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Hardness results for well-balanced orientations

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Abstract

In 1960 Nash-Williams proved his strong orientation theorem about the existence of *well-balanced* orientations. In this paper we show that it is NP-hard to find a minimum cost well-balanced orientation (given the cost for the two possible orientations of each edge) or a well-balanced orientation satisfying lower and upper bounds on the out-degrees at each node. Similar results are proved for best-balanced orientations and other related problems are considered, too.

1 Introduction

More than 45 years ago, in 1960 Nash-Williams announced his strong orientation theorem which states the following (see [9]):

Theorem 1 (Nash-Williams' Strong Orientation Theorem). *Every undirected graph has a best-balanced orientation.*

Let us give the necessary definitions and notations. If not specified otherwise, a graph will mean an undirected graph; we will use the term *digraph* for a directed graph. Our (di-)graphs might have parallel edges, but cannot contain loops. A multigraph is a graph with a function on its edge set expressing the multiplicities of the edges.

Definition 1. *Given two vertices x, y of an undirected (directed) graph G we denote by $\lambda_G(x, y)$ the maximum number of edge-disjoint undirected (directed) paths from x to y in G.*

Definition 2. *A* well-balanced orientation *of a graph $G = (V, E)$ is an orientation $\vec{G}$ satisfying*

$$\lambda_{\vec{G}}(x, y) \geq \lfloor \lambda_G(x, y)/2 \rfloor \text{ for all } x, y \in V.$$

If furthermore the in-degree and the out-degree of any node differs by at most one in $\vec{G}$ then we call it a best-balanced orientation.

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We have to mention that Nash-Williams originally (in [10]) used the term well-balanced for an orientation that we call here best-balanced, but we will follow the notations of [7].

In fact, Nash-Williams proved something stronger, the so called *odd-vertex pairing theorem*. Let us state this theorem here, too, along with the necessary definitions.

Notations: Given a graph $G = (V, E)$ and a vertex set $X \subseteq V$, the *degree* of this set (denoted by $d_G(X)$) is the number of edges with exactly one endpoint in X. If $X = \{x\}$ then we will simply write $d_G(x)$. The number of induced edges by a set $X \subseteq V$ will be denoted by $i_G(X)$. We will omit the subscripts if no confusion can arise. T_G will denote the set of odd degree vertices in G. Obviously, $|T_G|$ is even. For a set $X \subseteq V$ define $R_G(X) = \max\{\lambda_G(x, y) : x \in X, y \in V - X\}$ (let $R_G(\emptyset) = R_G(V) = 0$) and $b_G(X) = d_G(X) - 2\lfloor R_G(X)/2\rfloor$.

Definition 3. *An* odd-vertex pairing *(or shortly* pairing*) of an undirected graph $G = (V, E)$ is a new graph M on the vertex set of G such that $d_M(x) = 1$ for every $x \in T_G$ and $d_M(x) = 0$ for every $x \in V - T_G$. A pairing is called* feasible *if $d_M(X) \le b_G(X)$ for every $X \subseteq V$.*

Theorem 2 (Nash-Williams' Odd-Vertex Pairing Theorem). *Every undirected graph has a feasible odd-vertex pairing.*

It is not so hard to see that the odd-vertex pairing theorem implies the strong orientation theorem. However, the methods needed for the proof of the odd-vertex pairing theorem are so different from other methods in graph theory that no relation with other results could be found so far, though this question intrigued many mathematicians in the past 45 years: these two theorems form an isolated island in our knowledge of graph theory. In 1978 Mader [8] announced a new proof of the strong orientation theorem using his result on *admissible splitting-offs* (*admissible lifting* in the terminology of [8]). Partly based on these ideas a new and simpler proof of the odd-vertex pairing theorem was found by András Frank in 1993 (see [2]), but it still needs a sophisticated argument so it does not give a generalization of the odd-vertex pairing theorem.

