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Alternating paths revisited II: restricted b-matchings in bipartite graphs

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Abstract

We give a constructive proof for a min-max relation on restricted b-matchings in bipartite graphs, extending results of Hartvigsen [5, 6], Király [7], and Frank [3]. Restricted b-matching is a special case of covering pairs of sets, for which Benczúr and Végh [1, 2] constructed a polynomial time algorithm – this implies a polynomial time algorithm for restricted b-matchings, as well. In this paper we solve restricted b-matchings directly, in a conceptually simple way.

1 Introduction

The main result in this paper is a min-max formula characterizing the maximum cardinality of so-called $\mathcal{K}$-free b-matchings in bipartite graphs. This comes as a generalization of earlier results of D. Hartvigsen [5, 6], Z. Király [7], and A. Frank [3]. Our proof of the formula is constructive and thus implies a combinatorial algorithm to find a maximum $\mathcal{K}$-free b-matching. The first polynomial time algorithm follows from a recent work of Benczúr and Végh [1, 2], via the general framework of covering pairs of sets. In this paper we show that restricted b-matching is a special of covering pairs of sets which is easier to solve directly.

Throughout this paper, the expression "b-matching" stands for edge-sets in a given undirected graph obeying an upper-bound condition on the degrees. Such edge-sets are often referred to as "simple" b-matchings – we, however, omit the word "simple". A b-matching is called $\mathcal{K}$-free if it contains no subgraphs from a specific class $\mathcal{K}$ of forbidden subgraphs. We will prove equality in the min-max formula when the specific class $\mathcal{K}$ of forbidden subgraphs fulfills the condition (2). Let us give an overview of earlier results which have motivated and are generalized by our main theorem.

In a simple undirected graph, a 2-matching is the edge-set of the node-disjoint union of cycles and paths. A 2-matching is called square-free if it contains no cycle of length 4. Putting effort into square-free 2-matchings is motivated by the observation that the

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complements of $(n-3)$-connected graphs are exactly square-free 2-matchings. Hence one can establish a relation with a node-connectivity augmentation problem (see Frank [3]). It is worth mentioning that, replacing $(n-3)$ by $(n-2)$ we get a problem which is equivalent to the maximum matching problem. The first positive result on square-free 2-matchings is due to D. Hartvigsen [5], who provided a min-max formula in the bipartite case, i.e. a min-max formula to determine the maximum cardinality of a square-free 2-matching in a simple bipartite graph. Z. Király [7] proved a simplified version of this min-max formula in a purely combinatorial way. Hartvigsen's original paper proposed an algorithmic approach, as well. Recently, Hartvigsen [6] verified this proposed algorithm, and turned it into an algorithmic proof of Király's formula. These results provide a solution of the above mentioned connectivity augmentation problem when the complement of the graph is bipartite. The complexity of the non-bipartite case is still open.

A significant step towards a better understanding of restricted matchings is the interpretation of A. Frank [3], who represented square-free 2-matchings in bipartite graphs using the theory of covering pairs of sets. (For pairs of sets, see Frank, Jordán [4].) Using this interpretation, Frank also proposed other general min-max formulas. For example, he proved an analogous formula for the maximum cardinality of a K_{tt}-free t-matching in a simple bipartite graph, for any fixed $t \geq 2$. For $t = 2$ this specializes to square-free 2-matchings. M. Makai [8] observed that Frank's formula holds for the following generalization. Suppose we are given an arbitrary bipartite graph, a positive integer t, and a pre-specified list $\mathcal{K}$ of forbidden subgraphs isomorphic to K_{tt}. (Hence some K_{tt}-subgraphs may be forbidden, while others may be allowed.) Find a maximum cardinality t-matching which avoids having subgraphs from the forbidden list. Makai cited general results on pairs of sets to prove a min-max formula for this $\mathcal{K}$-free t-matching problem. It is straightforeward that lists in Makai's problem fulfill the condition (2), and thus Makai's observation follows from results in this section, as well. Using the theory of pairs of sets, Frank also proved a different kind of formula concerning large-bi-clique-free t-matchings. The class of large-bi-cliques does, however, not fulfill the condition (2), hence this result is not a special case of our main theorem.

