

Dress and Lovász said that the matroid $\mathcal{M}$ has the *double circuit property* (*DCP* for short) if

$$r_{\mathcal{M}/Z}\left(\bigcap_{1 \le i \le d} \mathrm{sp}_{\mathcal{M}/Z}(C_i)\right) > 0 \tag{1}$$

holds for each non-trivial double circuit U of each contraction $\mathcal{M}/Z$ of $\mathcal{M}$ (using the above notations). The following result is implicit in Lovász [8] and explicit in Dress and Lovász [3].

Theorem 1.1 (Lovász [8], see Dress and Lovász [3]). *If $\mathcal{M}$ is a matriod having the DCP and $A \subseteq \binom{E}{2}$, then*

$$\nu_{\mathcal{M}}(A) = \min r_{\mathcal{M}}(Z) + \sum_{j=1}^{t} \left\lfloor \frac{r_{\mathcal{M}/Z}(A_j)}{2} \right\rfloor,$$

where the minimum is taken for all flats $Z \subseteq E$ of $\mathcal{M}$ and for all partitions $A_1, A_2, \dots, A_t$ of A.

The min-max formula has the same form as in Lovász [9] for linear matroids, but in the case of Dress and Lovász it is stated in a more general setting.

If there exists a matroid $\mathcal{M}'$ with ground-set $E' \supseteq E$ s.t. the restriction of $\mathcal{M}'$ to E is $\mathcal{M}$, then $\nu_{\mathcal{M}}(A) = \nu_{\mathcal{M}'}(A)$. Thus, the natural question which arises is to explore the class of matroids which have the DCP, and matroids which have extensions having

DCP. Dress and Lovász proved that full linear, full algebraic, full transversal, and full graphic matroids have the DCP, for definitions see [3]. In other words, every linear, algebraic, transversal, and graphic matroid is a restriction of a matroid which has the DCP.

Based on lattice theoretic concepts, Björner and Lovász [2] introduced the class of pseudomodular matroids and they have shown that the above mentioned matroids are pseudomodular. Later, Hochstättler and Kern [4] proved that pseudomodular matroids have the DCP.

How can we exploit that $\mathcal{M}$ is a restriction of a member of one of the above classes? Most of the matroids and polymatroids which we meet in applications are linear, as these are constructed from free matroids by the operations of sum, series and parallel extensions, principal extensions, truncations, Dilworth truncations, restrictions, contractions and dualizations, and these operations keep representability over the field of reals. Some of the above operations, (e.g. Dilworth truncation) are polymatroid operations, however, this does not cause problem due to the well-known correspondance of matroids and polymatroids proved by Pym and Perfect [11]. However, if only linearity is known, then the matroid is extended to the full linear space and combinatorial min-max formula cannot be expected. Moreover, some of the matroids defined by Dilworth truncation are not known to be deterministically representable, which is a requirement for computational results.

Björner and Lovász [2] prove that the class of pseudomodular matroids is closed under most of the above operations, however, they put the question of finding pseudomodular lattices whose Dilworth truncations are pseudomodular. Hence, if the matroid is defined by Dilworth truncation, it can be difficult to extend it to a pseudomodular matroid.

Thus it remains a great challenge to explore combinatorially suggested tractable classes which give a more unified view of the solvable cases. Our main goal is to take a step in the way of better comprehension of the matching problem of matroids defined by Dilworth truncation. This is carried out by considering the matroid matching in the following class of purely combinatorially defined matroids. This may be a class where Dilworth truncation arises in the most simple way, but even this gives a more unified view of some solved cases and also contains previously unsolved problems.

Let $k \geq 1$ and $l \geq 0$ be fixed integers and let $H = (V, E)$ be a finite hypergraph. Let us define $b : 2^V \to \mathbb{Z}$ by $b(X) = k|X| - l$ if $k|X| - l \geq 0$ and 0 otherwise. For $X \subseteq V$ and $F \subseteq E$ let $F[X] = \{e \in F : e \subseteq X\}$. Finally, let $\mathcal{M}$ be the matroid (*called* (k,l)*-matroid*) with ground-set E s.t. $F \subseteq E$ is independent in $\mathcal{M}$ if and only if $|F[X]| \leq b(X)$ for each $X \subseteq V$. We may suppose that each hyperedge has size bigger than $\frac{l}{k}$ since the smaller hyperedges are loops. The hyperedges of size two will be called simply *edges*.

The particular interest of the matroid matching of this matroid is due to the following more special problem. Let $H' = (V, E')$ be a hypergraph, and we ask for the largest set $F' \subseteq E'$ s.t. $|F'[X]| \leq \frac{k|X|-l}{2}$ for every $X \subseteq V$, $k|X| - l \geq 0$. Notice that, if H' is a graph, $k = 1$, $l = 0$, then this is exactly the matching problem of the graph H'. For graphs and $k = l = 2$, we get back the maximum forest problem. If H' is a 3-regular hypergraph, $k = l = 1$ and $F' \subseteq E'$, then $|F'[X]| \leq \frac{|X|-1}{2}$ holds for every $\emptyset \neq X \subseteq V$ if and only if the components of F' are triangle cacti (forests of 3-regular hypergraphs). Hence, in 3-regular hypergraphs and $k = l = 1$ we have the maximum forest problem. Jackson and Jordán [5] proved that if H' is a graph and $k = 5$, $l = 7$, then the arising edge-sets are independent in the 3-dimensional rigidity matroid. The importance of this result is that the rank-function of the 3-dimensional rigidity matroid is not known. Later we will give more examples.

The (k, l)-matroids can be shown to be linear, but for getting computational results from the application of Lovász' linear matroid matching theorem, the matroid has to be represented. However, if H contains only edges, $k = 2$ and $l = 3$, then $\mathcal{M}$ is the 2-dimensional rigidity matroid (Laman, [7]) which is not known to be deterministically representable. Moreover, if we are looking for a combinatorial min-max relation, then we cannot recline upon the linearity.

Thus, the matroid will be extended to a relatively small, combinatorially defined matroid which has the DCP. This extension is obtained by adding further hyperedges to H. As this operation does not affect $\nu_{\mathcal{M}}(A)$, we may assume for simplicity that the new hyperedges are already in H. We have to note that the new hyperedges have no individual importance. In a matroid which have the DCP, the flats have a very special structure. The main goal of adding new hyperedges is to reach this desired structure. For this, we require that

$$r_{\mathcal{M}}(E[X]) = b(X) \quad \text{holds for every} \quad X \subseteq V. \tag{2}$$

Our main theorems are the following.

