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Hypo-matchings in directed graphs

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Abstract

We give a common generalization of results on hypo-matchings given in a sequence of papers by G. Cornuéjols, D. Hartvigsen and W. Pulleyblank in [2, 3, 4, 5] and results on even factors given by W.H. Cunningham and J.F. Geelen in [7] and by the author and L. Szegő in [13].

1 Introduction

The main result of this paper is a common generalization of the hypo-matching formula and the even factor formula.

The maximum hypo-matching problem is the following. Given an undirected graph G and a family $\mathcal{F}$ of factor-critical (hypo-matchable) subgraphs of G. A hypo-matching in G is a subgraph the components of which are some members of $\mathcal{F}$ and some components isomorphic to K_2. The problem is to maximize the number of nodes covered by a hypo-matching, for results on this problem, see papers of G. Cornuéjols, D. Hartvigsen and W. Pulleyblank [2, 3, 4, 5].

The maximum even factor problem is given in directed graphs. An even factor is the arc set M of a subgraph the weak components of which are directed cycles of even length and directed paths of arbitrary length. The problem is to maximize the cardinality of M. This problem can only be solved for a class of directed graphs called *odd-cycle-symmetric* – a directed graph is odd-cycle-symmetric if each directed odd cycle is symmetric, i.e. its arcs also exist in the opposite direction. W.H. Cunningham proposed this problem in [6] as a generalization of the optimum path-matching problem, the algebraic method of Cunningham and J.F. Geelen [7, 8] may be extended to even factors. The author and L. Szegő gave a simplified min-max formula in [13].

We present a theorem generalizing results in both topics, let us make some remarks here on the method of proof. In [16] M. Loebl and S. Poljak gave an elegant proof of the hypo-matching formula, which uses the Edmonds-Gallai decomposition of G (see [9, 10]). In [13], the author and Szegő used a non-constructive inductive method ("divide-and-conquer") to show the maximum even factor formula, where they also described

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an Edmonds-Gallai-type decomposition for even factors. It would be desirable to use this decomposition for the elegant method of Loebl and Poljak, but there is an obstacle. For hypo-matchings in undirected graphs the inclusion $D_G^{\mathcal{F}} \subseteq D_G$ holds, using widespread notation for the Edmonds-Gallai-type decomposition in graph G. However, the analogue inclusion does not hold for the concept in this paper for hypo-matchings in directed graphs.

The "divide-and-conquer" method of proof could only be extended to prove a special case of Theorem 3.1, see [14].

The proof of this paper uses the constructive method first presented in [15] by the author to solve the maximum even factor problem algorithmically. In fact, the proof of the paper is also constructive in the same sense as those of results on hypo-matchings. That is, given a digraph D and $\mathcal{H}$ as in Definition 2.1, and suppose we have an oracle for testing $\mathcal{H}$-criticality of subgraphs, we can solve the maximum $\mathcal{H}$-matching problem in polynomial time.

2 Definitions

Consider a digraph $D = (V, A)$ where we allow loops or parallel arcs. A *cycle (path)* is the arc-set of a closed (unclosed) directed walk without repetition of arcs or nodes. The emptyset is regarded as the zero-length path, any node may be regarded as the start and end of a zero-length path. A loop-arc gives a one-arc cycle. We call an arc $e = uv \in A$ *symmetric* (in D) if there is an arc $f = vu \in A$, otherwise e is *asymmetric* (in D). A cycle or a path is *even (odd)* if it consists of an even (odd) number of arcs. A cycle is *asymmetric* (in D) if it has at least one asymmetric arc. Loop-arcs are regarded as symmetric. The weakly connected components of a digraph (i.e. connected components in the undirected sense) are called *weak components*, for short. A *path-cycle-matching* is the arc-set of a subgraph the weak components of which are directed cycles and directed paths.

