Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2005-17. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Upgrading edge-disjoint paths in a ring

Jácint Szabó

December 2005

Upgrading edge-disjoint paths in a ring

Jácint Szabó*

Abstract

In this paper we introduce the *upgrading problem* for edge-disjoint paths. In the *off-line upgrading problem* a supply graph G and two demand graphs H_1 and H_2 are given on the same vertex set. What is the maximum size of a set $F \subseteq E(H_1) \cap E(H_2)$ such that F has a routing in G which can be extended to a routing of H_i in G, for $i = 1, 2$? In the *online upgrading problem* we are given a supply graph G, a demand graph H with a routing and another demand graph H_2 such that $E(H) \subseteq E(H_2)$. What is the maximum size of a set $F \subseteq E(H)$ such that the restriction of the given routing to F can be extended to routing of H_2? Thus, depending on whether the graphs are directed or undirected, we have four different versions. In this paper we give full solution for the case when G is a ring and the demand graphs are stars. All four versions are NP-complete in general.

1 Introduction

The following notions are meant both in the directed and in the undirected case. Let G be a supply graph with capacity function $c\colon E(G) \to \mathbb{N}$ and let H be a demand graph on the same vertex set V. A map $\mathcal{P}$ from $E(H)$ is a *routing of* H *in* G if, for each edge $f \in E(H)$ joining s to t, $\mathcal{P}(f)$ is an st path in G, moreover, for each edge $e \in E(G)$, at most $c(e)$ of these paths use e. The number of the paths in $\mathcal{P}$ using edge e is the *load of* e, denoted by $l_{\mathcal{P}}(e)$. For $F \subseteq E(H)$ we say that *the routing* $\mathcal{P}$ *of* H *extends the routing* $\mathcal{P}_F$ *of* F if $\mathcal{P}_F = \mathcal{P}|_F$. Now we introduce the two kinds of upgrading problems.

Definition. In the OFF-LINE UPGRADING PROBLEM we are given a supply graph G, a demand graph H_i on the same vertex set V, and a routing of H_i in G, for $i = 1, 2$. Let $\varphi_{\text{off}}(G; H_1, H_2)$ denote the maximum size of a set $F \subseteq E(H_1) \cap E(H_2)$ such that F has a routing in G which can be extended to a routing of H_i in G, for $i = 1, 2$. Determine $\varphi_{\text{off}}(G; H_1, H_2)$.

*MTA-ELTE Egerváry Research Group (EGRES), Department of Operations Research, Eötvös University, Budapest, Pázmány P. s. 1/C, Hungary H-1117. Research is supported by France Telecom R & D, by OTKA grants T037547, TS 049788 and by European MCRTN Adonet, Contract Grant No. 504438. e-mail: `jacint@elte.hu`.

Definition. In the ONLINE UPGRADING PROBLEM we are given a supply graph G, a demand graph H with a routing $\mathcal{P}$ in G, and another demand graph H_2 with a routing in G, such that $E(H) \subseteq E(H_2)$. Let $\varphi_{\text{on}}(G; \mathcal{P}; H_2)$ denote the maximum size of a set $F \subseteq E(H)$ such that $\mathcal{P}|_F$ can be extended to a routing of H_2 in G. Determine $\varphi_{\text{on}}(G; \mathcal{P}; H_2)$.

These problems are motivated by telecommunication networks. Assume that such a network is given and different set of demands arise one at a time. Every time when such a new demand graph arises, we have to route it online. Furthermore, if we prefer routings where as much already present paths are kept intact as possible then we arrive to the ONLINE UPGRADING PROBLEM. Note that here we may assume that the previous demand graph is a subgraph of the new one. This observation explains why our definition requires in the ONLINE UPGRADING PROBLEM that $E(H) \subseteq E(H_2)$.

The OFF-LINE UPGRADING PROBLEM may arise if there exists some time dependent structure of the demand graphs so that we have enough computational capacity to off-line route a sequence of demand graphs in such a way that we reroute as few already existing paths as possible. Actually, our OFF-LINE UPGRADING PROBLEM concerns with the case when there are only two demand graphs one after another, but we could also introduce the OFF-LINE k-UPGRADING PROBLEM when a sequence of k demand graphs is given and the goal is to minimize the total number of already existing paths which are rerouted. We do not consider the k-UPGRADING PROBLEM in this paper.

In the definition of the upgrading problems it is assumed that routings of H_1 and H_2 in G are given. We did this in order to exclude the NP-complete problem of finding a routing of a demand graph in G. However, in Section 4 we prove that all four versions of the UPGRADING PROBLEM is NP-complete. We do this by a reduction of the TWO-COMMODITY INTEGRAL FLOW PROBLEM of Even, Itai and Shamir [2].

In this paper we solve the UPGRADING PROBLEM in one special setting.

Definition. A *bidirected circuit* is a directed graph with vertex set $\{v_1, v_2, \ldots, v_n = v_0\}$ ($n \geq 3$) and edge set $\{v_i v_{i+1},\ v_{i+1} v_i : 0 \leq i \leq n-1\}$ (two oppositely directed circuits on the same vertex set). An undirected or a bidirected circuit is a *ring*.

Section 2 considers the directed case. We give a polynomial algorithm together with a minimax formula in the case when G is a bidirected circuit and all edges in the demand graphs have the same source vertex. We give a solution for both the ONLINE and the OFF-LINE UPGRADING PROBLEM.

In Section 3 we consider the undirected case, when G is a circuit and all edges in the demand graphs have a vertex in common. We give polynomial algorithms for both kinds of the problem, and a minimax formula for the off-line case. It seems that there exists no nice formula for the undirected online case. We will not count exact running times in this paper.

Note that if we have only one demand graph, which is a star as above, then the cut condition is necessary and sufficient for the existence of a routing in G by the max-flow-min-cut theorem. For the upgrading problem the answer is more involved even if the supply graph is a ring.

Due to its significance in telecommunication networks, many researchers studied the routing problem in rings, with only one, not necessarily star, demand graph. The cut condition is not sufficient for the existence of a fractional routing in a bidirected circuit, explaining why we can find a fractional routing in a bidirected circuit only by solving a linear program. On the contrary, in the undirected case the cut condition is sufficient, and the first combinatorial algorithm finding a fractional routing in a circuit was sketched by Schrijver, Seymour and Winkler [6]. Their method was further enhanced by Király [4]. Shepherd and Zhang [5] gave an algorithm finding a minimum weight fractional routing in an edge weighted undirected circuit. Wilfong and Winkler [7] described an algorithm finding a routing in a bidirected circuit with integer capacities, provided that a fractional routing exists. In the undirected case a routing can be given in polynomial time by the method of Frank [3]. When the value of the demands are integer not restricted to be 1, and we require them to be unsplittable, we get an NP-complete problem even if the supply graph is a ring (Cosares and Saniee [1]). In the undirected case, Schrijver, Seymour and Winkler [6] gave a combinatorial approximation algorithm which, provided that a fractional routing exists, returns an unsplittable routing requiring $\frac{3}{2}D$ additional capacity on each edge, where D is the maximum value of the demands. Their solution works for the directed case, too.