Theorem 1.2. *If $\mathcal{M}$ is a (k, l)-matroid and $r_{\mathcal{M}}(E[X]) = b(X)$ holds for every $X \subseteq V$, then $\mathcal{M}$ has the DCP.*

Theorem 1.3. *Let $\mathcal{M}$ be a (k, l)-matroid s.t. $r_{\mathcal{M}}(E[X]) = b(X)$ holds for every $X \subseteq V$ and $A \subseteq \binom{E}{2}$. Then for each contraction $\mathcal{N}$ of $\mathcal{M}$,*

$$\nu_{\mathcal{N}}(A) = \min r_{\mathcal{N}}(Z) + \sum_{j=1}^{t} \left\lfloor \frac{r_{\mathcal{N}/Z}(A_j)}{2} \right\rfloor, \tag{3}$$

where the minimum is taken for all flats $Z \subseteq E$ of $\mathcal{N}$ and for all partitions $A_1, A_2, \ldots, A_t$ of A.

Theorem 1.3 immediately follows from Theorem 1.2 and Theorem 1.1. We give a short study of some criteria which imply (2). First, it is easy to see that if each set $X \subseteq V$ of size bigger than $\frac{l}{k}$ is in E with multiplicity $k|X| - l$, then (2) holds. A weaker condition also assures (2), as it is described in the following theorem.

Theorem 1.4. *Let $l = ck + d$ where c, d are integers, $0 \leq c$ and $0 \leq d < k$. Suppose that E contains all the subsets of V of size $c + 1$ with multiplicity $k - d$. Suppose moreover that if $\frac{ck}{c+1} < d$, then E contains all the subsets of V of size $c + 2$ with multiplicity $cd + d - ck$. Then (2) holds.*

Although the conditions of Theorem 1.4 seem to be artificial, they translate to clear requirements in the case of each particular matroid of our class.

The rest of the paper is organized as follows. After detailing the technical preliminaries on the rank-function of $\mathcal{M}$, we discuss some special cases of Theorem 1.3. This is followed by the proofs.

2 Preliminaries

It is not difficult to prove that the above defined (k, l)-matroids are matroids. The correctness of the following claims can be seen immediately for the reader who is familiar with matroid theory and Dilworth truncation. We sketch the proof of the equality

$$r_{\mathcal{M}}(F) = \min\left\{|Y| + \sum_{X\in\mathcal{X}} b(X) : Y \subseteq F,\ \mathcal{X} \subseteq \binom{V}{> \frac{l}{k}},\ F \subseteq Y \cup \bigcup_{X\in\mathcal{X}} E[X]\right\}. \quad (4)$$

$\mathcal{X}_1 \subseteq \binom{V}{>\frac{l}{k}}$ is said to be a refinement of $\mathcal{X}_2 \subseteq \binom{V}{>\frac{l}{k}}$ if for each $X_1 \in \mathcal{X}_1$ there exists $X_2 \in \mathcal{X}_2$ s.t. $X_1 \subseteq X_2$.

Claim 2.1. *(i) The right hand side of (4) is a matroid rank-function.*

(ii) If $F \subseteq E$, then there exists a unique pair $(\mathcal{X}_F, Y_F)$, $\mathcal{X}_F \subseteq \binom{V}{>\frac{l}{k}}$, $Y_F \subseteq F$, s.t. $r_{\mathcal{M}}(F) = |Y_F| + \sum_{X\in\mathcal{X}_F} b(X)$, $F \subseteq Y_F \cup \bigcup_{X\in\mathcal{X}_F} E[X]$ and for every $(\mathcal{X}, Y)$ with the same properties, $\mathcal{X}$ is a refinement of $\mathcal{X}_F$ (and $Y_F \subseteq Y$).

(iii) If $F_1 \subseteq F_2 \subseteq E$, then $\mathcal{X}_{F_1}$ refines $\mathcal{X}_{F_2}$.

Proof. For *(i)*, the right hand side of (4) is monotone increasing and singletons get value at most one. Thus the only non-trivial thing is to prove that the right hand side of (4) is submodular. Let F_1 and F_2 be subsets of E and let resp. $(\mathcal{X}_1, Y_1)$ and $(\mathcal{X}_2, Y_2)$ give the corresponding minimum in (4). In what follows, the word *collection* stands for a family where multiplicities are counted. Algebraic operations with collections are defined by the corresponding operations with the multiplicity functions. Starting from $\mathcal{G}_0 = \mathcal{X}_1 + \mathcal{X}_2$, we apply a simple uncrossing procedure and a sequence of collections $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_l$ is computed. If for some $i \geq 0$, $\mathcal{G}_i$ has already been defined, $X_1, X_2 \in \mathcal{G}_i$, $|X_1 \cap X_2| \geq \frac{l}{k}$, $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$, then let us define $\mathcal{G}_{i+1}$ by $\mathcal{G}_i - \{X_1\} - \{X_2\} + \{X_1 \cap X_2\} + \{X_1 \cup X_2\}$. Clearly, $\sum_{X\in\mathcal{G}_i} \chi_{E[X]} \leq \sum_{X\in\mathcal{G}_{i+1}} \chi_{E[X]}$ and $\sum_{X\in\mathcal{G}_i} b(X) = \sum_{X\in\mathcal{G}_{i+1}} b(X)$. Next, this procedure is finite, since $\sum_{X\in\mathcal{G}_i} |X|^2 < \sum_{X\in\mathcal{G}_{i+1}} |X|^2$ and $\sum_{X\in\mathcal{G}_i} |X|^2 \leq |\mathcal{G}_i||V|^2 = |\mathcal{G}_0||V|^2$. When the uncrossing finishes in $\mathcal{G}_l$, $\{E[X] : X \in \mathcal{G}_l\}$ is a laminar family. Let $\mathcal{G}_{\max}$ contain one from each of the maximal members of $\mathcal{G}_l$. Then $F_1 \cup F_2 \subseteq (Y_1 \cap Y_2) \cup (Y_1 - F_2) \cup (Y_2 - F_1) \cup \bigcup_{X\in\mathcal{G}_{\max}} E[X]$ and $F_1 \cap F_2 \subseteq ((Y_1 \cup Y_2) \cap F_1 \cap F_2) \cup \bigcup_{X\in\mathcal{G}_l - \mathcal{G}_{\max}} E[X]$, thus $r_{\mathcal{M}}(F_1) + r_{\mathcal{M}}(F_2) = |Y_1| + |Y_2| + \sum_{X\in\mathcal{G}_0} b(X) = |(Y_1 \cap Y_2) \cup (Y_1 - F_2) \cup (Y_2 - F_1)| + \sum_{X\in\mathcal{G}_{\max}} b(X) + |(Y_1 \cup Y_2) \cap F_1 \cap F_2| + \sum_{X\in\mathcal{G}_l - \mathcal{G}_{\max}} b(X) \geq r_{\mathcal{M}}(F_1 \cup F_2) + r_{\mathcal{M}}(F_1 \cap F_2)$, which completes the proof.