An undirected graph $G = (V, E)$ is called *factor-critical* if for all $v \in V$ the graph $G - v$ has a perfect matching. (For a survey on matching theory, see L. Lovász and M.D. Plummer [12].) A digraph is called *symmetric-critical* if all its arcs are symmetric and the underlying undirected graph is factor-critical. Let $\mathcal{H}$ be a (possibly empty) family of symmetric-critical subgraphs in D. For example, single node subgraphs may be put in $\mathcal{H}$ as they are symmetric-critical.

Definition 2.1. An *$\mathcal{H}$-matching* in $D = (V, A)$ is a subset M of A so that for each weak component (V_0, M_0) of the digraph (V, M)

1. M_0 is a path, or
2. M_0 is an even cycle, or
3. M_0 is an asymmetric odd cycle, or
4. (V_0, M_0) is a member of $\mathcal{H}$.

The $size(M)$ is defined as the number of arcs in all these paths and cycles plus the number of nodes covered by members of $\mathcal{H}$ used in M. We say M *uses* a member H of $\mathcal{H}$ if H is a weak component in (V, M). Let $\nu^{\mathcal{H}}(D)$ denote the maximum size of an $\mathcal{H}$-matching in D.

Some further definitions, notation: for a set $X \subseteq V$ let $\Gamma_D^+(X) := \{x \in V - X : \exists y \in X, yx \in A\}$ and $\varrho_D(X) = |\{uv \in A : u \in V - X, v \in X\}|$ and $\delta_D(X) = |\{uv \in A : u \in X, v \in V - X\}|$. For $uv \in A$ we call v the *head* of arc uv, while u is called the *tail* of arc uv; uv *leaves* u and *enters* v.

A set $X \subseteq V$ *induces* the subgraph $D[X] = (X, A[X])$ where $A[X] = \{uv \in A : u, v \in X\}$. $D - X := D[V - X]$. Also, for a set $M \subseteq A$ we define the set of arcs in M induced in X as $M[X] := \{uv \in M : u, v \in X\}$. For a set $U \subseteq V$ we denote the *contracted graph* by D/U having node set $V/U = V - U + \{U\}$ and arc-set A/U given by deleting the arcs in $A[U]$ and identifying the nodes in U by $\{U\}$ (we will use contractions only if $D[U]$ is connected, thus this definition is equivalent with the usual contraction of the arcs in $A[U]$). Consider an induced subgraph $D[U]$, let $\mathcal{H}_U$ be the family of members of $\mathcal{H}$ that are subgraphs of $D[U]$. To make the notation simpler, an $\mathcal{H}_U$-matching in $D[U]$ will be called an $\mathcal{H}$-matching in $D[U]$.

For an $\mathcal{H}$-matching M let $V_{\mathcal{H}}(M)$ denote the set of nodes covered by a member of $\mathcal{H}$ used in M. Let $V^+(M) := \{v \in V : \delta_M(v) = 1\} \cup V_{\mathcal{H}}(M)$ and $V^-(M) := \{v \in V : \varrho_M(v) = 1\} \cup V_{\mathcal{H}}(M)$, hence $|V^+(M)| = |V^-(M)| = size(M)$. We say the set $V - V^-(M)$ consists of the *M-source-nodes* and $V - V^+(M)$ consists of the *M-sink-nodes*. M is defined to be *perfect* if it $size(M) = |V|$, or equivalently it has no sink-nodes or source-nodes.

Consider an induced subgraph $D[U]$ which is symmetric-critical. Let $D[U]$ be called *$\mathcal{H}$-critical* if there is no perfect $\mathcal{H}$-matching in $D[U]$. Here we remark that, a symmetric-critical subgraph is not necessarily induced, but only induced symmetric-critical subgraphs may be called $\mathcal{H}$-critical, by definition. (This definition of $\mathcal{H}$-critical subgraphs makes also sense because of the following observation. If some digraph has a symmetric-critical spanning subgraph and has an asymmetric arc, then there exists a perfect $\mathcal{H}$-matching – this follows from Claim 3.5.)

