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**Pin-collinear Body-and-Pin Frameworks
and the Molecular Conjecture**

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Pin-collinear Body-and-Pin Frameworks and the Molecular Conjecture

Bill Jackson* and Tibor Jordán**

Abstract

T-S. Tay and W. Whiteley independently characterized the multigraphs which can be realized as an infinitesimally rigid d -dimensional body-and-hinge framework. In 1984 they jointly conjectured that each graph in this family can be realized as an infinitesimally rigid framework with the additional property that the hinges incident to each body lie in a common hyperplane. This conjecture has become known as the Molecular Conjecture because of its implication for the rigidity of molecules in 3-dimensional space. Whiteley gave a partial solution for the 2-dimensional form of the conjecture in 1989 by showing that it holds for multigraphs $G = (V, E)$ in the family which have the minimum number of edges, i.e. satisfy $2|E| = 3|V| - 3$. In this paper, we give a complete solution for the 2-dimensional version of the Molecular Conjecture. Our proof relies on a new formula for the maximum rank of a pin-collinear body-and-pin realization of a multigraph as a 2-dimensional bar-and-joint framework.

1 Introduction

All graphs considered are finite and without loops. We will reserve the term graph for graphs without multiple edges and refer to graphs which may contain multiple edges as *multigraphs*. Given a multigraph G and a positive integer k , we use kG to denote the multigraph obtained by replacing each edge of G by k parallel edges.

Informally, a body-and-hinge framework in \mathbb{R}^d consists of large rigid bodies articulated along affine subspaces of dimension $d - 2$ which act as hinges i.e. bodies joined by pin-joints in 2-space, line-hinges in 3-space, plane-hinges in 4-space, etc. This notion may be formalized by using the facts that the infinitesimal motions of a rigid body in d -space can be coordinatized using screw centers (real vectors of length $\binom{d+1}{2}$ which represent $(d - 1)$ -tensors in projective d -space), and that rotations correspond to particular kinds of screw centers called $(d - 1)$ -extensors, see [1]. A *d-dimensional*

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body-and-hinge framework (G, q) is a multigraph $G = (V, E)$ together with a map q which associates a $(d - 2)$ -dimensional affine subspace $q(e)$ of \mathbb{R}^d with each edge $e \in E$. An *infinitesimal motion* of (G, q) is a map S from V to $\binom{d+1}{2}$ -space such that, for every edge $e = uv$, $S(u) - S(v)$ is a scalar multiple of $P(e, q)$, where $P(e, q)$ is a $(d - 1)$ -extensor which corresponds to a rotation about $q(e)$. An infinitesimal motion S is *trivial* if $S(u) = S(v)$ for all $u, v \in V$ and (G, q) is said to be *infinitesimally rigid* if all its infinitesimal motions are trivial.¹ We refer the reader to [18, 20] for a more detailed account of body-and-hinge frameworks in \mathbb{R}^d . We will only be concerned with the case $d = 2$ and specific details for this case will be given in Sections 2 and 5 of this paper.

Multigraphs which can be realized as infinitesimally rigid body-and-hinge frameworks are characterized by the following theorem, proved independently by Tay [14] and Whiteley [18].

Theorem 1.1. *A multigraph G can be realized as an infinitesimally rigid body-and-hinge framework in \mathbb{R}^d if and only if $((\binom{d+1}{2} - 1)G$ has $\binom{d+1}{2}$ edge-disjoint spanning trees.*

Tay and Whiteley jointly conjecture that the same condition characterizes when a multigraph can be realized as an infinitesimally rigid body-and-hinge framework in \mathbb{R}^d with the additional property that all the hinges incident to each body are contained in a common hyperplane.

Conjecture 1.2. [16] *Let G be a multigraph. Then G can be realized as an infinitesimally rigid body-and-hinge framework in \mathbb{R}^d if and only if G can be realized as an infinitesimally rigid body-and-hinge framework (G, q) in \mathbb{R}^d with the property that, for each $v \in V$, all of the subspaces $q(e)$, e incident to v , are contained in a common hyperplane.*

Conjecture 1.2 is known as the Molecular Conjecture because of its implications for the rigidity of molecules when $d = 3$.² It has been verified by Whiteley [19] when $d = 2$ for the special case when $2G$ is the union of three edge-disjoint spanning trees.