The idea of upgrading leads to other new questions of combinatorial optimization, which may prove to be interesting on their own. E.g. the edges of a bipartite graph are colored red, blue or both. Determine the maximum size of a matching consisting of red-blue edges which can be extended both to a red perfect matching and to a blue perfect matching. We do not know the complexity of this problem.

2 The directed case

Let G be a bidirected circuit with vertex set V. A directed demand graph is called a *star centered at* $s \in V$ if the source of each of its edges is s. In this section we give algorithmic proofs of minimax formulas for both kinds of the UPGRADING PROBLEM in the case when G is a bidirected circuit and both H_1 and H_2 are stars centered at the same $s \in V$. In this section G, H, H_1 and H_2 always denote such directed graphs. First we need some definitions.

Definition 2.1. From the two directions of the bidirected circuit G we choose one to be the *forward* and the other one to be the *backward* direction. Accordingly, an edge $e \in E(G)$ can be forward or backward, and from the two possible $u \to v$ paths for $u, v \in V$, $[u, v]$ denotes the forward and $\overleftarrow{[u, v]}$ the backward path (if $u = v$ then both consist of only this vertex). Let $(u, v] = [u, v] - u$. Finally, let $\overleftarrow{e} \in E(G)$ denote the reversely oriented pair of $e \in E(G)$.

For star demand graphs we may assume that each routing has a special structure.

Definition 2.2. We say that a routing $\mathcal{P}$ of H is *smooth* if there exists a vertex $z \in V - s$ such that for all demands $f \in E(H)$ with target $t \neq z$ it holds that if $t \in V[s, z]$ (resp., $t \in V[z, s]$) then $\mathcal{P}(f)$ is the forward (resp., backward) $s \to t$ path.

The demands with target z may be routed in either direction. z is called a *counter vertex* of $\mathcal{P}$.

Lemma 2.3. *For each routing $\mathcal{P}$ of a star demand graph H in a bidirected circuit, H has a smooth routing $\mathcal{P}'$ with $l_{\mathcal{P}'} \leq l_{\mathcal{P}}$.*

Proof: Assume that f_1 and f_2 are demands joining s to $t_1 \neq t_2$ resp., such that $\mathcal{P}(f_i)$ contains t_{3-i}, for $i = 1, 2$. Now rerouting both demands to the other paths we do not increase the load on any edge and we even decrease the load at some edges. So after a finite number of steps the modified routing $\mathcal{P}'$ contains no such demands f_1, f_2, implying that $\mathcal{P}'$ is smooth. $\square$

Note that this proof was algorithmic.

Definition 2.4. For a star demand graph H centered at s and for $u, v \in V$ let

$$d_H(u, v) = |\{f : f \in E(H) \text{ with target in } [u, v]\}|.$$

We say that the forward edge $e_1 \in E(G)$ with target t_1 and the backward edge $e_2 \in E(G)$ with target t_2 *face each other* if $t_1 \in V(s, t_2]$. Let $d_H(e_1, e_2) = d_H(t_1, t_2)$. Finally, for $e \in E(G)$ let

$$r_H(e) = \min\{c(\overleftarrow{e}) + c(e') - d_H(\overleftarrow{e}, e') : e' \in E(G) \text{ faces } \overleftarrow{e}\}.$$

Call a routing of some $F \subseteq E(H_1) \cap E(H_2)$ *extendible* if it can be extended to a routing of H_i in G, for $i = 1, 2$. Note that any extendible routing of $F \subseteq E(H_1) \cap E(H_2)$ has load at most $r_{H_i}(e)$ on edge $e \in E(G)$, for $i = 1, 2$. Now we prove a minimax formula for the OFF-LINE UPGRADING PROBLEM.

Theorem 2.5. *Let G be a bidirected circuit and H_1, H_2 be stars centered at $s \in V(G)$ with routings in G. We denote by H the graph with vertex set V and edge set $E(H_1) \cap E(H_2)$. Then*

$$\varphi_{\text{off}}(G; H_1, H_2) \leq |E(H)| - \max\{d_H(e_1, e_2) - r_{H_1}(e_1) - r_{H_2}(e_2)\}$$

taken over all facing pair of edges $e_1, e_2 \in E(G)$. Moreover, either equality is attained for some facing pair e_1, e_2 or $\varphi_{\text{off}}(G; H_1, H_2) = |E(H)|$.

Proof: The inequality is obvious. For the other assertion, assume that $\varphi_{\text{off}}(G; H_1, H_2) < |E(H)|$. Among all maximum size edge sets $F \subseteq E(H)$ with an extendible routing, choose one with an extendible routing $\mathcal{P}$ minimizing $\sum\{l_{\mathcal{P}}(e) : e \in E(G)\}$. Denote the extending routing of $H_i - F$ by $\mathcal{P}_i$, for $i = 1, 2$. By Lemma 2.3, we can assume that both $\mathcal{P}_1$ and $\mathcal{P}_2$ are smooth with counter vertices z_1, z_2. Assume that, say, $z_1 \in V(s, z_2]$ and redefine the counter vertex z_i of $\mathcal{P}_i$ such that the paths $[s, z_1]$ and $[z_2, s]$ are as short as possible. See Figure 1.

$|F| < |E(H)|$ so let $[s_1, s_2]$ be the minimal graph with $s_1 \in V(s, s_2]$ which contains the targets of all demands $f \in E(H) \setminus F$. By the maximality of F, the paths $\mathcal{P}_1(f)$ and $\mathcal{P}_2(f)$ are distinct for each demand $f \in E(H) \setminus F$. Thus $[s_1, s_2] \subseteq [z_1, z_2]$.

We show that we can assume that $\mathcal{P}_1(f)$ is a backward path for all $f \in E(H) \setminus F$. This clearly holds if $z_1 \neq z_2$. If $z_1 = z_2$ and there exist demands $f_1,\ f_2 \in E(H) \setminus F$ such that $\mathcal{P}_1(f_1)$ is a forward and $\mathcal{P}_1(f_2)$ is a backward path then we could reroute both paths in $\mathcal{P}_1$ hence adding f_1 and f_2 to F, a contradiction. So by possibly changing the role of H_1 and H_2, the above assumption holds.