The above mentioned two proofs of the odd-vertex pairing theorem (the original due to Nash-Williams and that of András Frank) both imply a polynomial algorithm to find an odd-vertex pairing, though it is not explicitly stated in either of them. An explicit algorithm for this problem is sketched in [3], it states that an odd-vertex pairing (and consequently a best-balanced orientation) can be found in $O(nm^2)$ time in a graph and in $O(n^6)$ time in a multigraph.

Recently, a different approach was used by Király and Szigeti [7]: they looked at the consequences of the odd-vertex pairing theorem on well-balanced orientations. They found very interesting results which demonstrate that the odd-vertex pairing theorem is *in fact stronger* than the strong orientation theorem.

Many related questions were attacked by the authors of [6]. Mostly, they found negative results, counter-examples for many related problems and they raised some interesting questions that are open at the moment.

It is a natural question whether one can find a well-balanced orientation of minimum cost (with costs given for the two orientations of every edge) or whether one can find a well-balanced orientation satisfying some other constraints, for example lower and upper bounds on the out-degrees at each node. In his 1993 survey paper [2] András Frank mentions these questions when he writes the following about his proof of the odd-vertex pairing theorem: *I keep feeling that there must be an even more illuminating proof which finally will lead to methods to solve the minimum cost and/or degree-constrained well-balanced orientation problem.* Here we present negative answers to these hopes: we prove the NP-completeness of these problems. Let us introduce the problems we want to consider and give some motivation.

Notations: If $\vec{G} = (V, A)$ is a directed graph and $X \subseteq V$ is a vertex-set then $\varrho_{\vec{G}}(X)$ denotes the number of edges entering X, while $\delta_{\vec{G}}(X)$ is the number of edges leaving X. We will write $\varrho_{\vec{G}}(v)$ instead of $\varrho_{\vec{G}}(\{v\})$ for a $v \in V$ and the same applies to the $\delta_{\vec{G}}$ function. We will omit $\vec{G}$ from the subscript if no confusion can arise.

For well-balanced orientations we look at the following problems:

Problem 1. *:* MinCostWellBalanced
Instance*: A graph G, nonnegative integer costs for the two orientations of each edge, integer K.*
Question*: Is there a well-balanced orientation of G with total cost not more than K?*

Problem 2. *:* BoundedWellBalanced
Instance*: A graph $G = (V, E)$, $l, u : V \mapsto \mathbb{Z}_+$ bounds with $l \leq u$.*
Question*: Is there a well-balanced orientation $\vec{G}$ of G with $l(v) \leq \delta_{\vec{G}}(v) \leq u(v)$ for every $v \in V$?*

Problem 3. *:* MinNodeCostWellBalanced
Instance*: A graph G, integer costs $c : V \mapsto \mathbb{Z}$, integer B.*
Question*: Is there a well balanced orientation $\vec{G}$ of G with $\sum_{v \in V}(c(v)\delta_{\vec{G}}(v)) \leq B$?*

For best-balanced orientations we consider the following problems:

Problem 4. *:* MinCostBestBalanced
Instance*: A graph G, nonnegative integer costs for the two orientations of each edge, integer K.*
Question*: Is there a best-balanced orientation of G with total cost not more than K?*

Problem 5. *:* BoundedBestBalanced
Instance*: A graph $G = (V, E)$, $l, u : V \mapsto \mathbb{Z}_+$ bounds with $\lfloor d_G(v)/2 \rfloor \leq l(v) \leq u(v) \leq \lceil d_G(v)/2 \rceil$ for each $v \in V$.*
Question*: Is there a well-balanced orientation $\vec{G}$ of G with $l(v) \leq \delta_{\vec{G}}(v) \leq u(v)$ for every $v \in V$ (i.e. a best-balanced orientation with these bounds)?*

Problem 6. *:* MinNodeCostBestBalanced
Instance*: A graph G, integer costs $c : V \mapsto \mathbb{Z}$, integer B.*
Question*: Is there a best-balanced orientation $\vec{G}$ of G with $\sum_{v \in V}(c(v)\delta_{\vec{G}}(v)) \leq B$?*

