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Packing non-returning A-paths

Gyula Pap

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Abstract

Chudnovsky et al. gave a min-max formula for the maximum number of node-disjoint non-zero A-paths in group-labeled graphs [1], which is a generalization of Mader's theorem on node-disjoint A-paths [3]. Here we present a further generalization with a shorter proof. The main feature of Theorem 2.1 is that parity is "hidden" inside $\widehat{\nu}$, which is given by an oracle for non-bipartite matching.

1 Introduction

W. Mader [3] gave a theorem on packing A-paths, which was re-stated in a more transparent form in two papers, in [5] by A. Sebő and L. Szegő, and independently in [1] by M. Chudnovsky, J. Geelen, B. Gerards, L. Goddyn, M. Lohman and P. Seymour. In [1] a min-max formula is given for the packing of non-zero A-paths in group-labeled graphs, which contains Mader's theorem as a special case. In this paper we show Theorem 2.1 on non-returning A-paths in "permutation-labeled" graphs, which contains the result on non-zero A-paths. The method of proof in this paper is also related to the short proof of Mader's theorem given by A. Schrijver [4]. We use an analogue of Berge's alternating paths' lemma in the proof of the main theorem of this paper.

Let $G = (V, E)$ be an oriented graph with node-set V, arc-set E and a fixed set $A \subseteq V$ of **terminals**. The orientation of arcs is needed only for reference. Let Ω be an arbitrary **set of "potentials"** and let $\omega : A \to \Omega$ define the **potential of origin** for the terminals. Let $\pi : E \to S(\Omega)$ where $S(\Omega)$ is the set of all permutations of Ω. For an arc $ab = e \in E$, let $\pi(e, a) := \pi(e)$ and $\pi(e, b) := \pi^{-1}(e)$ be the **mapping of potential** on arc ab. An **A-path** in G is a sequence of nodes and arcs which correspond to a path in the underlying undirected graph joining two distinct nodes of A, not using any other node in A. For an A-path $P = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$, let $\pi(P) := \pi(e_0, v_0) \circ \pi(e_1, v_1) \circ \ldots \circ \pi(e_{k-1}, v_{k-1})$ define the **mapping of potentials** on P. Let P be called **non-returning** if $\pi(P)(\omega(v_0)) \neq \omega(v_k)$. In other words, an

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⋆⋆Dept. of Operations Research, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). e-mail: `gyuszko@cs.elte.hu`

A-path is returning if it maps the potentials of origin onto each other. Notice that an A-path is non-returning if and only if its reverse is non-returning. A family of fully node-disjoint non-returning A-paths will be called a **non-returning family**. Let $\nu(G, A, \omega, \pi)$ denote the maximum cardinality of a non-returning family.

The problem of finding a maximum non-returning family is a slight generalization of the problem of packing non-zero A-paths in group-labeled graphs. To see this, we define the set of potentials as the set of elements of the group, the potential of origin as zero for each terminal, and we define $\pi(e)$ as the multiplication by the group-label on arc e.

2 The min-max formula

Consider a graph $G = (V, E)$ with a set $A \subseteq V$ of terminals. Let $\widehat{\nu}(G, A)$ denote the maximum number of fully node-disjoint A-paths. This is in fact a special case of packing non-returning A-paths, since it is easy to construct mappings of potentials such that every A-path is non-returning. T. Gallai [2] determined $\widehat{\nu}(G, A)$ by a reduction to non-bipartite matching.

Next we define a notion which we use in the main theorem to determine $\nu(G, A, \omega, \pi)$. Consider a set $F \subseteq E$ of arcs. Let $A' := A \cup V(F)$. F is called A-**balanced** if ω can be extended to a function $\omega' : A' \to \Omega$ such that $\pi(ab)(\omega'(a)) = \omega'(b)$ for each arc $ab = e \in F$. (Or equivalently, each arc in F gives a one-arc returning A'-path with respect to ω'. Notice that an A-path P is returning if and only if $E(P)$ is A-balanced.)