¹These definitions can be motivated by considering each vertex $v \in V$ as being represented by a large rigid body B_v in d -space and each edge $e = uv \in E$ as being represented by the ‘hinge’ $q(e)$ attached to B_u and B_v . Each body B_v can move continuously subject to the constraints that, for each edge $e = uv \in E$, the relative motion of B_u with respect to B_v is a rotation about the hinge $q(e)$. At any given instant, the motion of B_v is represented by the screw center $S(v)$. The constraint concerning the relative motion of B_u with respect to B_v is represented by the condition that $S(u) - S(v)$ is a scalar multiple of $P(e, q)$.

²It is known that infinitesimal rigidity is a projective invariant, and it is the projective dual of the case $d = 3$ of the Molecular Conjecture which has implications for the rigidity of molecules. Under projective duality in \mathbb{R}^3 , lines are mapped to lines, and planes are mapped to points. Thus the conjecture for $d = 3$ is equivalent to the statement that a graph G can be realized as an infinitesimally rigid body-and-hinge framework in \mathbb{R}^3 with all hinges incident to each vertex concurrent at a point, if and only if $5G$ has six edge-disjoint spanning trees. The application to molecules represents atoms as vertices and bonds between atoms as edges [21, 24]. The corresponding body-and-hinge framework will centre each atom at the point of concurrence of the bonds which are incident to it. For partial results on the 3-dimensional version of the Molecular Conjecture see [5, 6].

The purpose of this paper is to give a complete solution of the conjecture when $d = 2$. We will assume henceforth that $d = 2$ and refer to hinges as pins and to body-and-hinge frameworks as body-and-pin frameworks. We say that a body-and-pin framework is pin-collinear if the pins incident to each body lie on a common line. Our proof that Conjecture 1.2 holds when $d = 2$ uses an equivalent definition of body-and-pin structures as ‘bar-and-joint’ frameworks. These will be described in Sections 2 and 4 along with some basic results. We need two more sections to collect the notions and preliminary results we use in the main proof: in Section 3 we summarize some structural properties of forest covers of multigraphs, and in Section 5 we define body-and-pin frameworks and their associated rigidity matrices. Our main result, which solves the bar-and-joint version of the 2-dimensional Molecular Conjecture, is given in Section 6. We give some corollaries of this result in Section 7 and deduce, in particular, that Conjecture 1.2 holds when $d = 2$. We close by describing a conjectured characterization of when a general incidence structure can be realized as an infinitesimally rigid pin-collinear body-and-pin framework in Section 8.

2 Bar-and-joint frameworks

A (2-dimensional) *bar-and-joint framework* (G, q) is a graph $G = (V, E)$ together with a map $q : V \rightarrow \mathbb{R}^2$. We say that (G, q) is a *realization* of G . We consider each vertex to be represented by a universal joint at $q(e)$ and each edge $e = uv \in E$ to be represented by a rigid bar attached to the joints $q(u)$ and $q(v)$. The joints are free to move continuously in \mathbb{R}^2 , subject to the constraints that the bar-lengths $\|q(u) - q(v)\|$ remain constant for all $uv \in E$. The framework (G, q) is said to be *rigid* if each such motion preserves the distances $\|q(u) - q(v)\|$ for all $u, v \in V$.

The *rigidity matrix* of the framework is the matrix $R(G, q)$ of size $|E| \times 2|V|$, where, for each edge $v_i v_j \in E$, in the row corresponding to $v_i v_j$, the entries in the two columns corresponding to vertices v_i and v_j are given by the coordinates of $(q(v_i) - q(v_j))$ and $(q(v_j) - q(v_i))$, respectively, and the remaining entries are zeros. See [20] for more details. The rigidity matrix of (G, q) defines the *rigidity matroid* $\mathcal{R}(G, q)$ of (G, q) on the ground set E by linear independence of rows of the rigidity matrix. We denote the rank of $R(G, q)$ by $r(G, q)$.

The following lemma gives an upper bound on $r(G, q)$.