Next, we prove *(ii)*. Let $(\mathcal{X}_1, Y_1)$ and $(\mathcal{X}_2, Y_2)$ be two different pairs which give the minimum in the right hand side of (4). As there is a finite number of such pairs, it is sufficient to construct a pair $(\mathcal{X}, Y)$ with the same properties s.t. $\mathcal{X}_1$ and $\mathcal{X}_2$ refine $\mathcal{X}$. Let us apply the uncrossing procedure presented in the proof of part *(i)*. Then it constructs families $\mathcal{G}_{\max}$ for $F \cup F$ and $\mathcal{G}_l - \mathcal{G}_{\max}$ for $F \cap F$ s.t. $\mathcal{X}_1$ and $\mathcal{X}_2$ are refinements of $\mathcal{G}_{\max}$. It can be seen easily that $F \subseteq (Y_1 \cap Y_2) \cup \bigcup_{X \in \mathcal{G}_{\max}} E[X]$ and $E \subseteq (Y_1 \cup Y_2) \cup \bigcup_{X \in \mathcal{G}_l - \mathcal{G}_{\max}} E[X]$, thus $2r_{\mathcal{M}}(F) = |Y_1| + |Y_2| + \sum_{X \in \mathcal{G}_0} b(X) = |Y_1 \cap Y_2| + \sum_{X \in \mathcal{G}_{\max}} b(X) + |Y_1 \cup Y_2| + \sum_{X \in \mathcal{G}_l - \mathcal{G}_{\max}} b(X) \geq 2r_{\mathcal{M}}(F)$. Hence $r_{\mathcal{M}}(F) = |Y_1 \cap Y_2| + \sum_{X \in \mathcal{G}_{\max}} b(X)$, $\mathcal{X}_1$ and $\mathcal{X}_2$ are refinements of $\mathcal{G}_{\max}$ and $F \subseteq (Y_1 \cap Y_2) \cup \bigcup_{X \in \mathcal{G}_{\max}} E[X]$. Therefore both $\mathcal{X}_1$ and $\mathcal{X}_2$ refine $\mathcal{X} = \mathcal{G}_{\max}$.

Last we prove *(iii)*. If we apply the above uncrossing procedure for $\mathcal{X}_{F_1}$ and $\mathcal{X}_{F_2}$, then it produces families $\mathcal{G}_{\max}$ for $F_1 \cup F_2 = F_2$ and $\mathcal{G}_l - \mathcal{G}_{\max}$ for $F_1 \cap F_2 = F_1$ s.t. $\mathcal{X}_{F_1}$ and $\mathcal{X}_{F_2}$ are refinements of $\mathcal{G}_{\max}$. We can see that $F_2 \subseteq (Y_1 \cap Y_2) \cup (Y_2 - F_1) \cup \bigcup_{X \in \mathcal{G}_{\max}} E[X]$ and $F_1 \subseteq ((Y_1 \cup Y_2) \cap F_1) \cup \bigcup_{X \in \mathcal{G}_l - \mathcal{G}_{\max}} E[X]$, thus $r_{\mathcal{M}}(F_1) + r_{\mathcal{M}}(F_2) = |Y_1| + |Y_2| + \sum_{X \in \mathcal{G}_0} b(X) = |(Y_1 \cap Y_2) \cup (Y_2 - F_1)| + \sum_{X \in \mathcal{G}_{\max}} b(X) + |(Y_1 \cup Y_2) \cap F_1| + \sum_{X \in \mathcal{G}_l - \mathcal{G}_{\max}} b(X) \geq r_{\mathcal{M}}(F_2) + r_{\mathcal{M}}(F_1)$. *(ii)* implies $\mathcal{X}_{F_2} = \mathcal{G}_{\max}$, which completes the proof. □

Claim 2.2. *If $F \subseteq E$, then $\{Y_F\} \cup \{E[X] : X \in \mathcal{X}_F\}$ forms a subpartition of E.*

Proof. If $Y_F \cap E[X] \neq \emptyset$ for some $X \in \mathcal{X}_F$, then Y_F could be replaced by $Y_F - E[X]$. If $|X_1 \cap X_2| \geq \frac{l}{k}$ for some $X_1, X_2 \in \mathcal{X}_F$, then we could replace $\mathcal{X}_F$ by $\mathcal{X}_F - \{X_1\} - \{X_2\} + \{X_1 \cup X_2\}$. □

It can be proved easily that the matroid defined by the right hand side of (4) is identical with our (k, l)-matroid. In the sense of these Claims, (2) can be reformulated s.t. if $X \subseteq V$ and $b(X) > 0$, then $\mathcal{X}_{E[X]} = \{X\}$.

For some applications, let $\varrho_{\mathcal{M}}(A) = \min\{|B| : B \subseteq A, r_{\mathcal{M}}(B) = r_{\mathcal{M}}(A)\}$. For any matroid $\mathcal{M}$, the computation of $\varrho_{\mathcal{M}}(A)$ is equivalent to computing $\nu_{\mathcal{M}}(A)$. This is formalized more specifically as follows.

Theorem 2.3 (Lovász, [10]). *For any matroid $\mathcal{M}$ with ground-set E and $A \subseteq \binom{E}{2}$,*

$$\nu_{\mathcal{M}}(A) + \varrho_{\mathcal{M}}(A) = r_{\mathcal{M}}(A).$$

3 Special cases

3.1 Berge-Tutte formula

First, we show that if $k = 1$ and $l = 0$, then Theorem 1.3 implies the Berge-Tutte formula [1] as it was stated in Lovász [8]. In the Introduction we sketched a construction which contains the matching problem of graphs. The following, a bit different construction gives an easier way to derive the Berge-Tutte formula.

Theorem 3.1 (Berge and Tutte, [1]). *Let $G = (V, E')$ be an undirected graph. Then the maximum matching of G has cardinality*

$$\min |X| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|C|}{2} \right\rfloor,$$

where $\mathcal{C}$ denotes the set of vertex-sets of the components of $G[V - X]$.