A set $S \subseteq V$ is called a *source-component* in D if $D[S]$ is strongly connected and $\varrho_D(S) = 0$. Let $\sigma_{\mathcal{H}}(D[X])$ denote the number of those source-components in $D[X]$ which are $\mathcal{H}$-critical.

The *deficiency* of an $\mathcal{H}$-matching M is $\mathrm{def}_{D,\mathcal{H}} M := |V| - size(M)$, which is non-negative, of course. The *deficiency* of a set $X \subseteq V$ is defined by $\mathrm{def}_{D,\mathcal{H}} X := \sigma_{\mathcal{H}}(D[X]) - |\Gamma_D^+(X)|$. Let us use the notation $\tau_{D,\mathcal{H}}(X) := |V| + |\Gamma_D^+(X)| - \sigma_{\mathcal{H}}(D[X]) = |V| - \mathrm{def}_{D,\mathcal{H}} X$. Define $\tau^{\mathcal{H}}(D) := \min_{X \subseteq V} \tau_{D,\mathcal{H}}(X)$.

To prove the main theorem 3.1, in the next section we will use the following well-known statements about factor-critical undirected graphs.

Lemma 2.2. *Consider an undirected graph $G = (V, E)$. If for some $U \subseteq V$ the induced subgraph $G[U]$ is factor-critical and G/U is factor-critical, then G is factor-critical, too.* □

Lemma 2.3. *Suppose we are given a factor-critical undirected graph $G = (V, E)$ and nodes $s, t \in V$. Then there is an even path P from s to t and a perfect matching M in $G - V(P)$.* □

3 A min-max formula

In this section we prove the following theorem, the main result of the paper.

Theorem 3.1. *If $D = (V, A)$ is a digraph and $\mathcal{H}$ is a family of symmetric-critical subgraphs of D, then*

$$\nu^{\mathcal{H}}(D) = \tau^{\mathcal{H}}(D). \tag{1}$$

The easy part of the proof is to see that the left hand side in (1) is at most the right hand side. This follows from the following lemma.

Lemma 3.2. *For any $\mathcal{H}$-matching M and any set $X \subseteq V$ we have $|X - V^+(M)| \geq \sigma_{\mathcal{H}}(D[X]) - |\Gamma_D^+(X)|$.*

Proof. First we show that we may assume without loss of generality that M does not use any member H of $\mathcal{H}$ with $V(H) \cap X \neq \emptyset$, $V(H) - X \neq \emptyset$. Otherwise, if M uses some H split by X, then consider a node v in $V(H) \cap \Gamma_D^+(X)$, which is non-empty since H is strongly connected. Since H is symmetric-critical, there exists a perfect matching in the underlying undirected graph of $H - v$. We construct another $\mathcal{H}$-matching M' by replacing H in M by a subset of $A(H)$ of $|V(H)| - 1$ arcs which is the union of node-disjoint two-arc cycles covering $V(H) - v$: these two-arc cycles are constructed from a perfect matching in the underlying undirected graph of $H - v$. Then $X - V^+(M') = X - V^+(M)$.

Consider the set $M[X]$ of arcs of M induced in X. By the above assumption, $M[X]$ is an $\mathcal{H}$-matching. Consider an $\mathcal{H}$-critical source-component S in $D[X]$. Since S is a source-component, $M[S]$ is an $\mathcal{H}$-matching, too. There is no perfect $\mathcal{H}$-matching in S, hence $S - V^-(M[S]) \neq \emptyset$. Since S is a source-component, $S - V^-(M[S]) = S - V^-(M[X])$. Thus $|X - V^+(M[X])| = |X - V^-(M[X])| \geq \sigma_{\mathcal{H}}(D[X])$.