Let $f \in E(H) \setminus F$ be a demand with target t_f. We state that there exists no demand $g \in F$ with target $t_g \neq t_f$ such that $\mathcal{P}(g)$ contains t_f. Suppose otherwise and assume that, say, $P(g)$ is a forward path. Now the routing $\mathcal{P}'$ of $F - g + f$ with $\mathcal{P}'(f) = \mathcal{P}_2(f),\ \mathcal{P}'|_{F-g} = \mathcal{P}|_{F-g}$ is clearly extendible, contradicting to the choice that $\mathcal{P}$ minimized the sum of its loads. Hence any vertex in $[s_1,\ s_2]$ is a counter vertex of $\mathcal{P}$.

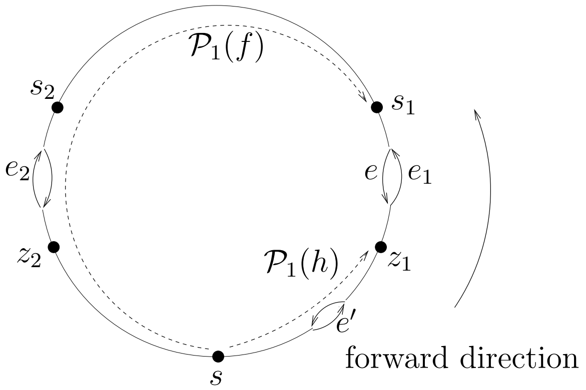

Figure 1: The off-line problem in the directed case

Let $f \in E(H) \setminus F$ be a demand with target s_1, see Figure 1. $\mathcal{P}_1(f)$ cannot be rerouted in $\mathcal{P}_1$ to the forward path by the maximality of F hence there exists a forward edge $e' \in E[s, s_1]$ such that $l_{\mathcal{P}_1}(e') + l_{\mathcal{P}}(e') = c(e')$. Note that $l_{\mathcal{P}_1}(e') > 0$ since $\mathcal{P}_2(f)$ loads e'. Thus $e' \in E[s, z_1]$ and we can choose a demand $h \in E(H_1) \setminus E(H)$ joining s to z_1 for which $\mathcal{P}_1(h)$ is a forward path. Now we cannot reroute both f and h in $\mathcal{P}_1$ by the maximality of F, thus there exists a backward edge $e \in E\overleftarrow{[s_1,\ z_1]}$ such that $l_{\mathcal{P}_1}(e) + l_{\mathcal{P}}(e) = c(e)$. As s_1 is a counter vertex of $\mathcal{P}$, we have $l_{\mathcal{P}}(e) = 0$. Let $e_1 = \overleftarrow{e}$. Summarizing, s_1 (resp. z_1) is a counter vertex of $\mathcal{P}$ (resp. $\mathcal{P}_1$), $e' \in E[s, z_1]$ and $e \in E\overleftarrow{[s_1,\ z_1]}$ so

$$r_{H_1}(e_1) \le c(e) + c(e') - d_{H_1}(e, e') = (l_{\mathcal{P}_1}(e) + l_{\mathcal{P}_1}(e')) + l_{\mathcal{P}}(e') - d_{H_1}(e, e') =$$

$$= d_{H_1 - F}(e, e') + (d_F(e, e') + l_{\mathcal{P}}(e_1)) - d_{H_1}(e, e') = l_{\mathcal{P}}(e_1).$$

Thus $l_{\mathcal{P}}(e_1) = r_{H_1}(e_1)$. Similarly, there exists a backward edge $e_2 \in E\overleftarrow{[z_2,\ s_2]}$ with $l_{\mathcal{P}}(e_2) = r_{H_2}(e_2)$. s_1 and s_2 are counter vertices of $\mathcal{P}$ hence $d_F(e_1, e_2) = l_{\mathcal{P}}(e_1) + l_{\mathcal{P}}(e_2)$. Finally,

$$d_H(e_1, e_2) - r_{H_1}(e_1) - r_{H_2}(e_2) =$$

$$= d_{H-F}(e_1, e_2) + d_F(e_1, e_2) - l_{\mathcal{P}}(e_1) - l_{\mathcal{P}}(e_2) = d_{H-F}(e_1, e_2) = |E(H) \setminus F|,$$

proving the assertion. □

One can observe that this proof is algorithmic. Starting from the empty routing $\mathcal{P}$ of $F = \emptyset$, in each step we either increase the cardinality of F or we change F keeping its cardinality such that the sum of the loads of $\mathcal{P}$ strictly decreases.

The following proof of a formula for the ONLINE UPGRADING PROBLEM is also algorithmic.

Theorem 2.6. *Let G be a bidirected circuit and H_2 be a star centered at $s \in V(G)$ with a routing in G. Let H be a subgraph of H_2 with a routing $\mathcal{P}$ in G. Then*

$$\varphi_{\text{on}}(G;\mathcal{P};\, H_2) \leq |E(H)| - \max\left\{d_{H_2-E(H)}(e_1,e_2) + l_{\mathcal{P}}(e_1) + l_{\mathcal{P}}(e_2) - c(e_1) - c(e_2)\right\}$$

taken over all facing pair of edges $e_1, e_2 \in E(G)$. Moreover, either equality is attained for some facing pair e_1, e_2 or $\varphi_{\text{on}}(G;\mathcal{P};\, H_2) = |E(H)|$.

Proof: The inequality is clear. For the second assertion, assume that $\varphi_{\text{on}}(G;\mathcal{P};\, H_2) < |E(H)|$. Let $F \subseteq E(H)$ be a maximum size edge set for which $\mathcal{P}' = \mathcal{P}|_F$ can be extended to a routing of H_2 in G. Denote this extending routing of $H_2 - F$ by $\mathcal{P}_2$. By Lemma 2.3, we may assume that $\mathcal{P}_2$ is smooth. Let z_1 and z_2 be counter vertices of $\mathcal{P}_2$ minimizing $[s, z_1]$ and $[z_2, s]$. Possibly $z_1 = z_2$. See Figure 2.