Proof. Let $k = 1$, $l = 0$ and let E be the set of singletons of V. It can be seen immediately that (2) holds. The set of pairs for the matroid matching is $A = \{\{\{u\}, \{v\}\} : uv \in E'\}$. Observe that M' is a matching of G if and only if $M = \{\{\{u\}, \{v\}\} : uv \in M'\}$ is a matroid matching of A w.r.t. $\mathcal{M}$.

Let $Z \subseteq E$ and $A_1, A_2, \ldots, A_t$ give equality in the min-max relation stated in Theorem 1.3 so that $\mathrm{sp}_{\mathcal{M}}(Z)$ is minimal, and subject to this, t is as small as possible. Clearly, $Y_F = \emptyset$ and $|\mathcal{X}_F| \leq 1$ for any $F \subseteq E$. Let us define X_F by $\mathcal{X}_F = \{X_F\}$ if $\mathcal{X}_F \neq \emptyset$ and let $X_F = \emptyset$ otherwise. Let $X = X_Z$. If $X = \emptyset$, then $\nu_{\mathcal{M}}(A) = \sum_{j=1}^{t} \left\lfloor \frac{1}{2} |X_{\bigcup A_j}| \right\rfloor \geq \sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2}|C| \right\rfloor$. If $X \neq \emptyset$, then $X_{Z \cup \bigcup A_j} \cap X_{Z \cup \bigcup A_{j'}} = X$ for every $1 \leq j < j' \leq t$. Then, $\nu_{\mathcal{M}}(A) = |X| + \sum_{j=1}^{t} \left\lfloor \frac{1}{2}|X_{Z \cup \bigcup A_j} - X_Z| \right\rfloor \geq |X| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2}|C| \right\rfloor$. This is exactly what we have needed. □

3.2 Transversal matroids

One of the usual interpretations of transversal matroids is that we have a hypergraph $H = (V, E)$ and $F \subseteq E$ is independent if and only if $|F[X]| \leq |X|$ holds for every $X \subseteq V$. Similarly to the case of Berge-Tutte formula, if each singleton hyperedge is in E with multiplicity one, then (2) holds.

We also have to note that the transversal matroid matching can be solved in an easier way. Tong, Lawler and Vazirani [14] showed that even the weighted case can be reduced to the weighted matching problem of graphs.

3.3 Hypergraphic matroid and rigidity matroid

We have mentioned in the Introduction that the maximum forest problem of 3-regular hypergraphs is also a special case. As in the case of the Berge-Tutte formula, this problem fits into our set-up in two different ways. We switch to that one which contains also the more general graphic matroid matching problem.

Let $1 \leq k \leq l \leq 2k - 1$. According to this choice, $c = 1$ and $d = l - k$. To satisfy the requirements of Theorem 1.4, E has to contain $2k - l$ parallel edges on each pair of vertices. If $d > \frac{ck}{c+1} = \frac{k}{2}$ or equivalently $l = k + d > \frac{3k}{2}$, then we have to put $cd + d - ck = 2d - k = 2l - 3k$ parallel hyperedges of size three to each triple of vertices. Next we consider the applications of this case.

If $k = l = 1$, then $\mathcal{M}$ is the hypergraphic matroid with ground-set E. As $l \leq \frac{3k}{2}$, then (2) holds if $\binom{V}{2} \subseteq E$. If E contains only edges, then Theorem 1.3 specializes to Lovász' theorem on the maximum graphic matroid matching [8].

Theorem 3.2 (Lovász, [8]). *Let $A \subseteq \binom{\binom{V}{2}}{2}$ and let $\mathcal{M}$ be the cycle matroid of the graph $\left(V, \binom{V}{2}\right)$. Then*

$$\nu_{\mathcal{M}}(A) = \min |V| - |\mathcal{P}| + \sum_{j=1}^{t} \left\lfloor \frac{r_{\mathcal{N}}(A_j)}{2} \right\rfloor ,$$

where the minimum is taken for all partitions $\mathcal{P} = \{P_1, P_2, \ldots, P_q\}$ *of* V *and for all partitions* $A_1, A_2, \ldots, A_t$ *of* A *and* $\mathcal{N}$ *is the cycle matroid of the graph obtained from* $(V, \bigcup A_j)$ *by contracting the members of* $\mathcal{P}$*.*

If $\bigcup A$ contains also hyperedges of size bigger than two, then Theorem 1.3 cannot be rewritten in such a special form. In this case, the contraction cannot be described by a partition of V. To see this, let $e_0, e_1, \ldots, e_m$, $m \geq 3$ be pairwise vertex-disjoint hyperedges of size three and $A = \{\{e_0, e_i\} : 1 \leq i \leq m\}$. Then the only possibility of obtaining equality in the min-max formula is $Z = \{e_0\}$ and $A_j = \{\{e_0, e_j\}\}$, $1 \leq j \leq t = m$.

If $k = 2$ and $l = 3$, then $F \subseteq E$ is independent in $\mathcal{M}$ if and only if $|F[X]| \leq 2|X| - 3$ for every $X \in \binom{V}{\geq 2}$. Notice that, if $\bigcup A$ contains only edges, then this is the "smallest" case when Theorem 1.3 gives a new result. Just as above, (2) is satisfied if $\binom{V}{2} \subseteq E$. If E contains only edges, then it is known that the bases of $\mathcal{M}$ are exactly the 2-dimensional minimally rigid graphs on V (Laman, [7]). Let $G = (V, E')$ be a 2-dimensional rigid graph and let A be a set of (not necessarily disjoint) pairs from E'. Then the maximum number of edge-pairs from A which are contained in a minimally rigid subgraph of G is $\nu_{\mathcal{M}}(A)$, which can be computed by Theorem 1.3. If $G = (V, E')$ is not a rigid graph but $(V, E' \cup \bigcup A)$ is rigid, where $A \subseteq \binom{\binom{V}{2}}{2}$, then $\varrho_{\mathcal{M}/E'}(A)$ is the minimum cardinality of a set $B \subseteq A$ s.t. $(V, E' \cup \bigcup B)$ is rigid. This problem can be solved by Theorem 1.3 and Theorem 2.3.

For larger k and l, (2) does not follow from $\binom{V}{2} \subseteq E$. Say, if $k = 3$ and $l = 5$, then a hyperedge of size three has to be put to each triple of vertices.