Problems MinCostWellBalanced and MinCostBestBalanced are quite natural weighted versions of the original problem, the problem of finding a well-balanced or a best-balanced orientation. The constrained versions BoundedWellBalanced and BoundedBestBalanced also arise naturally: problem BoundedBestBalanced was asked from me by András Frank and a related problem, when we have only bounds from one side (say, lower bounds) is still an open problem mentioned in [1]. The third approach is motivated by the following observation: in an orientation problem with edge-connectivity requirements, finding a good out-degree function is polynomially equivalent with finding a good orientation. The authors of [6] introduce the following polyhedron for a graph $G = (V, E)$ (see section 9 in [6]):

$$\begin{aligned} P := \ \{x \in \mathbb{R}^V : x(Z) \geq i_G(Z) + \lfloor R_G(Z)/2 \rfloor \ \forall Z \subseteq V, x(V) = |E|, \\ \lfloor d_G(v)/2 \rfloor \leq x(v) \leq \lceil d_G(v)/2 \rceil \ \forall v \in V\}. \end{aligned}$$

This polyhedron corresponds to the fractional relaxations of good out-degree functions of a best-balanced orientation. It is proved in [6] that this polyhedron is not necessarily integral: here we prove that optimization over the integer hull of this polyhedron (that is, problem MinNodeCostBestBalanced) is NP-complete. Problem MinNodeCostWellBalanced is just the counterpart of this problem for well-balanced orientations.

We conclude the section with some known results that will be needed later.

The well-known theorems of Menger imply the following: if G is a graph then

$$\lambda_G(x, y) = \min\{d_G(X) : X \subseteq V, x \in X, y \notin X\}.$$

Similarly, for a directed graph $\vec{G}$ we have

$$\lambda_{\vec{G}}(x, y) = \min\{\delta_{\vec{G}}(X) : X \subseteq V, x \in X, y \notin X\}.$$

The following is a simple observation: the proof is left to the reader.

Lemma 1. *If $\vec{G}$ and $\vec{G}'$ are two orientations of a graph $G = (V, E)$ with $\delta_{\vec{G}}(x) = \delta_{\vec{G}'}(x)$ for all $x \in V$ then $\vec{G}'$ can be obtained from $\vec{G}$ by reversing directed cycles.* □

Corollary 1. *If $\vec{G}$ and $\vec{G}'$ are two orientations of a graph $G = (V, E)$ with $\delta_{\vec{G}}(x) = \delta_{\vec{G}'}(x)$ for all $x \in V$ then*

$$\vec{G} \text{ is well-balanced} \iff \vec{G}' \text{ is well-balanced.}$$

Proof. Directly from lemma 1. Alternatively, we can show that $\lambda_{\vec{G}}(x, y) = \lambda_{\vec{G}'}(x, y)$ for all $x, y \in V$ using the fact $\delta_{\vec{G}}(X) = \sum_{x \in X} \delta_{\vec{G}}(x) - i_G(X) = \delta_{\vec{G}'}(X)$ for any $X \subseteq V$. □

2 Hardness results for well-balanced orientations

For well-balanced orientations we have the following result.

Theorem 3. *The problems* MinCostWellBalanced, BoundedWell-Balanced *and* MinNodeCostWellBalanced *are NP-complete.*