Each node a in $(X - V^+(M[X])) - (X - V^+(M))$ is covered by an arc $ab \in M$ with $b \in \Gamma^+(X)$. This arc ab can only be an arc on a cycle or a path of M, hence $|(X - V^+(M[X])) - (X - V^+(M))| \leq |\Gamma^+(X)|$. □

Definition 3.3. A set $X \subseteq V$ is a *verifying set for* an $\mathcal{H}$-matching M if $size(M) = \tau_{D,\mathcal{H}}(X)$, or equivalently $\mathrm{def}_{D,\mathcal{H}} M = \mathrm{def}_{D,\mathcal{H}} X$.

Lemma 3.2 implies $size(M) = |V^+(M) \cap X| + |V^+(M) - X| \leq |X| - \sigma_{\mathcal{H}}(D[X]) + |\Gamma_D^+(X)| + |V - X| = \tau_{D,\mathcal{H}}(X)$, so we can easily see the following "slackness" type condition.

Claim 3.4. *If M is an $\mathcal{H}$-matching with a verifying set X, then $V - X \subseteq V^+(M)$.*

The following claim is straightforward from Lemma 2.3.

Claim 3.5. *Suppose $D = (V, A)$ is symmetric-critical, and $s, t \in V$ are two not necessarily distinct nodes. Then there is an $\mathcal{H}$-matching M_{st} for which*

1. *$size(M_{st}) = |V| - 1$,*
2. *M_{st} consists of two-arc cycles and an even $s - t$ path P_{st} (in case of $s = t$ this path has length zero).* □

Claim 3.6. *Consider symmetric-critical induced subgraph $D[U]$, suppose J induces a symmetric-critical subgraph in D/U with $\{U\} \in J$. Let R be the pre-image of J in D. Then $D[R]$ is symmetric-critical, or there is a perfect $\mathcal{H}$-matching in $D[R]$ (or both).*

Proof. By Lemma 2.2, the underlying undirected graph of $D[R]$ is factor-critical, so either $D[R]$ is symmetric-critical, or there is an asymmetric arc $ab \in A[R]$. Then ab is not induced in U, so the images a' and b' in D/U are distinct. By Claim 3.5, in D/U there is a node-disjoint family of two-arc cycles and a $b' - a'$ path internally covering J, the union of these is denoted by N'. Then by Claim 3.5, there is an expansion N in D which internally covers R by even cycles and a $b - a$ path. So $N + ab$ covers R by even cycles and one asymmetric cycle. □

Definition 3.7. Suppose $D[U]$ is a symmetric-critical induced subgraph in D, let $D' := D/U$. We construct a family $\mathcal{H}'$ in D' as follows. We put a symmetric-critical induced subgraph $D'[Q']$ into $\mathcal{H}'$ if and only if the pre-image Q of Q' induces a subgraph $D[Q]$ which is not symmetric-critical, or a symmetric-critical subgraph $D[Q]$ which is not $\mathcal{H}$-critical.

By Claim 3.6 we make the following observation on this definition.

Observation 3.8. If $D'[J] \in \mathcal{H}'$ and R is the pre-image of J in D, then there is a perfect $\mathcal{H}$-matching in $D[R]$.

Definition 3.9. Suppose $D[U]$ is symmetric-critical. We say an $\mathcal{H}$-matching N *fits* U if $\delta_N(U) = 0$, $\varrho_N(U) \leq 1$, and $size(N[U]) = |U| - 1$.

Notice, it follows from $\delta_N(U) = 0$ that N does not use any member H of $\mathcal{H}$ split by U. Thus $N[U]$ is an $\mathcal{H}$-matching, so $size(N[U])$ also makes sense. If N fits U, then N/U will be an $\mathcal{H}'$-matching in D/U with $size(N) - size(N/U) = |U| - 1$. The following Lemma shows the key property of the definition of "N fitting U" which will be used later in an inductive fashion.