3.4 A connectivity augmentation problem

The problems discussed here were proposed by Zsolt Fekete. Let $G = (V, E')$ be an undirected graph, let $1 \leq k \leq l \leq 2k - 1$ and $\mathcal{M}$ be as above. Let moreover an other edge-set E'' on V and a set of packets $A \subseteq 2^{E''}$ be given. We ask for the minimum cardinality set $B \subseteq A$ s.t. the rank of $(V, E' \cup \bigcup B)$ is $k|V| - l$ in $\mathcal{M}$. Clearly, if A is composed by singletons, then this is a minimum cardinality spanning subset problem in a matroid.

Frank observed (personal communication) that if each packet is composed by p parallel edges, $p = k = l$ and k is part of the input, then then the problem is NP-hard. The graph on 2 vertices obtained from G after contracting $|V| - 2$ pairs of vertices consecutively contains k edge-disjoint spanning trees if and only if G has a cut of size at least k. Hence, this reduces the maximum cut problem to our problem.

If $p = 2$ and $k \geq 1$, $l \geq 0$ are arbitrary integers, then we just have to compute $\varrho_{\mathcal{M}/E'}(A)$. This contains the problem of adding a minimum number of capacity 2 edges to G (from a prescribed set) so that the resulting graph has k edge disjoint spanning trees ($k = l$). By Theorem 1.3 and Theorem 2.3, a combinatorial characterization is achieved.

4 Structure of double circuits

The key phenomenon in the background of Theorem 1.2 is the modular structure of double circuits. Let $Z \subseteq E$ be a flat of the (k,l)-matroid $\mathcal{M}$, and let U be a non-trivial double circuit of $\mathcal{M}/Z$ with principal partition $U = U_1 \dot\cup U_2 \dot\cup \ldots \dot\cup U_d$. For the positive integer n, let $[n] = \{1, 2, \ldots, n\}$, and for $T \subseteq [d]$, let $C(T)$ denote $\bigcap_{t \in T} \mathrm{sp}_{\mathcal{M}/Z}(C_t)$ (where $C(\emptyset)$ is defined to be $\mathrm{sp}_{\mathcal{M}/Z}(U)$).

Theorem 4.1. *If (2) holds, U is a non-trivial double circuit of $\mathcal{M}/Z$ with the above notations, and $T \subseteq [d]$, then*

$$\mathrm{sp}_{\mathcal{M}/Z}\left(C(T - \{i\}) \cup C(T - \{j\})\right) = C(T - \{i, j\}), \tag{5}$$

where $i, j \in T$, $i \neq j$, and

$$r_{\mathcal{M}/Z}(C(T)) = |U| - \sum_{t \in T} |U_t| + |T| - 2. \tag{6}$$

Proof. First, we have to take some observations on the structure of circuits of $\mathcal{M}/Z$.

Claim 4.2. *If C is a circuit of $\mathcal{M}/Z$, then $Y_{C \cup Z} \subseteq Y_Z$ and $|\mathcal{X}_{C \cup Z} - \mathcal{X}_Z| = 1$.*

Proof. Since $\mathcal{X}_Z$ is a refinement of $\mathcal{X}_{C \cup Z}$, then $\bigcup_{X \in \mathcal{X}_{C \cup Z}} E[X] \supseteq \bigcup_{X \in \mathcal{X}_Z} E[X]$ and $Y_{C \cup Z} \cap \bigcup_{X \in \mathcal{X}_{C \cup Z}} E[X] = \emptyset$ imply $Y_{C \cup Z} \cap \bigcup_{X \in \mathcal{X}_Z} E[X] = \emptyset$. If $Y_{C \cup Z} \not\subseteq Y_Z$, then let $e \in Y_{C \cup Z} - Y_Z$. In this case, $r_{\mathcal{M}/Z}(C - e) = r_{\mathcal{M}/Z}(C) - 1$, contradicting that C is a circuit. Hence,

$$|C| = |Y_{C \cup Z}| + \sum_{X \in \mathcal{X}_{C \cup Z}} b(X) - |Y_Z| - \sum_{X \in \mathcal{X}_Z} b(X) + 1 =$$
$$\sum_{X \in \mathcal{X}_{C \cup Z}} \left(b(X) - |Y_Z[X]| - \sum_{W \in \mathcal{X}_Z[X]} b(W) \right) + 1.$$

Thus, $|C[X]| \geq b(X) - |Y_Z[X]| - \sum_{W \in \mathcal{X}_Z[X]} b(W) + 1$ for some $X \in \mathcal{X}_{C \cup Z}$. If $|C[X]| \geq b(X) - |Y_Z[X]| - \sum_{W \in \mathcal{X}_Z[X]} b(W) + 2$ for some $X \in \mathcal{X}_{C \cup Z}$, then a hyperedge could be removed from $C[X]$ without making C independent, contradicting that C is a circuit. If there would be different $X_1, X_2 \in \mathcal{X}_{C \cup Z}$ with $|C[X_i]| = b(X_i) - |Y_Z[X_i]| - \sum_{W \in \mathcal{X}_Z[X_i]} b(W) + 1$, then $C[X_1]$ could be removed from C without making C independent, contradicting that C is a circuit, and finishing the proof. □

Claim 4.3. *If $\emptyset \neq T \subseteq [d]$, then $Y_{C(T)} \subseteq Y_Z$ and $|\mathcal{X}_{C(T)} - \mathcal{X}_Z| \leq 1$. If, moreover, $r_{\mathcal{M}/Z}(C(T)) > 0$, then $|\mathcal{X}_{C(T)} - \mathcal{X}_Z| = 1$.*

Proof. The statement is proved by induction on $|T|$. If $|T| = 1$, then we are done by Claim 4.2. Next, we suppose $|T| \geq 2$, and let $i \in T$.

By induction we have $\mathcal{X}_{C(\{i\})} - \mathcal{X}_Z = \{X_i\}$. Similarly, either $\mathcal{X}_{C(T - \{i\})} - \mathcal{X}_Z = \emptyset$ or $\mathcal{X}_{C(T - \{i\})} - \mathcal{X}_Z = \{X_{T - \{i\}}\}$. In the first case, $\mathcal{X}_{C(T)} = \mathcal{X}_Z$. In the second case, $\mathcal{X}_{C(T)} - \mathcal{X}_Z = \{X_i \cap X_{T - \{i\}}\}$ if $b(X_i \cap X_{T - \{i\}}) > 0$ and $\mathcal{X}_{C(T)} - \mathcal{X}_Z = \emptyset$ otherwise.