Proof. The problems are clearly in NP. In order to show their completeness we will give a reduction from Vertex Cover (see [5], Problem GT1). For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of the Vertex Cover problem consider the following undirected graph $G = (V, E)$. The node set V will contain one designated node s, $d_{G'}(v) + 1$ nodes $x_0^v, x_1^v, x_2^v, \ldots, x_{d_{G'}(v)}^v$ for every $v \in V'$, and one node x_e for every $e \in E'$ (in fact Vertex Cover remains NP-complete even if $d_{G'}(v) \leq 3$ for all $v \in V'$, as is shown in [4], but we will not use this fact). Let us fix an ordering of V', say $V' = \{v_1, v_2, \ldots, v_n\}$. The edge set E contains a circle on $s, x_0^{v_1}, x_0^{v_2} \ldots, x_0^{v_n}$ in this order, one edge from s to x_1^v for every $v \in V'$, edges between x_i^v and x_{i+1}^v for every $v \in V'$ and every i between 0 and $d_{G'}(v) - 1$, two parallel edges between s and x_e for every $e \in E'$ and finally for each $v \in V'$ take an arbitrary order of the $d_{G'}(v) = d$ edges of G' incident to v, say $e^1, e^2, \ldots, e^d$ and include the edge $(x_i^v, x_{e^{i-1}})$ for any $2 \leq i \leq d-1$ and the edges $(x_d^v, x_{e^{d-1}})$ and (x_d^v, x_{e^d}) (i.e. distribute the edges of G' incident to v arbitrarily among nodes $x_2^v, \ldots, x_d^v$ resulting $d_G(x_i^v) = 3$ for each $2 \leq i \leq d$). Let us call this the "first construction" (to distinguish from a modification of it to be given later).

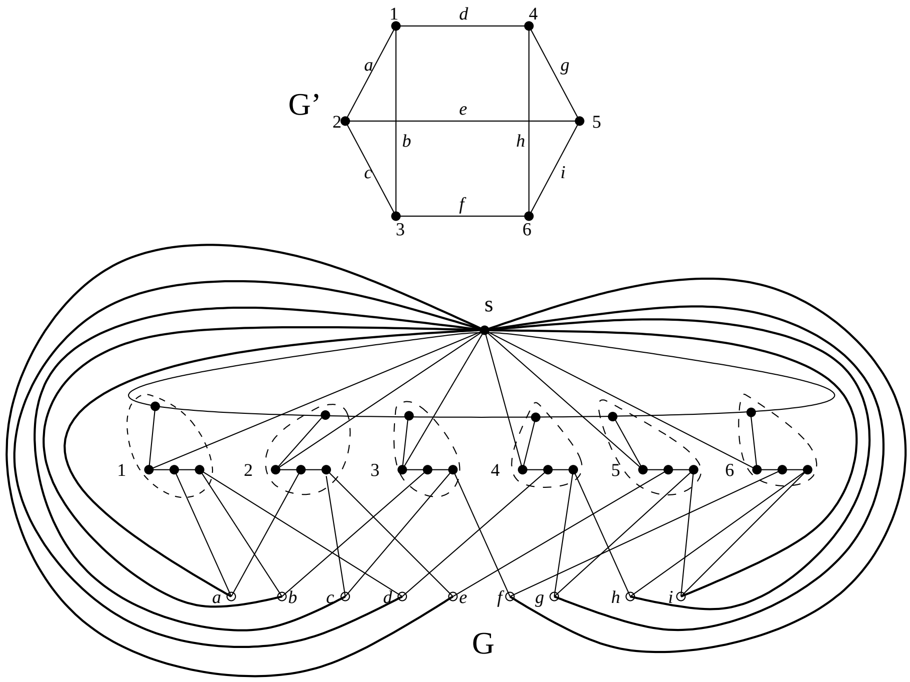

Figure 1: The construction of the graph G

The construction is illustrated in Figure 1. The edges drawn bold indicate a multiplicity of 2.

Notice, that for every $v \in V'$ and $0 \leq i \leq d_{G'}(v)$ we have $d_G(x_i^v) = 3$ and for every $e \in E'$ we have $d_G(x_e) = 4$. What is more, it is easy to check, that $\lambda_G(x, y) = \min(d_G(x), d_G(y))$ for every $x, y \in V$ (for example one can check that this is true if $y = s$ from which it follows for arbitrary x, y).

Define a partial orientation of G: orient the $s, x_0^{v_1}, x_0^{v_2} \ldots, x_0^{v_n}$ circle to become a directed circle in this order, orient the edges from x_i^v to x_{i+1}^v for every $v \in V'$ and every i between 0 and $d_{G'}(v) - 1$, orient the two parallel edges from x_e towards s for every $e \in E'$ and finally for each $v \in V'$, $2 \leq i \leq d_{G'}(v)$ and $e \in E'$ if there is an edge between x_i^v and x_e then orient this edge from x_i^v to x_e (so we have given the orientation of every edge except those of form (s, x_1^v) for $v \in V'$). Again, Figure 2 is an illustration.