Lemma 2.1. [20, Lemma 11.1.3] *Let (G, q) be a bar-and-joint framework with $n \geq 2$ vertices. Then $r(G, q) \leq 2n - 3$.*

The framework (G, q) is said to be *infinitesimally rigid* if $r(G, q) = 2n - 3$. We refer to the vectors in the null space, $Z(G, q)$, of $R(G, q)$ as *infinitesimal motions* of (G, q) , and define the *number of degrees of freedom* of (G, q) , $df(G, q)$, to be the dimension of $Z(G, q)$. Thus $df(G, q) = 2n - r(G, q)$. (These definitions are motivated by the fact that every continuous motion of (G, q) which preserves bar-lengths ‘induces’ an infinitesimal motion of (G, q) . In particular the rigid isometries of \mathbb{R}^2 corresponding to translation along either axis and rotation about the origin give rise to three linearly

independent infinitesimal motions of (G, q) . If (G, q) is infinitesimally rigid, then these three infinitesimal motions are a basis of $Z(G, q)$.

We say that (G, q) is a *generic* realization of G if the set of coordinates of all points $q(v)$, $v \in V$, is algebraically independent over \mathbb{Q} . (It follows from a result of Gluck [3] that if (G, q) is generic, then (G, q) is rigid if and only if (G, q) is infinitesimally rigid.) All generic realizations of G have the same rank and we denote this value by $r(G)$. The following lemma follows from the fact that the entries in $R(G, q)$ are polynomial (and hence continuous) functions of the components of $q(v)$, $v \in V$, with rational coefficients.

Lemma 2.2. *Let (G, q) be a bar-and-joint framework. Then*

(a) $r(G, q) \leq r(G)$.

(b) *There exists an $\epsilon > 0$ such that for all $q' : V \rightarrow \mathbb{R}^2$ with $\|q'(v) - q(v)\| \leq \epsilon$ for all $v \in V$, we have $r(G, q') \geq r(G, q)$.*

Let $G = (V, E)$ be a multigraph. For $X \subseteq V$, the *degree* of X , $d_G(X)$, is the number of edges of G from X to $V - X$. If $X = \{v\}$ for some $v \in V$ then we simply write $d_G(v)$ for the degree of v . The set of *neighbours* of X (i.e. the set of those vertices $v \in V - X$ for which there exists an edge $uv \in E$ with $u \in X$) is denoted by $N_G(X)$.

We shall need some basic results on the rigidity of frameworks.

Lemma 2.3. [20, Lemma 2.1.3] *Let $G_1 = (V_1, E_1)$ be a graph and v_1, v_2 be distinct vertices of G_1 . Let G be obtained from G_1 by adding a new vertex v and edges vv_1, vv_2 . Let (G_1, q_1) be a realization of G_1 such that $q_1(v_1) \neq q_1(v_2)$. Choose a point Q such that $q_1(v_1), q_1(v_2), Q$ are not collinear and let (G, q) be the realization of G obtained from (G_1, q_1) by putting $q(v) = Q$. Then $r(G, q) = r(G_1, q_1) + 2$.*

Lemma 2.4. [20, Lemma 2.2.2] *Let $G_1 = (V_1, E_1)$ be a graph, v_1, v_2, v_3 be distinct vertices of G_1 and v_1v_2 be an edge of G_1 . Let G be obtained from $G_1 - v_1v_2$ by adding a new vertex v and edges vv_1, vv_2, vv_3 . Let (G_1, q_1) be a realization of G_1 such that $q_1(v_1), q_1(v_2), q_1(v_3)$ are not collinear. Let $Q \in \mathbb{R}^2$ be a point on the line through $q_1(v_1), q_1(v_2)$ distinct from $q_1(v_1), q_1(v_2)$. Let (G, q) be the realization of G obtained from (G_1, q_1) by putting $q(v) = Q$. Then $r(G, q) = r(G_1, q_1) + 2$.*

The following result is given without proof in [20, Figure 2.9]. We include a proof for the sake of completeness.