$Y_{C(T)} \subseteq Y_Z$ can be seen easily, as for each hyperedge $e \in C(\{i\}) \cap C(T-\{i\})$, either $e \in E[X_i] \cap E[X_{T-\{i\}}]$, thus $e \notin Y_{C(T)}$, or $e \in Y_{C(\{i\})} \cup Y_{C(T-\{i\})} \subseteq Y_Z$. Last, if $\mathcal{X}_{C(T)} - \mathcal{X}_Z = \emptyset$, then $r_{\mathcal{M}/Z}(C(T)) = 0$, which proves the last statement. □

The following claims have crucial role in proving the lower bound for $r_{\mathcal{M}/Z}(C(T))$.

Claim 4.4. *If $X_1, X_2, X_3 \in \binom{V}{>\frac{l}{k}}$ are s.t. $b(X_i \cap X_j) > 0$ for every $1 \le i < j \le 3$, then*

$$\sum_{1\le i<j\le 3} b(X_i \cap X_j) + b(X_1 \cup X_2 \cup X_3) \le \sum_{i=1}^{3} b(X_i) + b(X_1 \cap X_2 \cap X_3).$$

Proof. If $k|\bigcap_{1\le i\le 3} X_i| - l \ge 0$, then the inequality holds with equality. If $k|\bigcap_{1\le i\le 3} X_i| - l < 0$, then the right hand side is greater by $l - k|\bigcap_{1\le i\le 3} X_i|$. □

Claim 4.5. *Let F_1, F_2 and F_3 be flats of $\mathcal{M}/Z$ s.t. $|\mathcal{X}_{F_i\cup Z} - \mathcal{X}_Z| = 1$ and $Y_{F_i \cup Z} \subseteq Y_Z$ for every $1 \le i \le 3$. Suppose, moreover, that $r_{\mathcal{M}/Z}(F_i \cap F_j) > 0$ for every $1 \le i < j \le 3$. Then*

$$\sum_{1\le i<j\le 3} r_{\mathcal{M}/Z}(F_i \cap F_j) + r_{\mathcal{M}/Z}(F_1 \cup F_2 \cup F_3) \le \sum_{i=1}^{3} r_{\mathcal{M}/Z}(F_i) + r_{\mathcal{M}/Z}(F_1 \cap F_2 \cap F_3).$$

Proof. According to the hypothesis, let $\{X_i\} = \mathcal{X}_{F_i\cup Z} - \mathcal{X}_Z$, $\mathcal{Z}_i = \mathcal{X}_Z - \mathcal{X}_{F_i\cup Z}$, and $Y_i = Y_Z - Y_{F_i\cup Z}$. Then

$$r_{\mathcal{M}/Z}(F_i) = |Y_{F_i\cup Z}| + \sum_{X\in\mathcal{X}_{F_i\cup Z}} b(X) - |Y_Z| - \sum_{X\in\mathcal{X}_Z} b(X) = \\ -|Y_i| + b(X_i) - \sum_{X\in\mathcal{Z}_i} b(X).$$

If $W \in \mathcal{X}_Z$ and $1 \le i \le 3$, then either $W \subseteq X_i$ or $|W \cap X_i| < \frac{l}{k}$. Therefore, if $W \in \mathcal{X}_Z$, $X \in \{X_1 \cap X_2, X_1 \cap X_3, X_2 \cap X_3, X_1 \cap X_2 \cap X_3\}$, then $W \subseteq X$ or $|W \cap X| < \frac{l}{k}$ also holds.

By Claim 2.1, $\mathcal{X}_Z$ refines $\mathcal{X}_{(F_i\cap F_j)\cup Z}$ which refines both $\mathcal{X}_{F_i\cup Z}$ and $\mathcal{X}_{F_j\cup Z}$. If $b(X_i \cap X_j) = 0$ for some $1 \le i < j \le 3$, then $\mathcal{X}_{(F_i\cap F_j)\cup Z}$ would refine $\mathcal{X}_Z$, contradicting $r_{\mathcal{M}/Z}(F_i \cap F_j) > 0$. Thus $b(X_i \cap X_j) > 0$. By (2), $\mathcal{X}_{E[X_i\cap X_j]} = \{X_i \cap X_j\}$ which refines $\mathcal{X}_{(F_i\cap F_j)\cup Z}$. This together implies

$$\mathcal{X}_{(F_i\cap F_j)\cup Z} = \{X_i \cap X_j\} \cup (\mathcal{X}_Z - (\mathcal{Z}_i \cap \mathcal{Z}_j)),$$

$$r_{\mathcal{M}}((F_i \cap F_j) \cup Z) = b(X_i \cap X_j) + \sum_{X\in\mathcal{X}_Z-(\mathcal{Z}_i\cap\mathcal{Z}_j)} b(X) + |Y_Z - (Y_i \cap Y_j)|,$$

and

$$r_{\mathcal{M}/Z}(F_i \cap F_j) = -|Y_i \cap Y_j| + b(X_i \cap X_j) - \sum_{X\in\mathcal{Z}_i\cap\mathcal{Z}_j} b(X).$$

Similar argument shows that if $b(X_1 \cap X_2 \cap X_3) > 0$, then

$$\mathcal{X}_{(F_1\cap F_2\cap F_3)\cup Z} = \{X_1 \cap X_2 \cap X_3\} \cup (\mathcal{X}_Z - (\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_3)).$$

If $b(X_1 \cap X_2 \cap X_3) = 0$, then $\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_3 = \emptyset$ and

$$\mathcal{X}_{(F_1\cap F_2\cap F_3)\cup Z} = \mathcal{X}_Z.$$

In both cases

$$r_{\mathcal{M}/Z}(F_1 \cap F_2 \cap F_3) = -|Y_1 \cap Y_2 \cap Y_3| + b(X_1 \cap X_2 \cap X_3) - \sum_{X\in\mathcal{Z}_1\cap\mathcal{Z}_2\cap\mathcal{Z}_3} b(X).$$

Last,

$$F_1 \cup F_2 \cup F_3 \cup Z \subseteq E[X_1 \cup X_2 \cup X_3] \cup \bigcup_{X\in\mathcal{X}_Z-(\mathcal{Z}_1\cup\mathcal{Z}_2\cup\mathcal{Z}_3)} E[X] \cup (Y_Z - (Y_1 \cup Y_2 \cup Y_3)),$$

and

$$r_{\mathcal{M}/Z}(F_1 \cup F_2 \cup F_3) \le -|Y_1 \cup Y_2 \cup Y_3| + b(X_1 \cup X_2 \cup X_3) - \sum_{X\in\mathcal{Z}_1\cup\mathcal{Z}_2\cup\mathcal{Z}_3} b(X).$$

Now we apply Claim 4.4 and the statement follows.