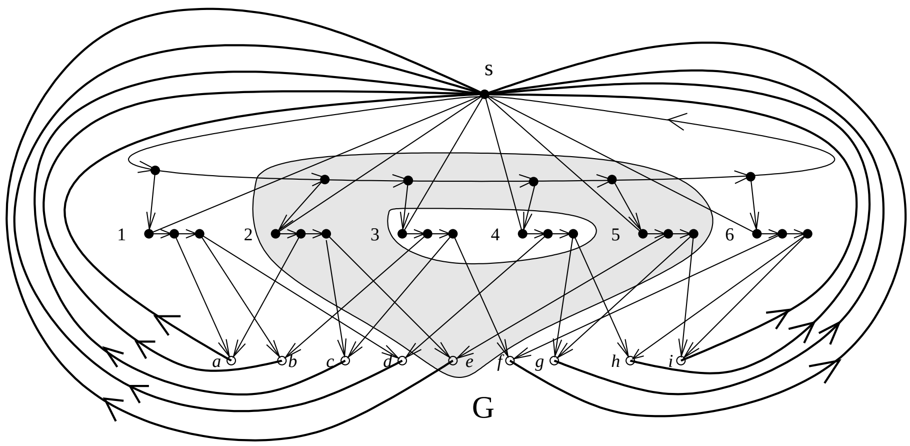

Figure 2: The partial orientation and the cut

Let us call the subgraph $G - \{(s, x_1^v) : v \in V'\}$ by G_1 and the above given orientation of this graph by $\vec{G}_1$. Observe that $\vec{G}_1$ is a strongly connected graph and $\lambda_{\vec{G}_1}(x_e, s) = 2$ for each $e \in E'$.

Claim 1. *Problem* MinCostWellBalanced *is NP-complete.*

Proof: For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinCostWellBalanced: let the graph G be as described above, let $K = k$ be the bound on the total cost and define the orientation-costs as follows. For the edges of G_1 orienting these edges costs nothing in the way as given in $\vec{G}_1$, but reversing any one will cost exactly $k + 1$. We only have to tell the costs of orientations of edges between s and x_1^v for each $v \in V'$: such an edge costs 1, if oriented from s to x_1^v and 0 in the other direction. So we only have freedom choosing the orientation of these edges, if we don't want to exceed the cost limit k.

First we claim that if there is a vertex cover $S \subseteq V'$ of size not more than k then there is a well-balanced orientation $\vec{G}$ of G of cost not more than k: for each $v \in S$ orient the edge (s, x_1^v) from s to x_1^v and orient the other edges in the direction which costs nothing. This has clearly cost at most k and it is easy to check that $\lambda_{\vec{G}}(s, x_e) = 2$ for each $e \in E'$ which together with the former observations gives that $\vec{G}$ is well-balanced.

On the other hand suppose that we have found a well-balanced orientation $\vec{G}$ of G of cost at most k: this is only possible if there are at most k vertices in V' such that the edges (s, x_1^v) are oriented from s to x_1^v exactly for these edges and all the other edges are oriented in the direction which costs 0. We claim that these vertices form a vertex cover of G': if edge $e = (v_j, v_k) \in E'$ was not covered (where $j < k$ are the indices of the vertices in the fixed ordering), then $\varrho_{\vec{G}}(X) = 1$ would contradict the well-balancedness of $\vec{G}$, where

$$\begin{aligned} X &= \{x_e\} \bigcup \{x_0^{v_i} : j \leq i \leq k\} \\ &\quad \bigcup \{x_i^{v_j} : 1 \leq i \leq d_{G'}(v_j)\} \bigcup \{x_i^{v_k} : 1 \leq i \leq d_{G'}(v_k)\} \end{aligned}$$

(Figure 2 illustrates the cut, too). □

Claim 2. *Problem* BoundedWellBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of BoundedWellBalanced: let the graph G be as described above and upper bound on the out-degree of s given by $u(s) = k + 1$, and lower bounds $l(x_i^v) = 2$ for each $v \in V'$ and $i \in \{0, 2, \ldots, d_{G'}(v)\}$ (observe that these are in fact exact prescriptions for these out-degrees, notice, that we excluded $i = 1$): the other bounds can be trivial, that is $l(x) = 0$ and $u(x) = d_G(x)$ if it was not specified otherwise.