The theory of covering pairs of sets, invented by A. Frank and T. Jordán [4], is a common generalization of a whole bunch of min-max formulas, including connectivity augmentation, Győri's theorem on intervals, and restricted matching problems. It was a great breakthrough when A. Benczúr and L. Végh [1, 2] constructed a polynomial running time algorithm for this general framework. Basic observations show that the problem considered in this paper is a special case of covering pairs of sets, and thus is solved by Benczúr and Végh's algorithm in polynomial time. Thus the goal of this paper is to show that restricted b-matchings can in fact be handled in an easier way than pairs of sets in general.

2 Preliminaries on b-matchings

To solve the maximum restricted b-matching problem, it will be useful to give an account on basic results concerning (unrestricted) b-matchings – as it is in fact a special case of the general problem. We will only define later what this general restricted b-matching problem is. For purposes in our general problem, we will require a tool for b-matchings which either provides an augmenting path, or provides a set which verifies optimality via the min-max formula. This section is devoted to the construction of this elementary tool.

Consider a bipartite graph $G = (A, B; E)$ with node set $V = A \cup B$ and a function $b : V \to \mathbb{N} = \{0, 1, 2, \cdots\}$. A **$b$-matching** is a set $M \subseteq E$ of edges such that $\delta_M(v) \leq b(v)$ holds for all $v \in V$. The following good characterization of the maximum cardinality of a b-matching is well-known.

Theorem 2.1. *The maximum cardinality of a b-matching in a bipartite graph $G = (A, B; E)$ is equal to*

$$\min_{Z \subseteq V} b(V - Z) + |E[Z]|. \tag{1}$$

It is straightforward that the maximum is at most the minimum in this formula. Let us call a set Z a verifying set for a given b-matching M, if equality $|M| = b(V - Z) + |E[Z]|$ holds. Equality in the min-max formula is equivalent with the existence of a b-matching with a verifying set. Next we will prove Lemma 2.2 on alternating paths – the proof of equality is then immediate. Later, the Lemma 2.2 will be useful in the proof for restricted b-matchings, as well.

Consider an arbitrary b-matching $M \subseteq E$. A node $v \in V$ is called **M-loose** if $\delta_M(v) < b(v)$ holds. A path in G is called an **augmenting (M-alternating) path**, if it starts in an M-loose node in A, ends in an M-loose node in B, and its edges are alternatingly in $E - M, M$ with the first and last being in $E - M$. Clearly, the symmetric difference of M and the edge-set of an augmenting path is a b-matching of cardinality $|M| + 1$. The essence of the following lemma is that an augmenting path is all we have to be looking for to find a larger b-matching.

Lemma 2.2. *Given a b-matching M in a bipartite graph $G = (A, B; E)$, one can find in linear time either an augmenting M-alternating path, or a set Z for which $|M| = b(V - Z) + |E[Z]|$ holds.*

Proof. Let us define a digraph $D = (V, A)$ by orienting edges of M towards A, and orienting edges of $E - M$ towards B. There is a one-to-one correspondance between edges in E and arcs in A. Notice that an augmenting path corresponds to a directed path in D starting in an M-loose node in A and ending in an M-loose node in B. Hence, if there exists an augmenting path, then one can construct an augmenting path in linear time by breadth first search. Otherwise, if there is no augmenting path, then we construct the set $S \subseteq V$ of nodes reachable in D from M-loose nodes in A. The construction of S can be performed in linear time by breadth first search starting at M-loose nodes in A. We claim that the set defined by $Z := S \Delta B$ verifies that M is a maximum b-matching, that is, $|M| = b(V - Z) + |E[Z]|$ holds. To prove this, we partition M into four disjoint sets defined by $M_1 := M \cap \delta(A \cap Z, B \cap Z)$, $M_2 := M \cap \delta(A - Z, B \cap Z)$, $M_3 := M \cap \delta(A \cap Z, B - Z)$, $M_4 := M \cap \delta(A - Z, B - Z)$. By definition of S we get that $\delta_D^{out}(S) = 0$, which implies $M_1 = \delta(A \cap Z, B \cap Z) = E[Z]$ and $M_4 = \emptyset$. We have assumed that there is no augmenting path, which implies that all M-loose nodes are in Z. Thus $|M_2 \cup M_3| = b(V - Z)$. □

Since maximum b-matchings do not admit augmenting paths, Theorem 2.1 follows by applying Lemma 2.2 to an arbitrary maximum b-matching. Recursive application of the lemma implies a polynomial time algorithm to find a b-matching with a verifying set.