$$\begin{aligned}
&\sum_{1\le i<j\le 3} r_{\mathcal{M}/Z}(F_i \cap F_j) + r_{\mathcal{M}/Z}(F_1 \cup F_2 \cup F_3) \le \\
&\sum_{1\le i<j\le 3}\left(-|Y_i \cap Y_j| + b(X_i \cap X_j) - \sum_{X\in\mathcal{Z}_i\cap\mathcal{Z}_j} b(X)\right) \\
&\qquad - \left|\bigcup_{i=1}^{3} Y_i\right| + b\left(\bigcup_{i=1}^{3} X_i\right) - \sum_{X\in\mathcal{Z}_1\cup\mathcal{Z}_2\cup\mathcal{Z}_3} b(X) \le \\
&\sum_{i=1}^{3}\left(-|Y_i| + b(X_i) - \sum_{X\in\mathcal{Z}_i} b(X)\right) - \left|\bigcap_{i=1}^{3} Y_i\right| + b\left(\bigcap_{i=1}^{3} X_i\right) - \sum_{X\in\mathcal{Z}_1\cap\mathcal{Z}_2\cap\mathcal{Z}_3} b(X) = \\
&\qquad \sum_{i=1}^{3} r_{\mathcal{M}/Z}(F_i) + r_{\mathcal{M}/Z}(F_1 \cap F_2 \cap F_3).
\end{aligned}$$

□

Claim 4.6. *For $i, j \in [d]$, $i \ne j$,*

$$\begin{aligned}
r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i)) &= |U| - |U_i| - 1, &(7)\\
r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i \cap C_j)) &= |U| - |U_i| - |U_j|, &(8)\\
r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i) \cup \mathrm{sp}_{\mathcal{M}/Z}(C_j)) &= |U| - 2, &(9)\\
\mathrm{sp}_{\mathcal{M}/Z}(C_i \cap C_j) &= \mathrm{sp}_{\mathcal{M}/Z}(C_i) \cap \mathrm{sp}_{\mathcal{M}/Z}(C_j). &(10)
\end{aligned}$$

Proof. (7) is clear since C_i is a circuit. $C_i \cap C_j$ is independent, hence (8) follows. For (9), $U \subseteq \mathrm{sp}_{\mathcal{M}/Z}(C_i) \cup \mathrm{sp}_{\mathcal{M}/Z}(C_j) \subseteq \mathrm{sp}_{\mathcal{M}/Z}(U)$. For (10), $\mathrm{sp}_{\mathcal{M}/Z}(C_i \cap C_j) \subseteq \mathrm{sp}_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i) \cap \mathrm{sp}_{\mathcal{M}/Z}(C_j)) = \mathrm{sp}_{\mathcal{M}/Z}(C_i) \cap \mathrm{sp}_{\mathcal{M}/Z}(C_j)$ and $r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i) \cap \mathrm{sp}_{\mathcal{M}/Z}(C_j)) \leq r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i)) + r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_j)) - r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i) \cup \mathrm{sp}_{\mathcal{M}/Z}(C_j)) = |U| - |U_i| - |U_j| = r_{\mathcal{M}/Z}(\mathrm{sp}_{\mathcal{M}/Z}(C_i \cap C_j))$. □

Now we turn to the proof of Theorem 4.1 by induction on $|T|$. Throughout the proof, the singleton $\{i\}$ is sometimes referred as i. For $|T| = 0$, (6) holds by definition. For $|T| = 1$, (6) only is to be proved, which follows from (7). For $|T| = 2$, (5) follows from (9), and (6) follows from (8) and (10).

So let us assume $|T| \geq 3$ and $T = [|T|]$ for sake of simplicity. First, (5) is proved. It can be seen immediately that

$$C(T - i) \cup C(T - j) \subseteq C(T - \{i, j\}).$$

By applying submodularity to $C(T - i) \cup C(T - j)$ and $C(i)$, using

$$\mathrm{sp}_{\mathcal{M}/Z}(C(T - i) \cup C(T - j) \cup C(i)) = C(\emptyset),$$

and

$$(C(T - i) \cup C(T - j)) \cap C(i) \supseteq C(T - j),$$

we get

$$\begin{aligned} &r_{\mathcal{M}/Z}(C(T - i) \cup C(T - j)) + r_{\mathcal{M}/Z}(C(i)) \geq \\ &\qquad r_{\mathcal{M}/Z}(C(T - j)) + r_{\mathcal{M}/Z}(C(\emptyset)) = r_{\mathcal{M}/Z}(C(T - \{i, j\})) + r_{\mathcal{M}/Z}(C(i)), \end{aligned}$$

where the last equality is obtained by using the induction hypothesis for (6). As $C(T - \{i, j\})$ is a flat, this implies

$$\mathrm{sp}_{\mathcal{M}/Z}(C(T - i) \cup C(T - j)) = C(T - \{i, j\}),$$

proving (5).

For (6), again, we begin with the easier part, using only submodularity and induction:

$$\begin{aligned} r_{\mathcal{M}/Z}(C(T)) + r_{\mathcal{M}/Z}(C(T - \{1, 2\})) &= \\ r_{\mathcal{M}/Z}(C(T - \{1\}) \cap C(T - \{2\})) + r_{\mathcal{M}/Z}(C(T - \{1\}) \cup C(T - \{2\})) &\leq \\ r_{\mathcal{M}/Z}(C(T - \{1\})) + r_{\mathcal{M}/Z}(C(T - \{2\})) &= \\ \left(|U| - \sum\nolimits_{i \in T - \{1\}} |U_i| + |T| - 3\right) + \left(|U| - \sum\nolimits_{i \in T - \{2\}} |U_i| + |T| - 3\right)&. \end{aligned}$$

By $r_{\mathcal{M}/Z}(C(T - \{1, 2\})) = |U| - \sum_{i \in T - \{1,2\}} |U_i| + |T| - 4$, this yields

$$r_{\mathcal{M}/Z}(C(T)) = |U| - \sum_{i \in T} |U_i| + |T| - 2.$$

For the reverse inequality, we apply Claim 4.5 for $F_1 = C(T - \{2,3\})$, $F_2 = C(T - \{1,3\})$ and $F_3 = C(T - \{1,2\})$. Then

$$r_{\mathcal{M}/Z}(C(T)) = r_{\mathcal{M}/Z}(C(T-\{2,3\}) \cap C(T-\{1,3\}) \cap C(T-\{1,2\})) \geq$$
$$\sum_{i=1}^{3} r_{\mathcal{M}/Z}(C(T-i)) + r_{\mathcal{M}/Z}(C(T-[3])) - \sum_{\{i,j\}\in\binom{[3]}{2}} r_{\mathcal{M}/Z}(C(T-\{i,j\}) =$$
$$\sum_{i=1}^{3}\Big(|U| - \sum\nolimits_{k\in T-i}|U_k| + |T| - 3\Big) + \Big(|U| - \sum\nolimits_{k\in T-[3]}|U_k| + |T| - 5\Big) -$$
$$\sum_{\{i,j\}\in\binom{[3]}{2}}\Big(|U| - \sum\nolimits_{k\in T-\{i,j\}}|U_k| + |T| - 4\Big) =$$
$$|U| - \sum_{i\in T}|U_i| + |T| - 2.$$