If there is a vertex cover $S \subseteq V'$ of size not more than k then there is a well-balanced orientation $\vec{G}$ of G satisfying the given lower and upper bounds: this is the same as was described in the previous claim.

Conversely, if there is a well-balanced orientation $\vec{G}$ of G that satisfies the given bounds, then the out-degrees of this orientation are necessarily the same as an orientation $\vec{G}'$ obtained by finishing the partial orientation given by $\vec{G}_1$: vertices of form x_e ($e \in E'$) will have out-degree necessarily 2 in every well-balanced orientation (of course vertices of degree 3 will have out-degree 1 or 2), so using $\sum_{x \in V} \varrho_{\vec{G}}(x) = \sum_{x \in V} \delta_{\vec{G}}(x)$ we get that there is a set $S \subseteq V'$ with $|S| \leq k$ and $(\delta_{\vec{G}}(x_1^v) = 1 \iff v \in S)$, so $\vec{G}'$ is the orientation which agrees with $\vec{G}_1$ on G_1 and the edges (s, x_1^v) are directed from s to x_1^v for $v \in S$ and from x_1^v to s for $v \in V' - S$. From Corollary 1 we get that $\vec{G}'$ is well-balanced and then the same reasoning, as in the previous claim gives that S is a vertex cover in G'. □

Claim 3. *Problem* MinNodeCostWellBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinNodeCostWellBalanced: let the graph G be as described above and node-costs the following: let $c(s) = 1$ and $c(x_i^v) = -k$ for each $v \in V'$ and $i \in \{0, 2, \ldots, d_{G'}(v)\}$ (and zero for the rest of the nodes). Finally, let $B = -4k|E'| + k + 1$.

As before, it is easy to see that a solution to the instance Vertex Cover gives rise to a well-balanced orientation of total node-cost at most B. Conversely, consider a well-balanced orientation $\vec{G}$ of G that has total node-cost at most B: if any of the nodes with node-cost $-k$ has out-degree at most one then the total node-cost is $\sum_{v \in V} c(v)\delta_{\vec{G}}(v) \geq -2k\left(\sum(d_{G'}(v) : v \in V')\right) + k + 2 = -4k|E'| + k + 2 > B$, a contradiction (using the fact that $\delta(s) \geq 2$ in any well-balanced orientation). So all those nodes have out-degree 2 and then s has out-degree at most $k+1$, so by the same reasoning as before we obtain a solution of VERTEX COVER. □

3 Hardness results for best-balanced orientations

For best-balanced orientations we have the following results.

Theorem 4. *The problems* MINCOSTBESTBALANCED, BOUNDEDBESTBALANCED *and* MINNODECOSTBESTBALANCED *are NP-complete.*

Proof. The problems are clearly in NP. To show completeness we reduce VERTEX COVER as before, but we need to change the construction a bit. For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of the VERTEX COVER problem, modify the construction of the graph $G = (V, E)$ as follows: add $2|E'| + |V'| - 2k = N$ new nodes $z_1, z_2, \ldots, z_N$ and connect each of these nodes with s. So these new nodes will have degree 1 and s will have degree $4|E'| + 2|V'| + 2 - 2k$ in G. Denote this modified graph with $G = (V, E)$ (we will call it "the second construction" to distinguish from the former).

Define again a partial orientation of G: this is the same as the one defined above in the first construction, with the addition that for each i between 1 and N orient the edge (s, z_i) from s to z_i.