3 Restricted b-matchings

Consider a simple bipartite graph $G = (A, B; E)$ with node set $V = A \cup B$ and a function $b : V \to \mathbb{N} = \{0, 1, 2, \cdots\}$. Let $\mathcal{K}$ be a family of some complete bipartite subgraphs (bi-cliques) of G with both classes consisting of at least two nodes. So for each $K \in \mathcal{K}$ we have $|A \cap V(K)| \geq 2$ and $|B \cap V(K)| \geq 2$. Let us point out that we only distinguish two subgraphs of G if their node-sets or edge-sets are not identical. Thus there may well be two isomorphic subgraphs only one of which is in $\mathcal{K}$. A b-matching M is called **$\mathcal{K}$-free** if no component of the subgraph $(A, B; M)$ is a member of $\mathcal{K}$. We consider the problem of maximizing the cardinality of a $\mathcal{K}$-free b-matching.

In Theorem 3.1, we propose a min-max formula to determine the maximum cardinality of a $\mathcal{K}$-free b-matching. It will be easy to check that the expression in the "minimum" is an upper bound. To prove equality, we make an assumption on the family $\mathcal{K}$ – equality does not follow without the assumption. We consider triples $G, b, \mathcal{K}$ enjoying the following property.

(2) Consider an arbitrary $K \in \mathcal{K}$. Let $k := |A \cap V(K)|$ and $l := |B \cap V(K)|$. Then $k, l \geq 2$, moreover $b(v) = l$ for any $v \in A \cap V(K)$ and $b(v) = k$ for any $v \in B \cap V(K)$.

We make the following observation concerning property (2). If (2) holds and M is a b-matching such that $E(K) \subseteq M$ for some $K \in \mathcal{K}$, then K is in fact a component of $(A, B; M)$. Furthermore, it is easy to check that those problems considered in the introduction obey (2). For example Makai's list problem corresponds to the case when $b \equiv t$, and $\mathcal{K}$ consists of a number of subgraphs isomorphic with K_{tt}.

To formulate the dual expression in our formula, we need some more notation. A component of an induced subgraph $G[Z]$ is called a **$\mathcal{K}$-component of** $G[Z]$ if it is a component of $G[Z]$, and member of $\mathcal{K}$. Let $c_{\mathcal{K}}(G[Z])$ denote the number of $\mathcal{K}$-components of $G[Z]$. Our main result of this section is the following characterization of the maximum cardinality of a $\mathcal{K}$-free b-matching.

Theorem 3.1. *Given a simple bipartite graph $G = (A, B; E)$, a function $b : A \cup B \to \mathbb{N}$ and a family $\mathcal{K}$ of complete bipartite subgraphs of G so that the triple $G, b, \mathcal{K}$ fulfills* (2)*. Then the maximum cardinality of a $\mathcal{K}$-free b-matching is equal to*

$$\min_{Z \subseteq V} b(V - Z) + |E[Z]| - c_{\mathcal{K}}(G[Z]). \tag{3}$$

Let us point out that this min-max formula is a particularly strong kind of characterization. To determine the value of the dual expression in (3), we need to test for membership in $\mathcal{K}$ only a small number (at most $|A|/2$) of subgraphs of G. In some sense, the formula is a good characterization only assuming that $\mathcal{K}$ is given by a membership oracle. Furthermore, assuming a membership oracle, we can in fact find a maximum $\mathcal{K}$-free b-matching in polynomial time, together with a certificate Z.

We remark that the assumption that the bipartite graph G is simple is of barely technical nature – an appropriate generalization of Theorem 3.1 holds for non-simple graphs. However, property (2) has to be reformulated for non-simple graphs, and that reformulation is not nearly as attractive as the original property (2). Furthermore, that result for non-simple graphs easily follows from Theorem 3.1, so we omit the discussion of the non-simple case.