Lemma 2.5. *Let $G = (V, E)$ be a graph, $v \in V$ and $E_G(v) = \{vv_1, vv_2, \dots, vv_k\}$ for some $k \geq 2$. Choose j such that $2 \leq j \leq k$ and let G' be the graph obtained from $G - v$ by adding two new vertices v', v'' and edges $v'v_1, v'v_2, \dots, v'v_j, v''v_1, v''v_2, v''v_{j+1}, \dots, v''v_k$. Suppose $q : V \rightarrow \mathbb{R}^2$. Define $q' : V(G') \rightarrow \mathbb{R}^2$ by $q'(u) = q(u)$ for all $u \in V - v$ and $q'(v') = q'(v'') = q(v)$. Suppose $q(v) - q(v_1)$ and $q(v) - q(v_2)$ are linearly independent. Then $r(G', q') \geq r(G, q) + 2$.*

Proof: Since $q(v) - q(v_1)$ and $q(v) - q(v_2)$ are linearly independent, we can include vv_1, vv_2 in a basis B of $\mathcal{R}(G, q)$. Let $I_1 = \{vv_i \in B : 3 \leq i \leq j\}$ and $I_2 = \{vv_i \in B : j + 1 \leq i \leq k\}$. Put

$$B' = (B - E_G(v)) \cup \{v'v_i : vv_i \in I_1\} \cup \{v''v_i : vv_i \in I_2\} \cup \{v'v_1, v'v_2, v''v_1, v''v_2\}.$$

We will show that B' is independent in $\mathcal{R}(G', q')$. Suppose not. Then the rows of $R(G', q')$ corresponding to B' are linearly dependent. Thus there exist constants $\alpha_{uw} \in \mathbb{R}$, $uw \in B'$ not all equal to zero, such that, for each $u \in V(G')$, we have

$$\sum_{uw \in B'} \alpha_{uw}(q(u) - q(w)) = \mathbf{0}. \quad (1)$$

Define constants $\beta_{uw} \in \mathbb{R}$, $uw \in B$, as follows: $\beta_{uw} = \alpha_{uw}$ when uw is not incident to v ; $\beta_{vv_i} = \alpha_{v'v_i}$ for $vv_i \in I_1$; $\beta_{vv_i} = \alpha_{v''v_i}$ for $vv_i \in I_2$; $\beta_{vv_1} = \alpha_{v'v_1} + \alpha_{v''v_1}$; $\beta_{vv_2} = \alpha_{v'v_2} + \alpha_{v''v_2}$. Then (1) implies that $\sum_{uw \in B} \beta_{uw}(q(u) - q(w)) = \mathbf{0}$ for all $u \in V(G)$. Since B is independent, we must have $\beta_{uw} = 0$ for all $uw \in B$. Thus $\alpha_{uw} = 0$ for all $uw \in B'$ with $uw \notin \{v'v_1, v'v_2, v''v_1, v''v_2\}$. Substituting into (1) with $u = v'$ we may also deduce that $\alpha_{v'v_1}(q(v) - q(v_1)) + \alpha_{v'v_2}(q(v) - q(v_2)) = \mathbf{0}$. Since $q(v) - q(v_1)$ and $q(v) - q(v_2)$ are linearly independent, we must have $\alpha_{v'v_1} = 0 = \alpha_{v'v_2}$. Similarly, $\alpha_{v''v_1} = 0 = \alpha_{v''v_2}$. This contradicts the assumption that not all of the constants α_{uw} are equal to zero. Thus B' is independent and $r(G', q') \geq |B'| = |B| + 2 = r(G, q) + 2$. \bullet

We refer to the operations in Lemmas 2.3, 2.4 and 2.5 as *0-extensions*, *1-extensions*, and *vertex-splits*, respectively. The next result is a non-generic extension of [20, Lemma 3.1.4 (1)]. It can be proved similarly.

Lemma 2.6. *Let (G, q) be a framework and let G_1, G_2 be subgraphs of G with $2 \leq |V(G_i)| \leq |V(G)| - 1$, $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = X$. Suppose that (G_i, q_i) are both infinitesimally rigid frameworks, where q_i denotes the restriction of q to $V(G_i)$, $i = 1, 2$. Then $df(G, q) = 6$ if $X = \emptyset$, $df(G, q) = 4$ if $|q(X)| = 1$, and $df(G, q) = 3$ (so (G, q) is infinitesimally rigid) if $|q(X)| \geq 2$.*