Again call the subgraph $G - \{(s, x_1^v) : v \in V'\}$ by G_1 and the above given orientation of this graph by $\vec{G}_1$. Again we have $\lambda_G(x, y) = \min(d_G(x), d_G(y))$ for every $x, y \in V$, $\lambda_{\vec{G}_1}(x, y) \geq 1$ for every $x, y \in V - \{z_1, z_2, \ldots, z_N\}$ and $\lambda_{\vec{G}_1}(x_e, s) = 2$ for each $e \in E'$.

Claim 4. *Problem* MINCOSTBESTBALANCED *is NP-complete, even for $1-0$ orientation costs.*

Proof: For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of VERTEX COVER consider the following instance of MINCOSTBESTBALANCED: let the graph G be as described above, let $K = 0$ be the bound on the total cost and define the orientation-costs as follows. For the edges of G_1 orienting these edges costs nothing in the way as given in $\vec{G}_1$, but reversing any one will cost exactly 1. We only have to tell the costs of orientations of edges between s and x_1^v for each $v \in V'$: these edges can be oriented in any direction with 0 cost. So we only have freedom choosing the orientation of these edges, if we want a best-balanced orientation of 0 cost. We claim that there is a best-balanced orientation of G with cost at most 0 if and only if there is a vertex cover $S \subseteq V'$ in G' with $|S| = k$ (notice that one can always increase the size of a vertex cover if it was smaller than k). The argument is the same as was before (notice that in any best-balanced orientation $\vec{G}$ we have $\varrho_{\vec{G}}(s) = \delta_{\vec{G}}(s)$ so if it has 0 cost then there exists an $S \subseteq V'$ with $|S| = k$ such that $[(s, x_1^v)$ is oriented from s to $x_1^v \iff v \in S])$.
□

Claim 5. *Problem* BOUNDEDBESTBALANCED *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of VERTEX COVER consider the following instance of BOUNDEDBESTBALANCED: let the graph G be as described above and bounds on the out-degrees of odd degree vertices of G given as follows (of course, for even-degree vertices $x \in V$ one has $l(x) = d_G(x)/2 = u(x)$):

- $l(x_i^v) = 2 = u(x_i^v)$ for each $v \in V'$ and $i \in \{0, 2, \ldots, d_{G'}(v)\}$ (exact prescriptions),
- $l(z_i) = 0 = u(z_i)$ for each $i = 1, 2, \ldots, N$ (exact prescriptions),
- $l(x_1^v) = 1$ and $u(x_1^v) = 2$ for each $v \in V'$ (so we only have freedom here).

We claim that there is a vertex cover in G' of size exactly k if and only if there exists a best-balanced orientation of G satisfying these bounds. The argument is again not new: if there is such an orientation $\vec{G}$ then there is a set $S \subseteq V'$ with $|S| = k$ and $(\delta_{\vec{G}}(x_1^v) = 1 \iff v \in S)$ so by reversing directed cycles we get an orientation $\vec{G}'$ which coincides with $\vec{G}_1$ on the edges of G_1 and hence see that S must be a vertex cover (the other direction is easy again). □

Claim 6. *Problem* MINNODECOSTBESTBALANCED *is* NP*-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of VERTEX COVER consider the following instance of MINNODECOSTWELLBALANCED: let the graph G be as described above and node-costs the following: let $c(z_i) = 1$ for each $i = 1, 2, \ldots, N$ and $c(x_i^v) = -1$ for each $v \in V'$ and $i \in \{0, 2, \ldots, d_{G'}(v)\}$ (and zero for the rest of the nodes). Finally, let $B = -2\left(\sum(d_{G'}(v) : v \in V')\right) = -4|E'|$.