$|F| < |E(H)|$ and F is maximal so there exists a demand $f \in E(H) \setminus F$ with target t_f such that, say, $\mathcal{P}(f)$ is a backward and $\mathcal{P}_2(f)$ is a forward path. Choose f such that $[t_f, z_1]$ is minimal. $\mathcal{P}_2(f)$ cannot be rerouted to the backward path by the maximality of F hence there exists a backward edge $e \in E\overleftarrow{[s, t_f]}$ such that $l_{\mathcal{P}_2}(e) + l_{\mathcal{P}'}(e) = c(e)$. Observe that $\mathcal{P}(f)$ shows that $l_{\mathcal{P}_2}(e) > 0$, implying that $e \in E\overleftarrow{[s, z_2]}$ and that there exists a demand $h \in E(H_2) \setminus F$ with target z_2 for which $\mathcal{P}_2(h)$ is a backward path. Now we cannot reroute both $\mathcal{P}_2(f)$ and $\mathcal{P}_2(h)$ in the routing $\mathcal{P}_2$ by the maximality of F so there exists a forward edge $e_1 \in E[t_f, z_2]$ such that $l_{\mathcal{P}_2}(e_1) + l_{\mathcal{P}'}(e_1) = c(e_1)$. Note that no path in $\mathcal{P}_2(E(H) \setminus F)$ loads e_1 by the choice of f.

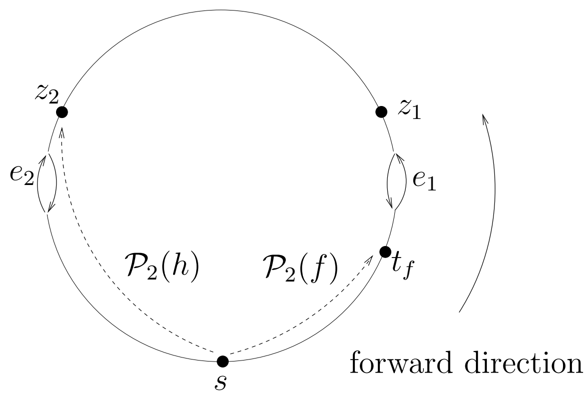

Figure 2: The online problem in the directed case

Now there are two cases. If no demand in $E(H) \setminus F$ is routed in the backward path in $\mathcal{P}_2$ then choose $e_2 = e$. Otherwise if $f' \in E(H) \setminus F$ is routed in the backward path in $\mathcal{P}_2$ then $\mathcal{P}(f')$ shows that $l_{\mathcal{P}_2}(e_1) > 0$ and hence that $e_1 \in E[t_f, z_1]$. Applying the above considerations in the backward sense we get that there exists an edge $e_2 \in \overleftarrow{[s, z_2]}$

such that $l_{\mathcal{P}_2}(e_2)+l_{\mathcal{P}'}(e_2)=c(e_2)$ and no path in $\mathcal{P}_2(E(H)\setminus F)$ loads e_2. Summarizing, in both cases $c(e_i)=l_{\mathcal{P}_2}(e_i)+l_{\mathcal{P}'}(e_i)$ $(i=1,2)$, z_1 and z_2 are counter vertices of $\mathcal{P}_2$ hence $l_{\mathcal{P}_2}(e_1)+l_{\mathcal{P}_2}(e_2)=d_{H_2-F}(e_1,e_2)$, moreover, no path in $\mathcal{P}_2(E(H)\setminus F)$ loads e_i thus $d_{H_2-F}(e_1,e_2)=d_{H_2-E(H)}(e_1,e_2)$ and

$$l_{\mathcal{P}}(e_1)-l_{\mathcal{P}'}(e_1)+l_{\mathcal{P}}(e_2)-l_{\mathcal{P}'}(e_2)\geq |E(H)\setminus F|.$$

So

$$d_{H_2-E(H)}(e_1,e_2)+l_{\mathcal{P}}(e_1)+l_{\mathcal{P}}(e_2)-c(e_1)-c(e_2)=$$

$$=d_{H_2-F}(e_1,e_2)+l_{\mathcal{P}}(e_1)+l_{\mathcal{P}}(e_2)-l_{\mathcal{P}_2}(e_1)-l_{\mathcal{P}_2}(e_2)-l_{\mathcal{P}'}(e_1)-l_{\mathcal{P}'}(e_2)\geq |E(H)\setminus F|,$$

proving the theorem. □

3 The undirected case

An undirected graph H is called a *star centered at the vertex* $s\in V$ if each edge of H is incident to s. In this section we consider the case when G is an undirected circuit and both H_1 and H_2 are stars centered at the same $s\in V$. We give an algorithmic proof of a minimax formula for the OFF-LINE UPGRADING PROBLEM. However, in the online case we give only an algorithm finding an extendible set of maximum size.

Forward and backward directions, smoothness and counter vertices are defined as in the directed case. An $s-t$ path is forward (backward) if orienting it from s to t results in a forward (backward) path.

Lemma 3.1. *For each routing $\mathcal{P}$ of a star demand graph H in a circuit, H has a smooth routing $\mathcal{P}'$ such that $l_{\mathcal{P}'}\leq l_{\mathcal{P}}$.*

Proof: It is easy to find an algorithmic proof exactly as in the directed case. □

Definition 3.2. For a star demand graph H centered at s and for $u,v\in V$ let

$$d_H(u,v)=|\{f: f\in E(H) \text{ joins } s \text{ to a vertex in } [u,v]\}|\,.$$

For the edges $e_i=u_iv_i\in E(G)$ $(i=1,2)$ we say that the ordered pair (e_1,e_2) is *facing* if the forward order of these vertices is s, u_1, v_1, u_2, v_2 (some of them may coincide). In this case let $d_H(e_1,e_2)=d_H(v_1,u_2)$. Finally, for the edge $e=uv\in E(G)$ with $u\in V[s,v]$ let

$$r^+_H(e)=\min\{c(e)+c(e')-d_H(e',e): e'\in E[s,u]\}, \text{ and}$$

$$r^-_H(e)=\min\{c(e)+c(e')-d_H(e,e'): e'\in E[v,s]\}.$$

Call a routing of some $F\subseteq E(H_1)\cap E(H_2)$ *extendible* if it can be extended to a routing of H_i in G, for $i=1,2$. Note that in any extendible routing of $F\subseteq E(H_1)\cap E(H_2)$ at most $\lfloor r^+_{H_i}(e)/2\rfloor$ forward paths and at most $\lfloor r^-_{H_i}(e)/2\rfloor$ backward paths load $e\in E(G)$, for $i=1,2$. Now we prove a minimax formula for the OFF-LINE UPGRADING PROBLEM.

Theorem 3.3. *Let G be a circuit and H_1, H_2 be stars centered at $s \in V(G)$ with routings in G. We denote by H the graph with vertex set V and edge set $E(H_1) \cap E(H_2)$. Then*

$$\varphi_{\text{off}}(G; H_1, H_2) \leq |E(H)| - \max\left\{ d_H(e_1, e_2) - \left\lfloor \frac{r^+_{H_1}(e_1)}{2} \right\rfloor - \left\lfloor \frac{r^-_{H_2}(e_2)}{2} \right\rfloor \right\}$$

taken over all facing pairs (e_1, e_2). Moreover, either equality is attained for some e_1, e_2 or $\varphi_{\text{off}}(G; H_1, H_2) = |E(H)|$.