Lemma 3.10. *Consider an $\mathcal{H}$-matching N in D which fits U. Define $D' = D/U$ and $N' = N/U$. For $\mathcal{H}'$, use Definition 3.7.*

1. *If N is a maximum $\mathcal{H}$-matching, then N' is a maximum $\mathcal{H}'$-matching.*
2. *If X' is a verifying set for N', then the pre-image X is a verifying set for N.*

We prove this lemma by showing Claim 3.11 and Claim 3.12 which imply the first and second assertions, respectively.

Claim 3.11. *Suppose $D[U]$ is symmetric-critical and M' is an $\mathcal{H}'$-matching in $D' = D/U$. Then*

1. *there is an $\mathcal{H}$-matching M in D of size $size(M') + |U| - 1$,*
2. *$\nu^{\mathcal{H}}(U) \geq \nu^{\mathcal{H}'}(D') + |U| - 1$.*

Proof. The second statement follows from the first, we need to prove the first. We construct M as the pre-image of M', except for the weak component of (V', M') incident with $\{U\}$.

If this component is a member $D'[J]$ of $\mathcal{H}'$, then by Observation 3.8 there is a perfect $\mathcal{H}$-matching in J's pre-image R. We add this perfect $\mathcal{H}$-matching in $D[R]$ to the pre-images of the other components of (V', M').

If this component is a cycle or a path, say Z', then Z' contains at most one arc entering $\{U\}$, and at most one arc leaving $\{U\}$. So the pre-image Z of the arcs in Z' contains at most one arc entering U, say $s's$, otherwise choose $s \in U$ arbitrarily. Similarly, Z contains at most one arc leaving U, say tt', otherwise choose $t \in U$ arbitrarily. Let M_{st} be the $\mathcal{H}$-matching in $D[U]$ from Claim 3.5. Adding M_{st} to the pre-image of M', we get an $\mathcal{H}$-matching M in D size $size(M') + |U| - 1$. □

Claim 3.12. *Suppose $D[U]$ is symmetric-critical and N is an $\mathcal{H}$-matching in D fitting U. Define $D' = D/U$ and $N' = N/U$. If X' is a verifying set in D' for N', then $\{U\} \in X'$ and the pre-image $X := X' - \{U\} \cup U$ is a verifying set for N.*

Proof. Since X' is a verifying set for N', N' is a maximum $\mathcal{H}'$-matching. By definition $\delta_{N'}(\{U\}) = 0$, thus by Claim 3.4 we get that $\{U\} \in X'$. Consider an $\mathcal{H}'$-critical source-component Q' in $D'[X']$. If $\{U\} \notin Q'$, then Q' is an $\mathcal{H}$-critical source-component in $D[X]$. If $\{U\} \in Q'$ then we claim that $Q := Q' - \{U\} \cup U$ induces an $\mathcal{H}$-critical source-component in $D[X]$, which can be seen as follows: $D[Q]$ is a source component in $D[X]$ – here we use that $D'[Q']$ and $D[U]$ are strongly connected, hence so is $D[Q]$. Moreover, by Definition 3.7 and Claim 3.6, $D[Q]$ is $\mathcal{H}$-critical.

Thus $D[X]$ has at least as many $\mathcal{H}$-critical source-components as the number $\mathcal{H}'$-critical source-components of $D'[X']$, that is $\sigma_{\mathcal{H}}(D[X]) \geq \sigma_{\mathcal{H}'}(D'[X'])$. Furthermore $\{U\} \in X'$ implies $\Gamma^+_D(X) = \Gamma^+_{D'}(X')$.

$$size(N) = size(N') + |U| - 1 = |V/U| + |\Gamma^+_{D'}(X')| - \sigma_{\mathcal{H}'}(D'[X']) + |U| - 1 \geq$$
$$\geq |V| + |\Gamma^+_D(X)| - \sigma_{\mathcal{H}}(D[X]) \geq size(N).$$

□

The following definition presents the auxiliary graph which will be used as a tool in the proof to find N fitting U.