Lemma 2.7. *Let $G = (V, E)$, $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be connected graphs such that $G = G_1 \cup G_2$, and $G_1 \cap G_2 = K$ is a complete graph on t vertices, for $t \in \{0, 1, 2\}$. Let (G, q) be a realization of G and let q_i be the restriction of q to V_i , $i \in \{1, 2\}$. Suppose the affine hull of $q_i(V_i)$ is 2-dimensional for each $i \in \{1, 2\}$ and that $q(u) \neq q(v)$ if u, v are distinct vertices of K . Then $r(G, q) = r(G_1, q_1) + r(G_2, q_2) - |E(K)|$.*

Proof: Choose a base B_i of $\mathcal{R}(G_i, q_i)$ with $E(K) \subseteq B_i$. Since the affine hull of $q_i(V_i)$ is 2-dimensional, we can add edges to the graph (V_i, B_i) such that the resulting graph $H_i = (V_i, T_i)$ satisfies $|T_i| = 2|V_i| - 3$ and (H_i, q_i) is infinitesimally rigid. Let $H = H_1 \cup H_2$. Lemma 2.6 now implies that $r(H, q) = |E(H)|$ and hence $T_1 \cup T_2$ is independent in $\mathcal{R}(H, q)$. Since $B = B_1 \cup B_2 \subseteq T_1 \cup T_2$, B is independent in $\mathcal{R}(G, q)$. On the other hand, the fact that B_i spans E_i in $\mathcal{R}(G_i, q_i)$ implies that B spans E in $\mathcal{R}(G, q)$. Thus B is a base of $\mathcal{R}(G, q)$ and

$$r(G, q) = |B| = |B_1| + |B_2| - |E(K_t)| = r(G_1, q) + r(G_2, q) - |E(K_t)|. \quad \bullet$$

As noted in the Introduction, the space of infinitesimal motions of an infinitesimally rigid bar-and-joint framework in \mathbb{R}^d can be coordinatized using screw centers, which

are real vectors of length $\binom{d+1}{2}$, see [1]. We will use this coordinatization for the special case $d = 2$ in Section 5 to define 2-dimensional body-and-hinge frameworks and their associated rigidity matrices. We derive the required properties of screw centers in 2-dimensional space in the remainder of this section.

For $S = (w_1, w_2, w_3)^T \in \mathbb{R}^3$, let $M_S = w_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{v}_S = \begin{pmatrix} w_2 \\ w_1 \end{pmatrix}$ and define $f_S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f_S(\mathbf{x}) = M_S \mathbf{x} + \mathbf{v}_S$. We use $\langle (x, y, z) \rangle$ to denote the subspace of \mathbb{R}^3 spanned by a vector $(x, y, z) \in \mathbb{R}^3$. The following lemma is straightforward to check.

Lemma 2.8. *Suppose $S, T \in \mathbb{R}^3$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Then*

- (a) $f_{S+T}(\mathbf{x}) = f_S(\mathbf{x}) + f_T(\mathbf{x})$.
- (b) $f_S(\mathbf{x}) = \mathbf{0}$ if and only if $S \in \langle (x_1, -x_2, 1) \rangle$.

Let $G = (V, E)$ be a graph and (G, q) be a realization of G as a 2-dimensional bar-and-joint framework. Since the columns of the rigidity matrix $R(G, q)$ are indexed by V we can consider the vectors in the null space of $R(G, q)$, i.e the infinitesimal motions of (G, q) , as maps $q' : V \rightarrow \mathbb{R}^2$ with the property that $(q(u) - q(v)) \cdot (q'(u) - q'(v)) = 0$ for all $uv \in E$. We adopt this convention in our next result.

Lemma 2.9. *Let $G = (V, E)$ be a graph with at least two vertices, $q : V \rightarrow \mathbb{R}^2$, and $S = (w_1, w_2, w_3) \in \mathbb{R}^3$. Let $f_S \circ q : V \rightarrow \mathbb{R}^2$ by $f_S \circ q(v) = f_S(q(v))$ for all $v \in V$. Then*

- (a) $f_S \circ q$ belongs to the null space, $Z(G, q)$, of $R(G, q)$.
- (b) If (G, q) is infinitesimally rigid, then the map $f : \mathbb{R}^3 \rightarrow Z(G, q)$ by $f(S) = f_S \circ q$ for all $S \in \mathbb{R}^3$ is a vector space isomorphism.