As before, it is easy to see that a solution to the instance VERTEX COVER gives rise to a best-balanced orientation of total node-cost at most B. Conversely, consider a best-balanced orientation $\vec{G}$ of G that has total node-cost at most B: if any of the nodes with node-cost -1 has out-degree one or any of the nodes with node-cost 1 has out-degree 1 then the total node-cost is bigger than B. So the same reasoning as in the previous claim gives a vertex cover of size exactly k in G'. □

4 Further remarks

Let us make some remarks about the claims proved above. First we mention that problem BOUNDEDBESTBALANCED can be formulated the following way:

Problem 7. *Given a graph G and two disjoint subsets T^+ and T^- of T_G, decide whether there exists a best-balanced orientation $\vec{G}$ of G satisfying the following:*

$$\delta_{\vec{G}}(v) = \lfloor d_G(v)/2 \rfloor \ \forall v \in T^- \text{ and } \delta_{\vec{G}}(v) = \lceil d_G(v)/2 \rceil \ \forall v \in T^+.$$

So, by Claim 5, this is an NP-complete problem, too. For the next remark we need a definition.

Definition 4. *A* mixed graph *is determined by the triple* (V, E, A) *where* V *is the set of nodes,* E *is the set of undirected edges and* A *is the set of directed edges. The* underlying undirected graph *is obtained by deleting the orientation of the arcs in* A*. An* orientation of a mixed graph *means that we orient the undirected edges (and leave the directed ones).*

A possible way to prove the well-balanced orientation theorem could be to characterize mixed graphs whose undirected edges can be oriented to have a well-balanced orientation of the underlying undirected graph. The following problem was mentioned in Section 4.2 of [6]:

Problem 8. *Given a mixed graph, decide whether it has an orientation that is a well-balanced orientation of the underlying undirected graph.*

The following question is an open problem also raised in [6]:

Question 1. *Is Problem 8* NP*-complete?*

While we don't know the answer to Question 1, the proof of Claim 4 immediately gives the NP-completeness of the following, related problem.

Problem 9. *Given a mixed graph, decide whether it has an orientation that is a best-balanced orientation of the underlying undirected graph.*

Using the "second construction" a variant of problems MinNodeCostWellBalanced and MinNodeCostBestBalanced can also be shown to be NP-complete (these questions were raised by Zoltán Király).

Problem 10. *:* MinNodeCost2BestBalanced
Instance*: A graph* G*, integer costs* $c : V \mapsto \mathbb{Z}$*, integer* B*.*
Question*: Is there a best-balanced orientation* $\vec{G}$ *of* G *with* $\sum_{v\in V} c(v)\,[\delta_{\vec{G}}(v) - \varrho_{\vec{G}}(v)] \leq B$*?*

Claim 7. *Problem* MinNodeCost2BestBalanced *is* NP*-complete.*

Proof. Let the graph G be as described in the second construction. The node-costs are as follows: let $c(z_i) = 1$ for each $i = 1, 2, \dots N$ and $c(x_i^v) = -1$ for each $v \in V'$ and $i \in \{0, 2, \dots, d_{G'}(v)\}$ (and zero for the rest of the nodes). Finally, let $B = -\left(\sum(d_{G'}(v) : v \in V')\right) - N = -|V'| - 4|E'| + 2k$. The rest of the argument is left to the reader. □

Problem 11. *:* MinNodeCost2WellBalanced
Instance*: A graph* G*, integer costs* $c : V \mapsto \mathbb{Z}$*, integer* B*.*
Question*: Is there a well balanced orientation* $\vec{G}$ *of* G *with* $\sum_{v\in V} c(v)\,[\delta_{\vec{G}}(v) - \varrho_{\vec{G}}(v)] \leq B$*?*

Claim 8. *Problem* MinNodeCost2WellBalanced *is* NP*-complete.*

Proof. Let the graph G be as given in the second construction and node-costs $c(x_i^v) = -M$ for each $v \in V'$ and $i \in \{0, 2, \dots, d_{G'}(v)\}$, $c(z_i) = M$ for each $i = 1, 2, \dots, N$ and $c(s) = 1$, where M is big enough. Finally, let $B = (-4|E'| - |V'| + 2k)M$. If $M > d_G(s)$ then in any well-balanced orientation $\vec{G}$ of cost at most B the nodes with cost $-M$ will have out-degree 2, those with node-cost M will have out-degree 0 and s will have $\delta_{\vec{G}}(s) \leq \varrho_{\vec{G}}(s)$. The rest is left to the reader. □

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