Definition 3.13. Let M be a fixed $\mathcal{H}$-matching in $D = (V, A)$, let $\{H_1, H_2, \cdots, H_m\}$ be the family of symmetric-critical subgraphs used by M. Let $D^* = (V^*, A^*)$ be the graph with each $V(H_i)$ contracted to a single node $\{H_i\}$, a loop l_i added on this node, moreover a loop is added on a node $a \in V^*$ if $\{a\} \in \mathcal{H}$ (these nodes a are called pseudonodes). M^* is defined by replacing H_i by l_i, i.e.

$$\begin{aligned} V^* &:= V/V(H_1)/V(H_2)/\cdots/V(H_m), \\ A^* &:= A/V(H_1)/V(H_2)/\cdots/V(H_m)+ \\ &\quad + \{l_1, l_2, \cdots, l_m\} + \{f : f \text{ a loop on a pseudonode }\}, \\ M^* &:= M - H_1 - H_2 - \cdots - H_m + l_1 + l_2 + \cdots + l_m \end{aligned}$$

So M^* is a path-cycle-matching with $|M^*| = size(M) - |V| + |V^*|$. Let $K^+ = K^+(M^*) := V^* - V^+(M^*)$ be the set of sink-nodes for M^*. A sequence $W = (v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n)$ is called an M^*-*alternating walk* if

1. $v_0 \in K^+$ and $v_i \in V^*$,
2. if i is even then $e_i = v_i v_{i+1} \in A^*$,
3. if i is odd then $e_i = v_{i+1} v_i \in M^*$.

Here n is called the *length* of W, v_0 is the *first node* of W and v_n is the *last node* of W. W is called *even/odd* by the parity of its length. Let $A^*(W) = \{e_i : 0 \le i \le n-1\}$ denote the *set of arcs in* W. A node v_i with i even/odd is called an *even/odd node* of W.

Definition 3.14. An M^*-alternating walk $W = (v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n)$ is called *special* if its even nodes are pairwise distinct, and its odd nodes are pairwise distinct. The *starting segment* of a walk $(v_0, e_0, v_1, e_1, \ldots, v_n)$ of length k is $(v_0, e_0, v_1, e_1, \ldots, v_k)$. Notice, the starting segment of a special M^*-alternating walk is a special M^*-alternating walk, too.

Claim 3.15. *If for some nodes $w, z \in V^*$ there is an even M^*-alternating walk from w to z, then there is a special even M^*-alternating walk from w to z, too.*

Proof. If an M^*-alternating walk W is not special, then $v_i = v_j$ for some $i < j$, $i \equiv j \mod 2$. A shorter M^*-alternating walk is constructed by deleting the section from v_i to v_j. So a shortest even M^*-alternating walk from w to z must be special. □

Claim 3.16. *For a special even M^*-alternating walk W, $M^* \Delta A^*(W)$ is a path-cycle-matching in D^*.*

Proof. Path-cycle-matchings are exactly those arc-sets having in- and out-degree at most one in any node. A special M^*-alternating walk has the property that it traverses any arc at most once. The in-degree of nodes is only inflicted for nodes v_i with i odd. The symmetric difference $M^* \Delta A^*(W)$ is constructed in such a way that we replace an arc in M with head v_i (i odd) by another arc with head v_i. So the in-degree of nodes does not change. Similar reasoning shows that the out-degree of nodes does not change, except for v_0 and v_n. There the out-degree is 0 and 1 in M, and is 1 and 0 in $M^* \Delta A^*(W)$, respectively. □

A cycle which is symmetric in D^* is called D^*-symmetric, for short. Let us call a D^*-symmetric odd cycle C in D^* *feasible* if the pre-image of $V^*(C)$ is not $\mathcal{H}$-critical.