Why don't we use an augmentation structure? For unrestricted b-matchings, the augmentation structure amounts to finding directed paths connecting loose nodes in an auxiliary digraph. An attractive approach to restricted b-matchings would be looking for directed paths in the same auxiliary digraph, trying to avoid that squares appear in the symmetric difference. D. Hartvigsen [6] constructed an algorithm from this idea. However, Hartvigsen's algorithm has to deal with complicated issues, which yields a quite complicated augmentation structure. In our approach, we don't deal with those complicated issues, and instead we go a different way to avoid squares in the b-matching.

Our approach differs from Hartvigsen's quite significantly in the respect that we do not directly try to avoid squares to appear. Instead, whenever a square – the "first" square – appears, we stop there, and use that constellation for a reduction. The actual way and the circumstances of finding this specific constellation is described in the 3-Way Lemma 4.1. This constellation – a "fitting bi-clique" – is useful in the reducucction of our problem to a smaller instance. In Lemma 4.2 we will prove that this reduction is an "equivalent reduction" in some sense – playing a similar role as the contraction of an odd cycle in Edmonds' matching algorithm.

4 The constructive proof

In this subsection we prove Theorem 3.1 in such a way that our key lemmas will directly imply a polynomial running time algorithm. To prove the easy part max $\leq$ min, consider a $\mathcal{K}$-free b-matching M and a set $Z \subseteq V$. Then $|M[Z]| \leq |E[Z]| - c_{\mathcal{K}}(G[Z])$, since M does not contain all the edges of any member of $\mathcal{K}$. Moreover, $|M - M[Z]| \leq b(V - Z)$, since the edges in $M - M[Z]$ are incident with a node in $V - Z$. These two inequalities sum up to $|M| \leq b(V - Z) + |E[Z]| - c_{\mathcal{K}}(G[Z])$. We obtain the following slackness conditions – that is, equality $|M| = b(V - Z) + |E[Z]| - c_{\mathcal{K}}(G[Z])$ implies:

(4) $\delta_M(v) = b(v)$ for all nodes $v \in V - Z$.

(5) $M[Z]$ contains $E[Z]$ except for exactly one edge from each $\mathcal{K}$-component of $G[Z]$.

To formulate our key lemma, let us introduce the notion of a fitting bi-clique which, in some sense, is the analogue of alternating odd cycles in Edmonds' matching algorithm. Consider a $\mathcal{K}$-free b-matching N. A bi-clique $K \in \mathcal{K}$ is called a **bi-clique fitting** N, if there is an edge $uw \in E(K)$ with $u \in A, w \in B$ such that $E(K) - uw \subseteq N$ and $\delta_N(u, B - V(K)) = \emptyset$. In words: N contains all but one of the edges of K, and at most one edge leaving the nodeset of K. A fitting bi-clique will be given by the four-tuple N, K, u, w. Our key lemma provides a method to find some certificate of optimality, or an augmentation, or a fitting bi-clique.

Lemma 4.1 (3-Way Lemma for $\mathcal{K}$-free b-matchings). *Consider an arbitrary $\mathcal{K}$-free b-matching M. At least one of the following assertions holds.*

(6) *There is a set Z such that $|M| = b(V - Z) + |E[Z]|$ holds.*

(7) *There is a $\mathcal{K}$-free b-matching N such that $|N| = |M| + 1$.*

(8) *There is a $\mathcal{K}$-free b-matching N such that $|N| = |M|$, and there is a bi-clique fitting N.*

Moreover, one can find in polynomial time either Z, or N, or the four-tuple N, K, u, w for one of the respective assertions.

Proof. If M is a maximum b-matching, then (6) follows from Theorem 2.1. Otherwise, by Lemma 2.2, there is an augmenting path P say with nodes in order $V(P) = \{v_0, z_0, v_1, z_1, \cdots, v_k, z_k\}$. (This means that the path P has an edge-set of $2k+1$ edges $E(P) = \{v_0z_0, z_0v_1, \cdots, v_kz_k\}$ such that $v_i \in A, z_i \in B, v_iz_i \in E - M, z_iv_{i+1} \in M$, where v_0, z_k are M-loose.) If the b-matching $M\Delta E(P)$ is $\mathcal{K}$-free, then assertion (7) holds. Otherwise we consider for $i = 0, 1, \cdots, k$ the subpath P_{2i} on the nodes $V(P_{2i}) = \{v_0, z_0, \cdots, v_i\}$, which is the subpath of P starting in $v_0 \in A$ and having $2i$ edges. Then $M_i := M\Delta E(P_{2i})$ is a b-matching for any i. Let j be the maximal index for which M_j is $\mathcal{K}$-free.