Theorem 2.1. *If G, A, ω, π is given as above then the equation*

$$\nu(G, A, \omega, \pi) = \min \widehat{\nu}(G - F, A \cup V(F)) \tag{1}$$

holds, where the minimum is taken over A-balanced arc-sets F.

We observe that in Theorem 2.1 the dual is based on two well separated "ingredients". Firstly, checking whether F is indeed A-balanced may be done by using a depth-first search. Secondly, determining $\widehat{\nu}(G - F, A \cup V(F))$ reduces to maximum matching in an auxiliary graph, see Gallai [2]. However, we will not refer to details of this reduction or to Gallai's result.

Let us point out that there is only a formal difference between the right hand side in (1) and those given in [5, 1]. It is quite easy to transform the "dual solutions" into each other by using Gallai's result. So, when we apply Theorem 2.1 to special cases, we obtain some of the results of [5, 1].

3 Proof of Theorem 2.1

First we prove the easy inequality, i.e. for an A-balanced arc-set F we show that $\nu(G, A, \omega, \pi) \leq \widehat{\nu}(G - F, A \cup V(F))$. Consider a non-returning A-path P. One can easily see that if $E[P] \subseteq F$ then P would be returning. Hence some section of P must be an $A \cup V(F)$-path in $G - F$. Given any non-returning family, the family of these sections gives a same-size family of node-disjoint $A \cup V(F)$-paths in $G - F$.

To show equality in (1) we use the following notion. For a non-returning family $\mathcal{P}$ let $A(\mathcal{P})$ denote the set of terminals covered by the paths in $\mathcal{P}$. A set $Z \subseteq A$ is called **exactly coverable** if there is a non-returning family $\mathcal{P}$ with $Z = A(\mathcal{P})$.

Lemma 3.1. *If Z is exactly coverable with $|Z| < 2\nu(G, A, \omega, \pi)$ (i.e. it is not maximum), then there is an exactly coverable set $Z + s + t$ with $s, t \notin Z$.*

Proof. We prove this by induction on $|V|$. Let $\mathcal{P}$ be a non-returning family with $Z = A(\mathcal{P})$. Consider a non-returning family $\mathcal{R}$ with $|\mathcal{R}| = |\mathcal{P}| + 1$. If $Z \subseteq A(\mathcal{R})$, then we are done. Otherwise there is a node $r \in Z - A(\mathcal{R})$.

Case I. Suppose that r is covered by a one-arc path rr' in $\mathcal{P}$. Then $Z' := Z - r - r'$ is exactly coverable for $G - r - r', A - r - r', \omega, \pi$. To see that Z' is not maximal, delete from $\mathcal{R}$ a path incident to r', if any. So, by induction, there is an exactly coverable set $Z' + s + t \subseteq A - r - r'$. It is easy to see that $Z + s + t$ is exactly coverable for G, A, ω, π.

Case II. Suppose that r is covered by a path with first arc $rq \in E$, $q \in V - A$. Define $\omega' : A - r + q \to \Omega$ by ω on $A - r$ and by $\omega'(q) := \pi(rq, r)(\omega(r))$. Then $Z' := Z - r + q$ is exactly coverable for $G - r, A - r + q, \omega', \pi$. We claim that Z' is not maximal, which can be seen as follows. If the paths in $\mathcal{R}$ are disjoint from q, then $A(\mathcal{R})$ is exactly coverable for $G - r, A - r + q, \omega', \pi$. Otherwise, if there is a path $R \in \mathcal{R}$ with $q \in V(R)$ joining nodes $q_1, q_2 \in A(\mathcal{R})$, then some $q - q_i$ section of R must be non-returning. Thus $A(\mathcal{R}) - q_{3-i} + q$ is exactly coverable for $G - r, A - r + q, \omega', \pi$. So, by induction, there is an exactly coverable set $Z' + s + t \subseteq A - r + q$. It is easy to see that $Z + s + t$ is exactly coverable for G, A, ω, π. □

Let $\alpha = \alpha(G, A, \omega, \pi)$ denote the number of terminals where at least one returning A-path starts. Consider a counterexample with $|V| + |E|$ minimal, and then α minimal.