Proof: The inequality is clear. For the other assertion, assume that $\varphi_{\text{off}}(G; H_1, H_2) < |E(H)|$. Among all maximum size edge sets $F \subseteq E(H)$ with an extendible routing, choose one with an extendible routing $\mathcal{P}$ minimizing $\sum\{l_{\mathcal{P}}(e) : e \in E(G)\}$. Denote the extending routing of $H_i - F$ by $\mathcal{P}_i$, for $i = 1, 2$. By Lemma 3.1 we can choose $\mathcal{P}_i$ to be smooth with counter vertex z_i, for $i = 1, 2$. Assume that, say, $z_1 \in V(s, z_2]$ and choose $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) to be smooth having the most number of forward (resp. backward) paths. Redefine z_1 and z_2 such that the paths $[s, z_1]$, $[z_2, s]$ are as short as possible. See Figure 3.

$|F| < |E(H)|$ so let $[s_1, s_2]$ be the minimal graph with $s_1 \in V(s, s_2]$ containing the vertices t for all demands $f \in E(H) \setminus F$ joining s to t. The maximality of F implies that $[s_1, s_2] \subseteq [z_1, z_2]$. As in the directed case, we may assume that $\mathcal{P}_1(f)$ is a backward path for all $f \in E(H) \setminus F$. Also, exactly as in the directed case, it cannot happen that there exists a demand $f \in E(H) \setminus F$ joining s to t_f and a demand $g \in F$ joining s to $t_g \neq t_f$ such that $\mathcal{P}(g)$ contains t_f, because $\mathcal{P}$ minimized the sum of its loads. Hence any vertex in $[s_1, s_2]$ is a counter vertex of $\mathcal{P}$.

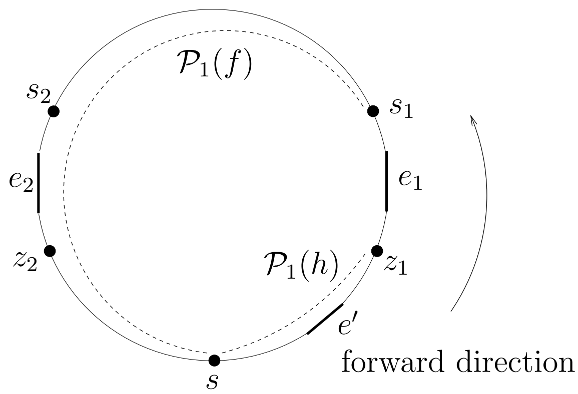

Figure 3: The off-line problem in the undirected case

Let $f \in E(H) \setminus F$ be a demand joining s to s_1, see Figure 3. $\mathcal{P}_1(f)$ cannot be rerouted in $\mathcal{P}_1$ to the forward path by the maximality of F hence there exists an edge $e' \in E[s, s_1]$ such that $l_{\mathcal{P}_1}(e') + l_{\mathcal{P}}(e') = c(e')$. We show that we can choose $e' \in E[s, z_1]$. Indeed, $\mathcal{P}_2(f)$ loads the edges of $[z_1, s_1]$ so otherwise we could reroute in $\mathcal{P}_1$ the longest backward path of $\mathcal{P}_1$, contradicting that $\mathcal{P}_1$ maximized the number of its forward paths. Now let $h \in E(H_1) \setminus E(H)$ be a demand joining s to z_1 for which $\mathcal{P}_1(h)$ is a forward path. Now we cannot reroute both f and h in $\mathcal{P}_1$ by the maximality of F. Thus there exists an edge $e_1 \in E[z_1,\, s_1]$ such that $l_{\mathcal{P}}(e_1) + l_{\mathcal{P}_1}(e_1) \geq c(e_1) - 1$. Summarizing, s_1 (resp. z_1) is a counter vertex of $\mathcal{P}$ (resp. $\mathcal{P}_1$), $e' \in E[s, z_1]$ and $e_1 \in E[z_1,\, s_1]$ so

$$r^+_{H_1}(e_1) \leq c(e_1)+c(e')-d_{H_1}(e', e_1) \leq (l_{\mathcal{P}_1}(e_1)+l_{\mathcal{P}_1}(e'))+(l_{\mathcal{P}}(e_1)+l_{\mathcal{P}}(e'))-d_{H_1}(e', e_1)+1 =$$

$$= d_{H_1-F}(e', e_1) + (d_F(e', e_1) + 2 \cdot l_{\mathcal{P}}(e_1)) - d_{H_1}(e', e_1) + 1 = 2 \cdot l_{\mathcal{P}}(e_1) + 1.$$

Thus $l_{\mathcal{P}}(e_1) = \lfloor r^+_{H_1}(e_1)/2 \rfloor$. Similarly, there exists an edge $e_2 \in E[s_2,\, z_2]$ with $l_{\mathcal{P}}(e_2) = \lfloor r^-_{H_2}(e_2)/2 \rfloor$. s_1 and s_2 are counter vertices of $\mathcal{P}$ hence $d_F(e_1, e_2) = l_{\mathcal{P}}(e_1) + l_{\mathcal{P}}(e_2)$. Finally,

$$d_H(e_1, e_2) - \lfloor r^+_{H_1}(e_1)/2 \rfloor - \lfloor r^-_{H_2}(e_2)/2 \rfloor =$$

$$= d_{H-F}(e_1, e_2) + d_F(e_1, e_2) - l_{\mathcal{P}}(e_1) - l_{\mathcal{P}}(e_2) = d_{H-F}(e_1, e_2) = |E(H) \setminus F|,$$

proving the theorem. □

The above proof is algorithmic.

Now we turn to the ONLINE UPGRADING PROBLEM. We are given a circuit G, a star H_2 centered at $s \in V(G)$ with a routing in G and a subgraph H of H_2 with a routing $\mathcal{P}$ in G. We say that $F \subseteq E(H)$ is *extendible* if $\mathcal{P}|_F$ can be extended to a routing of H_2 in G. We present an algorithm returning an extendible set $F \subseteq E(H)$ of size $\varphi_{\mathrm{on}}(G; \mathcal{P};\, H_2)$. In Theorem 3.5 possibly no facing pair (e_1, e_2) gives equality, this is why we cannot prove a nice minimax formula here. Nevertheless, if $\varphi_{\mathrm{on}}(G; \mathcal{P};\, H_2) < |E(H)|$ then the gap is at most 1.