□

Proof of Theorem 1.2. Let $Z \subseteq E$ and let U be a non-trivial (i.e. $d \geq 3$) double circuit of $\mathcal{M}/Z$ with principal partition $U = U_1 \dot\cup U_2 \dot\cup \ldots \dot\cup U_d$. Using the above notations, and by applying (6) to $T = [d]$,

$$r_{\mathcal{M}/Z}(C([d])) = |U| - \sum_{t\in[d]}|U_t| + |[d]| - 2 = d - 2 > 0.$$

□

Proof of Theorem 1.4. For $X \subseteq V$ with $|X| \leq c$ we have $b(X) = 0$, hence suppose next $|X| = c+1$. Then $b(X) = k(c+1) - (ck+d) = k - d$, and the condition that each hyperedge of size $c+1$ is present with multiplicity $k-d$ gives the proof.

Suppose now that $|X| \geq c+2$, and let $\mathcal{X} \subseteq \binom{V}{>\frac{l}{k}}$, $Y \subseteq E$ s.t. $Y \cup \bigcup_{W\in\mathcal{X}} E[W] \supseteq E[X]$, $r_{\mathcal{M}}(E[X]) = |Y| + \sum_{W\in\mathcal{X}} b(W)$. Let us choose $\mathcal{X}$ and Y s.t. $|Y|$ is minimal and to minimize $|\mathcal{X}|$ with the above primary conditions. If Y contains a hyperedge e of size $c+1$, then it contains all the $k-d$ parallel copies of e. By removing e and its copies from Y and adding e to $\mathcal{X}$, we get a new Y and $\mathcal{X}$ contradicting the extreme choice.

Suppose that for each $X' \subseteq X$, $|X'| = c+2$, there exists $X' \subseteq W' \in \mathcal{X}$. If $|X| = c+2$, then we are done. If $|X| \geq c+3$, then there exists $X', X'' \subseteq X$, $|X'| = |X''| = c+2$, $|X' \cap X''| = c+1$, $X' \subseteq W' \in \mathcal{X}$ and $X'' \subseteq W'' \in \mathcal{X}$. Then $k|W'| - l + k|W''| - l > k|W' \cup W''| - l$, hence we could replace $\mathcal{X}$ by $\mathcal{X} - W' - W'' + \{W' \cup W''\}$.

Thus there exists an X' having no such W'. But the hyperedges of size $c+1$ contained in X' are covered by $\mathcal{X}$ i.e. for each $i \in X'$ there exists $W_i \in \mathcal{X}$, $i \notin W_i$, $X' - i \subseteq W_i$. Then setting $W = \bigcup_{i\in X'} W_i$, $|W| = \sum_{i\in X'}|W_i| - c(c+2)$. If $d \leq \frac{ck}{c+1}$, then $\sum_{i\in X'}(k|W_i| - l) \geq k|W| - l$, thus we could remove each W_i from $\mathcal{X}$ and insert W, contradicting the extreme choice of Y and $\mathcal{X}$. If $d > \frac{ck}{c+1}$, then X' is in E with multiplicity $cd + d - ck$, and $\mathcal{X}$ does not cover these elements, they are included in Y. In this case, $\sum_{i \in X'}(k|W_i| - l) + cd + d - ck \geq k|W| - l$, we could remove each W_i from $\mathcal{X}$, insert W, and remove X' and its parallel copies from Y, which yields again a contradiction. □

5 Open Questions

The first polynomial matroid matching algorithm to solve problems which are not known to be reduced to the matroid intersection and to the matching problem of graphs was presented to linear matroids by Lovász [10]. Later, Dress and Lovász [3] noticed that pursuing the layout of this algorithm might yield a polynomial algorithm for the class of matroids having the DCP, provided that we are able to perform some algorithmic manipulations which handle flats and double circuits.

We have to notice that in a certain crucial point, this algorithm heavily relies on modularity. At the meantime, the author does not know how to extend Lovász' algorithm to this more general case.

In proving the DCP for full graphic and full transversal matroids, Dress and Lovász [3] put an intermediate step, they proved that these matroids have the weak series reduction property, and that matroids having weak series reduction property have the DCP.

The set S is said to be *in series* in U if S is a circuit of $\mathcal{M}/(U - S)$. The matroid $\mathcal{M}$ is said to have the weak series reduction property if for all $S \subseteq U \subseteq E$ s.t. S is in series in U and $\mathrm{sp}_{\mathcal{M}}(U - S)$ is connected, there is an element $\beta \in E$ s.t. for each $U \subseteq S$, $S \cup T$ is a circuit if and only if $\{\beta\} \cup T$ is a circuit.

For filling the gap in the hierarchy of matroid classes, Tan [13] proved that matroids having weak series reduction property are pseudomodular. We give an example for a (k, l)-matroids with (2) that does not have the weak series reduction property. Therefore, we have to put the question weather (k, l)-matroids with (2) are pseudomodular or not.

Let, $k = 2$ and $l = 3$, $V = \{x, y, u, v, z\}$, $E = \binom{V}{2}$, i.e. $\mathcal{M}$ is a 2-dimensional rigidity matroid on 5 vertices. Let $S = \{xy, xu, xv\}$ and $U = S \cup \binom{\{y,u,v,z\}}{2}$. Then, S is a circuit of $\mathcal{M}/(U - S)$ and $\mathrm{sp}_{\mathcal{M}}(U - S)$ is connected. Setting $T_1 = \binom{\{y,u,v,z\}}{2} - \{uv\}$ and $T_2 = \binom{\{y,u,v,z\}}{2} - \{uy\}$, we can see that $S \cup T_1$ and $S \cup T_2$ are circuits. The only $\beta_i \in E$ s.t. $\{\beta_i\} \cup T_i$ are a circuits, $\beta_1 = uv$ and $\beta_2 = uy$. Thus, there is no $\beta \in E$ which satisfies the requirement of the weak series reduction property.

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