Proof: Choose $u, v \in V$. Then

$$(q(u) - q(v)) \cdot (f_S \circ q(u) - f_S \circ q(v)) = w_3(q(u) - q(v))M_S(q(u) - q(v))^T = 0.$$

Thus $f_S \circ q \in Z(G, q)$ and (a) holds. To prove (b) we assume that (G, q) is infinitesimally rigid, and hence that $\dim Z(G, q) = 3$. It is straightforward to check that f is a linear map. Suppose $f(S) = \mathbf{0}$. Since (G, q) is infinitesimally rigid, we may choose $v_1, v_2 \in V$ such that $(x_1, y_1) = q(v_1) \neq q(v_2) = (x_2, y_2)$. Lemma 2.8(b) and the fact that $f_S(q(v_1)) = \mathbf{0} = f_S(q(v_2))$ now imply that $S \in \langle (x_1, -y_1, 1) \rangle \cap \langle (x_2, -y_2, 1) \rangle = \{\mathbf{0}\}$. Thus $S = \mathbf{0}$ and hence f is an injection. Since \mathbb{R}^3 and $Z(G, q)$ both have the same dimension, f is an isomorphism. \bullet

3 Bricks and Superbricks

In this section we summarize the structural results on forest covers of multigraphs that we shall use. Let $H = (V, E)$ be a multigraph. For a family \mathcal{F} of pairwise disjoint subsets of V let $E_H(\mathcal{F})$ denote the set, and $e_H(\mathcal{F})$ the number, of edges of H connecting distinct members of \mathcal{F} .

The following theorem is well-known.³

Theorem 3.1. [10, 11, 17] Let $H = (V, E)$ be a multigraph and let k be a positive integer. Then

(a) the maximum size of the union of k forests in H is equal to the minimum value of

$$e_H(\mathcal{P}) + k(|V| - |\mathcal{P}|) \quad (2)$$

taken over all partitions \mathcal{P} of V ;

(b) H contains k edge-disjoint spanning trees if and only if

$$e_H(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$$

for all partitions \mathcal{P} of V ;

(c) the edge set of H can be covered by k forests if and only if

$$|E(H[X])| \leq k(|X| - 1)$$

for each nonempty subset X of V .

In this paper we shall be concerned with the case when $H = 2G$, for some multigraph G , and $k = 3$. Let $G = (V, E)$ be a multigraph. For a partition \mathcal{Q} of V let

$$\text{def}_G(\mathcal{Q}) = 3(|\mathcal{Q}| - 1) - 2e_G(\mathcal{Q})$$

denote the *deficiency* of \mathcal{Q} in G and let

$$\text{def}(G) = \max\{\text{def}_G(\mathcal{Q}) : \mathcal{Q} \text{ is a partition of } V\}.$$

Note that $\text{def}(G) \geq 0$ since $\text{def}_G(\{V\}) = 0$. We say that a partition \mathcal{Q} of V is a *tight partition* of G if $\text{def}_G(\mathcal{Q}) = \text{def}(G)$.

A multigraph G is *strong* if $2G$ has three edge-disjoint spanning trees. Equivalently, by Theorem 3.1(b), G is strong if $\text{def}(G) = 0$. A subgraph H of a multigraph G is said to be a *brick* of G if H is a maximal strong subgraph of G . Thus bricks are induced subgraphs.

We say that a multigraph $G = (V, E)$ is *superstrong* if $2G - e$ has three edge-disjoint spanning trees for all $e \in E(2G)$. Equivalently, by Theorem 3.1(b), G is superstrong if $\text{def}(G) = 0$ and the only tight partition of V is $\{V\}$ itself. A subgraph H of G is said to be a *superbrick* of G if H is a maximal superstrong subgraph of G . Thus superbricks are induced subgraphs.

The following properties of bricks and superbricks were verified in [5]⁴.

³Theorem 3.1(a) appears in [13, Chapter 51]. It follows easily from the matroid union theorem of Nash-Williams [12] and Edmonds [2], which determine the rank function of the union of k matroids, by applying this theorem to the matroid $\mathcal{M}_k(H)$ which is the union of k copies of the cycle matroid of H . Part (a) implies parts (b) and (c), which are well-known results of Tutte [17] and Nash-Williams [10], and Nash-Williams [11], respectively. The minimum value of (2) is equal to $r_k(E)$, where r_k denotes the rank function of $\mathcal{M}_k(H)$.