Definition 3.4. For $e_1,\, e_2 \in E(G)$ let $\mathcal{P}_{e_1,e_2}$ denote the set of demands $f \in E(H)$ for which $\mathcal{P}(f)$ contains both e_1 and e_2.

Theorem 3.5. *Let G be a circuit and H_2 be a star centered at $s \in V(G)$ with a routing in G. Let H be a subgraph of H_2 with a routing $\mathcal{P}$ in G. Then*

$$\varphi_{\mathrm{on}}(G; \mathcal{P};\, H_2) \leq \mu := |E(H)| - \max\left\{ |\mathcal{P}_{e_1,e_2}| - \left\lfloor \frac{c(e_1) + c(e_2) - d_{H_2}(e_1, e_2)}{2} \right\rfloor \right\}$$

taken over all facing pairs (e_1, e_2). Moreover, $\varphi_{\mathrm{on}}(G; \mathcal{P};\, H_2) \in \{|E(H)|, \mu, \mu - 1\}$. We can decide which case occurs in polynomial time.

Proof: The inequality is clear. We present an algorithm showing the two other statements. The algorithm maintains an extendible set $F \subseteq E(H)$ and a routing $\mathcal{P}|_F \cup \mathcal{P}_2$ of H_2. Let $\mathcal{P}' = \mathcal{P}|_F$. In the beginning we can choose $F = \emptyset$ and $\mathcal{P}_2$ to be the given routing of H_2. In each step we either increase the size of F or find an extendible set $F^- \subseteq E(H)$ such that $|F^-| = |F|$ and the sum of the loads of $\mathcal{P}|_{F^-}$ is less than that of $\mathcal{P}'$. If $F = E(H)$ then we are done so assume otherwise.

Choose the routing $\mathcal{P}_2$ of $H_2 - F$ to be smooth and let z_1 and z_2 be counter vertices of $\mathcal{P}_2$ such that the paths $[s, z_1]$, $[z_2, s]$ are as short as possible. Whenever a demand $f \in E(H) \setminus F$ shows up with $\mathcal{P}(f) = \mathcal{P}_2(f)$ then we can add f to F, so we will assume otherwise.

First suppose that $\mathcal{P}_2(f)$ is a forward path for all demands $f \in E(H) \setminus F$. In this case choose $\mathcal{P}_2$ to be a smooth routing such that no forward path of $\mathcal{P}_2$ can be rerouted to the backward path. This means that we have an edge $e_2 \in E[z_1, s]$ such that $l_{\mathcal{P}_2}(e_2) + l_{\mathcal{P}'}(e_2) = c(e_2)$. If still $|F| < |E(H)|$ then consider a demand $f \in E(H) \setminus F$ joining s to t_f minimizing $[t_f, z_1]$. Now $\mathcal{P}(f)$ shows that actually $e_2 \in E[z_2, s]$. Let $h \in E(H_2) - E(H)$ be a demand joining s to z_2 such that $\mathcal{P}_2(h)$ is a backward path. If we can reroute both $\mathcal{P}_2(f)$ and $\mathcal{P}_2(h)$ in $\mathcal{P}_2$ then we increased F. Otherwise there exists an edge $e_1 \in E[t_f, z_2]$ such that $l_{\mathcal{P}_2}(e_1) + l_{\mathcal{P}'}(e_1) \geq c(e_1) - 1$. Now $E(H) \setminus F \subseteq \mathcal{P}_{e_1,e_2}$ by the choice of f. Hence

$$c(e_1) + c(e_2) - d_{H_2}(e_1, e_2) \leq$$

$$\leq (l_{\mathcal{P}_2}(e_1) + l_{\mathcal{P}_2}(e_2)) + (l_{\mathcal{P}'}(e_1) + l_{\mathcal{P}'}(e_2)) - d_H(e_1, e_2) - d_{H_2-E(H)}(e_1, e_2) + 1 = \quad (1)$$

$$= d_{H_2-E(H)}(e_1, e_2) + (2|F \cap \mathcal{P}_{e_1,e_2}| + d_H(e_1, e_2)) - d_H(e_1, e_2) - d_{H_2-E(H)}(e_1, e_2) + 1 =$$

$$= 2|F \cap \mathcal{P}_{e_1,e_2}| + 1,$$

showing that $|F| = \mu$, and we are done.

Now suppose that there exist demands f_1, $f_2 \in E(H) \setminus F$ joining s to t_1, t_2 resp., such that $\mathcal{P}_2(f_1)$ is a forward and $\mathcal{P}_2(f_2)$ is a backward path. See Figure 4. Choose f_1 minimizing $[t_1, z_1]$ and f_2 minimizing $[z_2, t_2]$. Let $h_i \in E(H_2) \setminus F$ be a demand joining s to z_i $(i = 1, 2)$, such that $\mathcal{P}_2(h_1)$ is a forward and $\mathcal{P}_2(h_2)$ is a backward path (possibly $h_i = f_i$). If we can reroute both $\mathcal{P}_2(f_1)$ and $\mathcal{P}_2(h_2)$ in $\mathcal{P}_2$ then we increased F. Otherwise there exists an edge $e_1 \in E[t_1, z_2]$ such that $l_{\mathcal{P}_2}(e_1) + l_{\mathcal{P}'}(e_1) \geq c(e_1) - 1$. Both $\mathcal{P}(f_1)$ and $\mathcal{P}(f_2)$ load the edges of $[z_1, z_2]$ so $e_1 \in E[t_1, z_1]$. Similar considerations give an edge $e_2 \in E[z_2, t_2]$ such that $l_{\mathcal{P}_2}(e_2) + l_{\mathcal{P}'}(e_2) \geq c(e_2) - 1$. If we can choose e_1 or e_2 such that strict inequality occurs then we are done. Indeed, $E(H) \setminus F \subseteq \mathcal{P}_{e_1,e_2}$ by the choice of f_1 and f_2, thus we can argue as in (1).