Claim 3.17. *If N^* is a path-cycle-matching in D^* such that all D^*-symmetric odd cycles in N^* are feasible, then there is an $\mathcal{H}$-matching N in D of size $|N^*|+|V|-|V^*|$ such that the contraction of N gives N^*.*

Proof. Notice, by Claim 3.5 for an even cycle C in D^*, there is a node-disjoint family of one even cycle and some two-arc cycles in D partitioning the pre-image of $V^*(C)$. For an asymmetric odd cycle C in D^*, there is a node-disjoint family of one asymmetric odd cycle and some two-arc cycles in D partitioning the pre-image of $V^*(C)$. For a path P in D^* there is a family of one path and some two-arc cycles in D partitioning the pre-image of $V^*(P)$. By Claim 3.6, if C is a feasible symmetric odd cycle, then there is a perfect $\mathcal{H}$-matching in the pre-image of $V^*(C)$. □

Definition 3.18. Let $L = L(D, M) \subseteq V^*$ be the set of nodes $v \in V^*$ for which there exists an even alternating walk with last node v.

Notice, a node in K^+ sets up a zero-length alternating walk, thus $K^+ \subseteq L$. So if M is not a perfect $\mathcal{H}$-matching, then L will be non-empty.

Proof of Theorem 3.1. We prove Theorem 3.1 by induction on $|V| + |A|$. Let M be a maximum $\mathcal{H}$-matching. By Lemma 3.2 it is enough to present a verifying set for M. Let D^* be the contracted graph defined in 3.13.

Case I. Suppose there is an arc $ab = e \in A^*$ with $a \in L = L(D, M)$ and $b \in V^* - V^-(M^*)$. (Arc ab may be a loop!) In this case we will find a maximum $\mathcal{H}$-matching N which fits a symmetric-critical subgraph, the proof will be completed using Theorem 3.10.

By Claim 3.15 there is a special even M^*-alternating walk W with last node a. Suppose $q = v_0 \in K^+$ is the first node, i.e.

$$W = (q = v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n = a).$$

Let $n = 2l$ be the length of W, let W_i be the starting segments of W of length $2i$ (for $i = 0, \ldots, l$). Of course, each W_i is a special even M^*-alternating walk. By Claim 3.16 $M_i^* := M^* \Delta A^*(W_i)$ are path-cycle-matchings in D^*. Notice, the only D^*-symmetric odd cycles in $M_0^* = M^*$ are the loops l_i, which are are feasible. From this fact we will only use later that D^*-symmetric odd cycles in M_0^* are feasible. It is easy to see that for $i \leq n - 1$

$$M_{i+1}^* = M_i^* + v_{2i}v_{2i+1} - v_{2i+2}v_{2i+1}, \tag{2}$$

i.e. M_{i+1}^* is obtained from M_i^* by replacing an arc entering v_{2i+1} by a different arc entering v_{2i+1}.

Subcase Ia. Suppose each D^*-symmetric odd cycle in M_l^* is feasible. It is easy to see that $M_l^* + ab$ is a path-cycle-matching. If each D^*-symmetric odd cycle in $M_l^* + ab$ is feasible, then by Claim 3.17 one could construct an $\mathcal{H}$-matching larger than M. So $M_l^* + ab$ has a unique D^*-symmetric odd cycle C which is not feasible, and we have

$ab \in C$. Let U be the pre-image of $V^*(C)$. Then $D[U]$ is symmetric-critical since C is not feasible. By Claim 3.17 applied for $N^* = M_l^*$ there is an $\mathcal{H}$-matching N of size $size(M)$ which fits U.

Subcase Ib. Suppose there is a D^*-symmetric odd cycle in M_l^* which is not feasible. Consider the smallest index $0 \leq i < l$ for which M_{i+1}^* has a D^*-symmetric odd cycle which is not feasible. So each D^*-symmetric odd cycle in M_i^* is feasible. Then by (2) there is a unique D^*-symmetric odd cycle C in M_{i+1}^* which is not feasible, and we have $v_{2i}v_{2i+1} \in C$. Let U be the pre-image of $V^*(C)$. By Claim 3.17 applied for $N^* = M_l^*$ there is an $\mathcal{H}$-matching N of size $size(M)$ which fits U.