⁴In [5] these notions and the following lemmas were formulated for graphs and for a different count: with six edge-disjoint spanning trees in $5G$. However, as was noted in [5], they hold in a much more general context, which implies the results for multigraphs and for both counts.

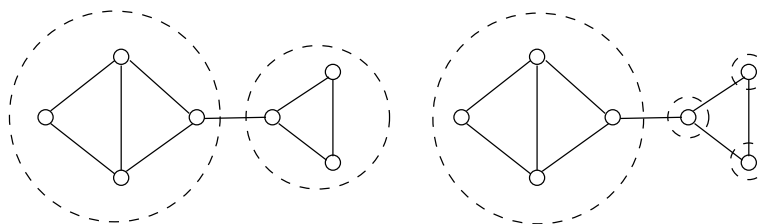


Figure 1: The brick partition \mathcal{B}_1 and the superbrick partition \mathcal{B}_2 of a graph G . We have $\text{def}(G) = \text{def}_G(\mathcal{B}_1) = \text{def}_G(\mathcal{B}_2) = 1$.

Lemma 3.2. [5] *Let $G = (V, E)$ be a multigraph. Then the vertex sets of the bricks (resp. superbricks) of G partition V .*

The term *brick partition* (resp. *superbrick partition*) of G refers to the partition of V given by the vertex sets of the bricks (resp. superbricks) of G , see Figure 1. We shall frequently use the fact that the brick and superbrick partitions of G are both tight. This follows from the next lemma.

Lemma 3.3. [5] *Let $G = (V, E)$ be a multigraph and \mathcal{P} be a tight partition of V .*

(a) *If \mathcal{P} is chosen so that $|\mathcal{P}|$ is as small as possible then \mathcal{P} is the brick partition of G .*

(b) *If \mathcal{P} is chosen so that $|\mathcal{P}|$ is as large as possible then \mathcal{P} is the superbrick partition of G .*

4 Body-and-Pin Realizations of Graphs as Bar-and-Joint Frameworks

Let $G = (V, P)$ be a multigraph. For $v \in V$ let $E_G(v)$ be the set of all edges of G incident to v . The *body-and-pin graph* of G is the graph G^* with $V(G^*) = V \cup P$ and

$$E(G^*) = \{vp : v \in V \text{ and } p \in E_G(v)\} \cup \{p_1p_2 : v \in V \text{ and } p_1, p_2 \in E_G(v)\}.$$

See Figure 2.

We first show that $\text{def}(G)$ can be used to obtain an upper bound on $r(G^*)$.

Lemma 4.1. *Let $G = (V, P)$ be a multigraph with no isolated vertices. Then $r(G^*) \leq 2(|V| + |P|) - 3 - \text{def}(G)$.*

Proof: Since $|V(G^*)| = |V| + |P|$, we have $r(G^*) \leq 2(|V| + |P|) - 3$ by Lemma 2.1. Thus we may assume that $\text{def}(G) \geq 1$. Let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_t\}$ be a tight partition of V . Since $\text{def}(G) \geq 1$, we must have $t \geq 2$.

For $v \in V$ let $B^*(v) = \{v\} \cup E_G(v) \subset V(G^*)$ and let $X_i = \cup_{v \in Q_i} B^*(v)$, for $1 \leq i \leq t$. Then every edge of G^* is induced by some X_i , and since G has no isolated vertices,

we have $|X_i| \geq 2$, for $1 \leq i \leq t$. Furthermore, $\sum_{i=1}^t |X_i| = |V| + |P| + e_G(\mathcal{Q})$. Now we can use Lemma 2.1 to deduce that

$$\begin{aligned} r(G^*) &\leq \sum_{i=1}^t (2|X_i| - 3) = 2(|V| + |P|) + 2e_G(\mathcal{Q}) - 3t \\ &= 2(|V| + |P|) - 3 - \text{def}(G). \end{aligned}$$

•

We shall see later, in Corollary 6.12, that equality holds in Lemma 4.1. (This could also be proved directly using the characterization of the rank function of the rigidity matroid of a graph given by Lovász and Yemini [9].)