Claim 3.2. *There is a node-disjoint family of $\nu(G, A, \omega, \pi) + 1$ A-paths, $\nu(G, A, \omega, \pi)$ of which are non-returning.*

Proof. In case of $\nu(G, A, \omega, \pi) = \widehat{\nu}(G, A)$ the formula (1) obviously holds with $F = \emptyset$. Otherwise, suppose some returning A-path starts in a terminal $t \in A$. Let $\Omega' := \Omega + \bullet$ for some $\bullet \notin \Omega$. Let π' be identical to π, let $\bullet$ be always mapped onto $\bullet$, and let us redefine $\omega'(t) := \bullet$. So all paths starting in t will be non-returning, and the status of paths disjoint from t does not change. We define $\alpha' := \alpha(G, A, \omega', \pi')$. Clearly, $\alpha' < \alpha$, hence formula (1) holds for the instance G, A, ω', π'. Consider a minimal A-balanced set F with respect to ω', π' with $\nu(G, A, \omega', \pi') = \widehat{\nu}(G - F, A \cup V(F))$.

We claim that F is A-balanced with respect to ω, π. This follows from $t \notin V(F)$, which can be seen as follows. By the definition of ω', π', there is no A-path in (V, F) starting in t, thus there is a component C of (V, F) with $\{t\} = V(C) \cap A$. It is easy to see that for $F' := F - E(C)$ we have $\widehat{\nu}(G - F', A \cup V(F')) \leq \widehat{\nu}(G - F, A \cup V(F))$, which by the minimality of F implies that $E(C) = \emptyset$.

So, by the choice of G, A, ω', π' we must have $\nu(G, A, \omega', \pi') > \nu(G, A, \omega, \pi)$. Consider $\nu(G, A, \omega, \pi) + 1$ non-returning A-paths with respect to ω', π'. Only one path incident with t can be returning with respect to ω, π. □

Consider a family $\mathcal{P}$ of A-paths given by Claim 3.2, and let $P \in \mathcal{P}$ be the returning path. Here $E(P)$ is A-balanced. Let ω' be the $A \cup V(P) \to \Omega$ function from the definition. Let $G' := G - E(P)$ and $A' := A \cup V(P)$. Since the paths in $\mathcal{P} - P$ are non-returning we have

$$\nu(G', A', \omega', \pi') \geq \nu(G, A, \omega, \pi) = \nu. \tag{2}$$

By the choice of G, A, ω, π, there is an A'-balanced arc set F' with respect to ω', π', with $\nu(G', A', \omega', \pi') = \widehat{\nu}(G' - F', A' \cup V(F'))$. It is easy to see that $F := F' \cup E(P)$ is A-balanced with respect to ω, π. This gives $\widehat{\nu}(G - F, A \cup V(F)) = \widehat{\nu}(G' - F', A' \cup V(F'))$. If in (2) we have equality, then we are done.

Otherwise, there is a non-returning family $\mathcal{P}'$ in G', A', ω', π' with $|\mathcal{P}'| > \nu$. By Lemma 3.1 we may choose $\mathcal{P}'$ with $A(\mathcal{P}') = A(\mathcal{P} - P) + s + t$ for some s, t. We will get to a contradiction by constructing a $|\mathcal{P}'|$-size non-returning family. To this end, a path ending in a node in $V(P)$ can be extended by a section of P to get a non-returning A-path. Since $\mathcal{P}'$ covers at most two nodes in $V(P)$ we need at most two such extending sections, and clearly, two extending sections fit into P. This contradiction completes the proof. □□

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