In both subcases we have a maximum $\mathcal{H}$-matching N which fits U. By part 1 of Lemma 3.10 $N' = N/U$ is a maximum $\mathcal{H}'$-matching in $D' = D/U$. By induction, there is a verifying set X' for N' in D'. By part 2 of Lemma 3.10 there is a verifying set for N in D, which completes the proof in case I.

Case II. Suppose there is no arc $ab = e \in A^*$ with $a \in L = L(D, M)$ and $b \in V^* - V^-(M^*)$. Let $X \subseteq V$ be the pre-image of L, we will prove that X is a verifying set for M. Let $M_1 := M[X] = \{vz \in M : v \in X, z \in X\}$, $M_2 := \{vz \in M : v \in X, z \in V - X\}$ and $M_3 := \{vz \in M : v \in V - X\}$. Let S be the set of nodes v in L for which there is no arc uv with $u \in L$ – these are exactly the source-nodes in $D^*[L]$ without a loop. Notice, by definition, the pre-images of the nodes in S are $\mathcal{H}$-critical source-components in $D[X]$.

Claim 3.19. *In Case II we have* $size(M_3) = |V| - |X|$, $size(M_2) = |\Gamma_D^+(X)|$ *and* $size(M_1) = |X| - |S|$.

Proof. The sizes defined in the claim make sense because M_1, M_2, M_3 are $\mathcal{H}$-matchings, since M uses no H split by X.

The first equality follows from $K^+ \subseteq L$.

Consider a node b in $\Gamma_D^+(X)$. We claim that b is covered by a cycle or a path in M. First suppose for contradiction that $b \in V(H_i)$ for some $H_i \in \mathcal{H}$ used by M. Then $\{H_i\} \in \Gamma_{D^*}^+(L)$, so there is an arc $a\{H_i\} \in A^*$ with $a \in L$. By the definition of L there is an even M^*-alternating walk W with last node a. The extension of W by arcs $a\{H_i\}$ and l_i gives an even M^*-alternating walk with last node $\{H_i\}$, thus $\{H_i\} \in L$, a contradiction. Hence a cycle or a path in M covers b. Then $b \in V^*$, i.e. b is not a contracted node. So $b \in \Gamma_{D^*}^+(L)$, and there is an arc $ab \in A^*$ with $a \in L$. From the assumption of case II we get that $b \in V^-(M^*)$, i.e. there must be an arc $cb \in M^*$. By the definition of L there is an even M^*-alternating walk W with last node a. The extension of W by arcs ab and cb gives an even M^*-alternating walk with last node c, thus $c \in L$. But $b \in \Gamma_{D^*}^+(L)$, so $b \neq c$. Hence arc cb is not equal to any loop l_i. Then c is also a non-contracted node in V^*, so $c \in X$, $cb \in A$. Thus, each node b in $\Gamma_D^+(X)$ is covered by an arc cb of a cycle or a path in M leaving X, this proves $size(M_2) = |\Gamma_D^+(X)|$.

For the third equality, consider a node $b \in L - S$. Then there is an arc $ab \in A^*$ with $a \in L$. From the assumption of case II we get that $b \in V^-(M^*)$, i.e. there must be an arc $cb \in M^*$. (Here cb may be one of the loops l_i.) There is an even

M^*-alternating walk W with last node a. The extension of W by arcs ab and cb gives an even M^*-alternating walk with last node c, thus $c \in L$. We get that each node in $b \in L - S$ is covered by an arc $cb \in M^*$. This implies the third equality. □

By definition, there is no $H \in \mathcal{H}$ used by M split by X, which explains the first part of the following calculation, completing the proof of Theorem 3.1.

$$size(M) = size(M_1) + size(M_2) + size(M_3) =$$
$$= |V| + |\Gamma^+_D(L)| - |S| \geq |V| + |\Gamma^+_D(L)| - \sigma(D[L]) \geq size(M).$$

□□

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