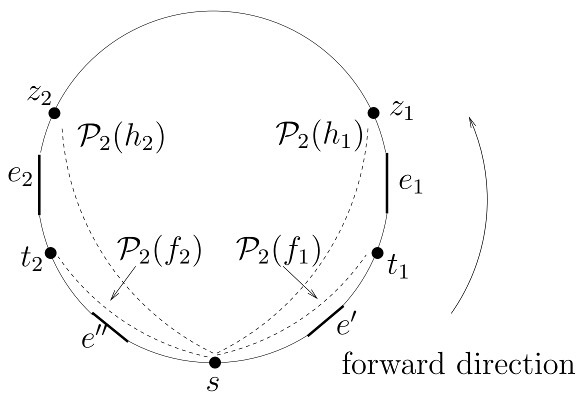

Figure 4: The online problem in the undirected case

So assume that $l_{\mathcal{P}_2}(e) + l_{\mathcal{P}'}(e) \leq c(e) - 1$ holds for all $e \in E[t_1, z_1] \cup E[z_2, t_2]$. $\mathcal{P}(f_1)$ and $\mathcal{P}(f_2)$ shows that this holds for all $e \in E[z_1, z_2]$, too. If it also holds for all $e \in E[t_2, s]$ then we can reroute $\mathcal{P}_2(f_1)$ increasing F. Hence assume that we have an edge $e'' \in E[t_2, s]$ such that $l_{\mathcal{P}_2}(e'') + l_{\mathcal{P}'}(e'') = c(e'')$, and similarly, we have an edge $e' \in E[s, t_1]$ such that $l_{\mathcal{P}_2}(e') + l_{\mathcal{P}'}(e') = c(e')$.

We say that $F' \subseteq H$ is *nice* if there exist no two demands $f' \in F' \cap \mathcal{P}_{e_1,e_2}$, $f'' \in \mathcal{P}_{e_1,e_2} \setminus F'$ such that $\mathcal{P}(f'')$ is a proper subpath of $\mathcal{P}(f')$. We can assume that F is nice since otherwise the routing $\mathcal{P}|_{F-f'+f''}$ of $F - f' + f''$ can be extended to a routing of H_2 and the sum of its loads is less than that of $\mathcal{P}'$.

$E(H) \setminus F \subseteq \mathcal{P}_{e_1,e_2}$ so exactly as in (1) we get that

$$c(e_1) + c(e_2) - d_{H_2}(e_1, e_2) \leq 2|F \cap \mathcal{P}_{e_1,e_2}| + 2. \tag{2}$$

Thus $\varphi_{\text{on}}(G; \mathcal{P}; H_2) \leq |F| + 1$. Assume that $F^* \subseteq E(H)$ is extendible and $|F^*| = |F| + 1$. Then clearly equality holds in (2). Let $\mathcal{P}^* = \mathcal{P}|_{F^*}$ and let $\mathcal{P}^* \cup \mathcal{P}_2^*$ be a routing of H_2. Now

$$2|F^* \cap \mathcal{P}_{e_1,e_2}| \geq 2|F \cap \mathcal{P}_{e_1,e_2}| + 2 = c(e_1) + c(e_2) - d_{H_2}(e_1, e_2) \geq$$

$$\geq (l_{\mathcal{P}_2^*}(e_1) + l_{\mathcal{P}_2^*}(e_2)) + (l_{\mathcal{P}^*}(e_1) + l_{\mathcal{P}^*}(e_2)) - d_H(e_1, e_2) - d_{H_2 - E(H)}(e_1, e_2) \geq$$

$$\geq d_{H_2 - E(H)}(e_1, e_2) + (2|F^* \cap \mathcal{P}_{e_1,e_2}| + d_H(e_1, e_2)) - d_H(e_1, e_2) - d_{H_2 - E(H)}(e_1, e_2) =$$

$$= 2|F^* \cap \mathcal{P}_{e_1,e_2}|,$$

so equality holds throughout. First it follows that $E(H) \setminus F^* \subseteq \mathcal{P}_{e_1,e_2}$. Second, for $i = 1, 2$, $l_{\mathcal{P}_2^*}(e_i) + l_{\mathcal{P}^*}(e_i) = c(e_i)$, thus $l_{\mathcal{P}_2}(e_i) = l_{\mathcal{P}_2^*}(e_i)$ holds, because $l_{\mathcal{P}^*}(e_i) = l_{\mathcal{P}'}(e_i) + 1$. Third, we get that for all $f' \in E(H_2) - E(H)$ the path $\mathcal{P}_2^*(f')$ does not contain both e_1 and e_2. With the notation $\mathcal{P}_- = (\mathcal{P}' \cup \mathcal{P}_2)|_{E(H_2) \setminus \mathcal{P}_{e_1,e_2}}$ and $\mathcal{P}_-^* = (\mathcal{P}^* \cup \mathcal{P}_2^*)|_{E(H_2) \setminus \mathcal{P}_{e_1,e_2}}$ we get that $l_{\mathcal{P}_-}(e) = l_{\mathcal{P}_-^*}(e)$ for all $e \in E[s, t_1] \cup E[t_2, s]$. We can assume that also F^* is nice so if, say, $|\{f \in F^* \setminus F : \mathcal{P}(f) \text{ is a forward path}\}| = g > 0$ then $|\{f \in F \setminus F^* : \mathcal{P}(f)$ is a backward path$\}| = g - 1$. But then $l_{\mathcal{P}^* \cup \mathcal{P}_2^*}(e') \geq l_{\mathcal{P}' \cup \mathcal{P}_2}(e') + g - (g-1) = c(e') + 1$, a contradiction. So F is already maximum. □

4 Complexity issues

In this section we prove that the UPGRADING PROBLEM is NP-complete in all four versions for general graphs. Even, Itai and Shamir [2] proved that the following TWO-COMMODITY INTEGRAL FLOW PROBLEM is NP-complete in both the directed and in the undirected version. Given a graph G with vertices s_1, t_1, s_2, $t_2 \in V(G)$ and integers k_1, k_2. Decide if G has a collection of edge-disjoint paths consisting of k_1 paths joining s_1 to t_1 and k_2 paths joining s_2 to t_2.

The following definition is meant both in the directed and in the undirected case.

Definition 4.1. Given a supply graph G and a demand graph H on the same vertex set. We say that we add a new demand edge f to H *blocking* the edge set $\{e_1, \ldots, e_l\} \subseteq E(G)$, where $l = 1$ or $l = 2$, if we add f to $E(H)$ and modify G as shown in Figure 5 (in the undirected case forget the orientations in the figure). We say that the routing sa_ib_it of f *forbids* e_i, $1 \leq i \leq l$, see Figure 5.