Firstly, suppose $j = k$. That is, M_k is a $\mathcal{K}$-free b-matching, but $M\Delta E(P) = M_k + v_kz_k$ is a b-matching which is not $\mathcal{K}$-free. Let $N := M_k$, $u := v_k$ and $w := z_k$, and let K be the unique $K \in \mathcal{K}$ such that $E(K) \subseteq M_k + v_kz_k$. Since $M_k + v_kz_k$ is a b-matching, and $\mathcal{K}$ fulfills property (2), we get that $\delta_N(V(K)) = \emptyset$. This implies $\delta_N(u, B - V(K)) = \emptyset$, hence (8) holds.

Secondly, suppose $j < k$. That is, M_j is a $\mathcal{K}$-free b-matching, but $M_{j+1} = M_j + v_jz_j - z_jv_{j+1}$ is a b-matching which is not $\mathcal{K}$-free. Let $N := M_j$, and let K be the unique $K \in \mathcal{K}$ which is subgraph of M_{j+1}. Let $u := v_j$ and $w := z_j$. Since M_{j+1} is a b-matching, and $\mathcal{K}$ fulfills property (2), we get that $\delta_{M_{j+1}}(V(K)) = \emptyset$. This implies $\delta_N(u, B - V(K)) = \emptyset$, hence (8) holds. □

To formulate our reduction lemma, suppose there is a $\mathcal{K}$-free b-matching N, and a fitting bi-clique given by N, K, u, w. We define a "contracted graph" $G' = (A', B'; E')$ by deleting the edges in $E(K)$, identifying the nodes in $K \cap A$ by a new node k_A and identifying the nodes in $K \cap B$ by a new node k_B, only keeping one edge from each bunch of parallel edges to retain a simple bipartite graph. By definition, there is no edge joining the two new nodes. We define a function b' on V' which returns the value of b for the old nodes, and for the new nodes we define $b'(k_A) := b'(k_B) := 1$. Let $\mathcal{K}'$ be the family of subgraphs in $\mathcal{K}$ which are disjoint from $V(K)$. The above definitions make sense, since $G', b', \mathcal{K}'$ fulfills condition (2). Let N' denote the image of N in G', i.e. we get N' by deleting all the $kl-1$ edges in $E(K) \cap N$. It is easy to see that N' is a $\mathcal{K}'$-free b'-matching in G' of cardinality $|N| - kl + 1$.

Lemma 4.2. *If $G', b', \mathcal{K}'$ is constructed as above, then both of the following claims hold.*

(9) *Given a $\mathcal{K}'$-free b'-matching M' in G', one can construct in polynomial time a $\mathcal{K}$-free b-matching in G of cardinality $|M'| + (kl-1)$.*

(10) *If Z' is an inclusionwise minimal verifying set for $N', G', b', \mathcal{K}'$, then the pre-image Z of Z' is a verifying set for $N, G, b, \mathcal{K}$.*

Proof. To prove the first claim, let M'' denote the pre-image of M'. By definition, $b'(k_A) = b'(k_B) = 1$. Hence $|\delta_{M''}(A \cap V(K), B - V(K))| \le 1$ and $|\delta_{M''}(A - V(K), B \cap V(K))| \le 1$. Thus, by adding a properly chosen set of $kl-1$ edges of $E(K)$ to M'', we obtain a b-matching M in G. It is easy to check that M is $\mathcal{K}$-free. This completes the proof of the first claim.