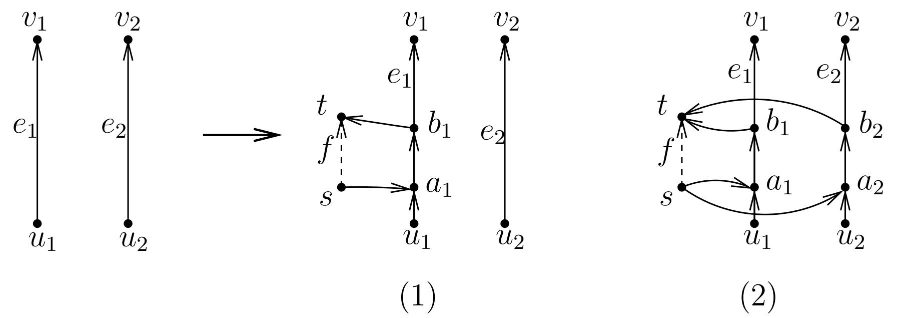

Figure 5: Blocking the edge sets (1) $\{e_1\}$ and (2) $\{e_1, e_2\}$

Observe that the addition of a demand to H which blocks $\{e_1\}$ is equivalent to deleting e_1 from G both in the directed and in the undirected setting. Similarly, adding a demand f blocking a pair $\{e_1, e_2\}$ is tantamount to that at most one of e_1 and e_2 can be used in any routing of H. This edge is the one which is not forbidden by the routing of f. This is clear for directed graphs and also easy to see for the undirected case.

Theorem 4.2. *All four versions of the* UPGRADING PROBLEM *are NP-complete.*

Proof: We detail only the directed version, since the undirected case is analogous. Let G' be a directed graph with vertices s_1, t_1, s_2, $t_2 \in V(G')$ and integers k_1, k_2, an instance of the DIRECTED TWO-COMMODITY INTEGRAL FLOW PROBLEM. Let $k = \max\{k_1, k_2\}$.

We construct an auxiliary graph for later reference. Add two vertices s and t to $V(G')$, and add k_i parallel ss_i edges and k_i parallel t_it edges for $i = 1, 2$ to $E(G')$, resulting in the directed graph I. Let the capacity of each edge be 1. We will need that one can present an integer $s - t$ flow in I of value $k_1 + k_2$ in polynomial time, if a fractional one exists.

We construct another auxiliary graph G'', which will be used as a skeleton of the supply graph in our reductions. Add four new vertices u_1, u_2, v_1, v_2 to $V(G')$, and add k parallel u_is_j and t_jv_i edges, for $i, j \in \{1, 2\}$, to $E(G')$, as shown in Figure 6 with solid edges.

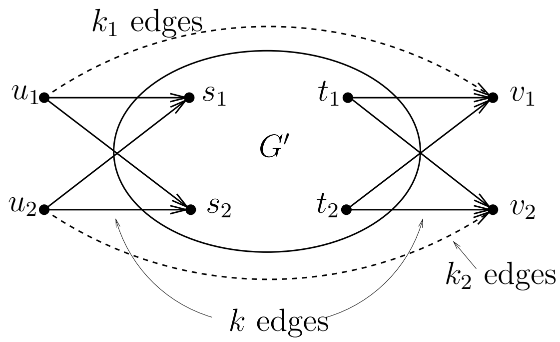

Figure 6: The auxiliary graph G''

Now we turn to the OFF-LINE UPGRADING PROBLEM. Let $E(H_1) \cap E(H_2)$ consist of k_i parallel u_iv_i edges, as shown in Figure 6 by dashed lines. Construct $E(H_1) - E(H_2)$

by blocking $\{e\}$ for each edge e joining u_i to s_{3-i}, for $i = 1, 2$. Also, construct $E(H_2) - E(H_1)$ by blocking $\{e\}$ for each edge e joining t_{3-i} to v_i, for $i = 1, 2$. G is defined to be the modified supply graph, with all capacities 1. From an integer $s-t$ flow in I of value $k_1 + k_2$ one can easily construct a routing of both H_1 and H_2 in G. On the other hand, if no such flow exists in I then the DIRECTED TWO-COMMODITY INTEGRAL FLOW PROBLEM in G' has clearly no solution. Finally, observe that $E(H_1) \cap E(H_2)$ has a routing in G which can be extended both to a routing of H_1 and to a routing of H_2 if and only if G' has a collection of edge-disjoint paths consisting of k_1 paths joining s_1 to t_1 and k_2 paths joining s_2 to t_2.

In the ONLINE UPGRADING PROBLEM let $E(H_2) - E(H)$ consist of k_i parallel $u_i v_i$ edges, for $i = 1, 2$, as in Figure 6. First let $E(H) = \emptyset$. Now for $i = 1, 2$ take a perfect matching between the k edges joining u_i to s_1 and the k edges joining u_i to s_2, and for each pair e, e' in this matching add a demand edge to H blocking $\{e, e'\}$. Similarly, for $i = 1, 2$ take a perfect matching between the k parallel $t_1 v_i$ and $t_2 v_i$ edges, and block each pair in this matching in H. These blocking demand edges form $E(H)$, and the modified supply graph G'' is denoted by G, with all capacities 1. Finally, let $\mathcal{P}$ be the routing of H in G which forbids exactly the edges of the form $u_i s_{3-i}$ and $t_{3-i} v_i$, for $i = 1, 2$. From an integer $s-t$ flow in I of value $k_1 + k_2$ one can easily construct a routing of H_2 in G. On the other hand, if no such flow exists in I then the DIRECTED TWO-COMMODITY INTEGRAL FLOW PROBLEM in G' has no solution. Now we only have to observe that $\mathcal{P}$ can be extended to a routing of H_2 in G if and only if G' has a collection of edge-disjoint paths consisting of k_1 paths joining s_1 to t_1 and k_2 paths joining s_2 to t_2. $\square$

Acknowledgements

The author wishes to thank Zsolt Csizmadia for raising the issue of upgrading, and also László Szegő for helpful discussions.

References

[1] S. COSARES, I. SANIEE, An optimization problem related to balancing loads on SONET rings. *Telecom. Systems* (1994) **3** 165–181.

[2] S. EVEN, A. ITAI, A. SHAMIR, On the complexity of timetable and multicommodity flow problems. *SIAM J. Comput.* (1976) **5** 691–703.

[3] A. FRANK, Edge-disjoint paths in planar graphs. *J. Combin. Theory, Ser. B.* (1985) **38** 164–178.

[4] Z. KIRÁLY An $O(n^2)$ algorithm for ring routing. *EGRES Technical Report* TR-2005-10, `www.cs.elte.hu/egres`

[5] B. SHEPHERD, L. ZHANG A cycle augmentation algorithm for minimum cost multicommodity flows on a ring. *Discrete Appl. Math.* (2001) **110** 301–315.

[6] A. Schrijver, P. Seymour, P. Winkler, The ring loading problem. *SIAM J. Discrete Math.* (1998) **11** 1–14.

[7] G. Wilfong, P. Winkler, Ring routing and wavelength translation. *Proceedings of the Ninth Annual ACM-SIAM Symposium on Discrete Algorithms (San Francisco, CA)*, (1998) 333–341.