To prove the second claim, we make use of complementary slackess conditions with respect to N', Z'. By definition, $\delta_{N'}(k_A) = 0$. Hence, condition (4) implies $k_A \in Z'$. No member of $\mathcal{K}'$ contains k_A, hence by criterion (5), a maximum $\mathcal{K}'$-free b'-matching must contain all edges of $E'[Z']$ incident with k_A. However, the maximum $\mathcal{K}'$-free b'-matching N' contains no edge incident with k_A, thus there is no edge $k_A t \in E'$ with $t \in Z'$. Next we distinguish the two cases whether k_B is in Z' or not. Firstly, if $k_B \notin Z'$, then it is easy to see that $b(V - Z) = b'(V' - Z') + kl - 1$, $c_{\mathcal{K}}(G[Z]) = c_{\mathcal{K}'}(G'[Z'])$, and $|E[Z]| = |E'[Z']|$. Secondly, if $k_B \in Z'$, then $b(V - Z) = b'(V' - Z')$ and $|E[Z]| = |E'[Z']| + kl$ follow easily. We also claim $c_{\mathcal{K}}(G[Z]) = c_{\mathcal{K}'}(G'[Z']) + 1$, which needs a bit of reasoning: we must prove that there is no edge $k_A t \in E'$ or $k_B t \in E'$ such that $t \in Z'$. We have already seen that there is no edge $k_A t \in E'$ with $t \in Z'$. Suppose for contradiction that there is an edge $k_B t \in E'$ with $t \in Z'$. But then it is easy to see that $Z' - k_B$ is a verifying set, as well. This contradicts the minimal choice of Z', thus we have proved $c_{\mathcal{K}}(G[Z]) = c_{\mathcal{K}'}(G'[Z']) + 1$. The following calculation can be verified in both of our cases. $|N| \le b(V - Z) + |E[Z]| - c_{\mathcal{K}}(G[Z]) = b'(V' - Z') + |E'[Z']| - c_{\mathcal{K}'}(G'[Z']) + (kl - 1) = |N'| + (kl - 1) = |N|$. □

Finally, let us put pieces together to prove Theorem 3.1. We prove sufficiency in formula (3) by induction on $|V|$. Consider a maximum $\mathcal{K}$-free b-matching M, and apply the 3-Way Lemma 4.1 for $G, b, \mathcal{K}, M$. Now, assertion (7) is impossible. If assertion (6) holds, then we are done, since Z is a verifying set for M. If assertion

(8) holds, then we use N, K, u, w to construct $G', b', \mathcal{K}', N'$, and apply Lemma 4.2. By the first claim of Lemma 4.2, N' is a maximum $\mathcal{K}$-free b'-matching. Thus, by induction, there is a verifying set Z' for N'. By the second claim of Lemma 4.2 there is a verifying set Z for N. This proves Theorem 3.1.

A polynomial time algorithm is given just as easily. We maintain $G, b, \mathcal{K}, M$ throughout the algorithm, and apply 3-Way Lemma 4.1 for $G, b, \mathcal{K}, M$. When (6) holds, then optimality is achieved and certified by set Z. When (7) holds, then we replace $G, b, \mathcal{K}, M$ by $G, b, \mathcal{K}, N$, which is an augmentation. When (8) holds, then we apply the procedure recursively for $G', b', \mathcal{K}', N'$, that is, we will either find an augmentation, or a verifying set with respect to $G', b', \mathcal{K}', N'$. Lemma 4.2 implies that an augmentation with respect to $G', b', \mathcal{K}', N'$ can be used to augment with respect to $G, b, \mathcal{K}, M$, moreover, a verifying set with respect to $G', b', \mathcal{K}', N'$ can be used to construct a verifying set with respect to $G, b, \mathcal{K}, M$. (For the latter notice that, given a verifying set, one can easily find an inclusionwise minimal verifying set just by dropping some nodes as long as we retain a verifying set. Also, we have to remark that, a membership oracle for $\mathcal{K}'$ is easy to derive from a membership oracle for $\mathcal{K}$ – we just have to check whether the specific bi-clique is disjoint from k_A, k_B, or not.)

Let us sketch the complexity analysis of the algorithm. We initiate the algorithm from a maximum $(b-1)$-matching, which is guaranteed to be $\mathcal{K}$-free. Then the above algorithm performs at most $|V|$ augmentations. Between two augmentations, we find at most $|V|$ fitting bi-cliques. We construct $|V|^2$ reduced graphs, and there we perform $|V|^2$ breadth-first-searches. Thus we have to test at most $|V|^3$ subgraphs for membership in $\mathcal{K}$. Thus the above algorithm has a total running time of $O(|V|^4)$. (The most time-consuming operation is the construction of the reduced graphs, hence one may hope for a slightly better running-time by applying a nice data-structure representing reduced graphs. Here we omit the discussion of such data-structures.)

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