Let $G = (V, P)$ be a multigraph and G^* be the body-and-pin graph of G . Given a map $q : V(G^*) \rightarrow \mathbb{R}^2$, we say that the bar-and-joint framework (G^*, q) is a *body-and-pin realization* of G if q acts injectively on $\{v\} \cup E_G(v)$ for all $v \in V$ and the subframework, B_v , of (G^*, q) induced by $\{v\} \cup E_G(v)$ is infinitesimally rigid. The realization (G^*, q) is said to be *pin-collinear* if the points $q(p)$, $p \in E_G(v)$, are collinear for each $v \in V$. In this case, the line through $q(p)$, $p \in E_G(v)$, is unique when $d_G(v) \geq 2$, and we denote it by $L_{(G^*, q)}(v)$. We extend this notation for vertices $v \in V$ with $d_G(v) = 1$, say $E_G(v) = \{p\}$, by putting $L_{(G^*, q)}(v)$ equal to the line through $q(p)$ which is orthogonal to the line containing $q(v)$, $q(p)$. We refer to the lines $L_{(G^*, q)}(v)$, $v \in V$, as the *pin-lines* of (G^*, q) . Note that the infinitesimal rigidity of B_v implies that $q(v)$ must not lie on $L_{(G^*, q)}(v)$ for each $v \in V$. The pin-collinear body-and-pin realization (G^*, q) of G is said to be *non-degenerate* if $L_{(G^*, q)}(u) \neq L_{(G^*, q)}(v)$ for all $uv \in P$. Note that if G has a non-degenerate pin-collinear body-and-pin realization (G^*, q) then G must be a graph, since if $u, v \in V$ are joined by two or more edges in G then we must necessarily have $L_{(G^*, q)}(u) = L_{(G^*, q)}(v)$.

Lemma 4.2. *Let $G = (V, E)$ be a graph and (G^*, q_1) be a pin-collinear body-and-pin realization of G . Then there exists a non-degenerate pin-collinear body-and-pin realization (G^*, q_2) of G such that $r(G^*, q_2) \geq r(G^*, q_1)$.*

Proof: Suppose $L_{(G^*, q_1)}(u) = L_{(G^*, q_1)}(v)$ for some $uv \in E$. If $d_G(u) = 1$ then we may move $q_1(u)$ by Lemma 2.2(b) so that the line through $q_1(u)$, $q_1(uv)$ is no longer perpendicular to $L_{(G^*, q_1)}(v)$. Thus we may suppose that $E_G(u) = \{uv, uv_1, \dots, uv_t\}$ for some $t \geq 1$. By Lemma 2.2(b), there exists a neighborhood S_i around each point $q_1(uv_i)$, $i \in \{1, 2, \dots, t\}$, such that $r(G^*, q_1)$ does not decrease if we move $q_1(uv_i)$ within S_i . Thus we may modify (G^*, q_1) by moving each point $q_1(uv_i)$ slightly, in such a way that it continues to lie on $L_{(G^*, q_1)}(v_i)$ and belong to S_i , and also such that $q_1(uv)$, $q_1(uv_1)$, \dots , $q_1(uv_t)$ all lie on a line L_0 which is not parallel to $L_{(G^*, q_1)}(v)$. (We may imagine L_0 is obtained by a small rotation of $L_{(G^*, q_1)}(u)$ about the point $q_1(uv)$.) Repeating this process for all such pairs of pin-lines we obtain a non-degenerate pin-collinear body-and-pin realization (G^*, q_2) of G such that $r(G^*, q_2) \geq r(G^*, q_1)$. •

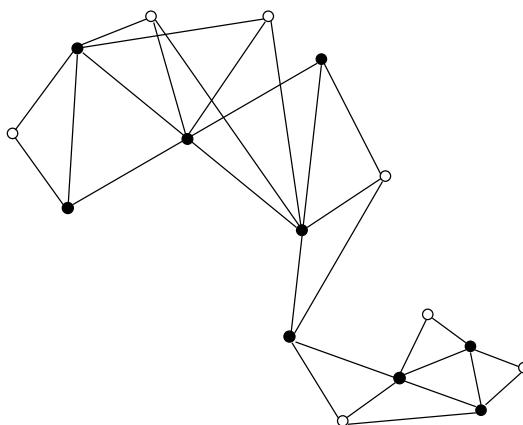


Figure 2: A non-degenerate pin-collinear body-and-pin realization (G^*, q) of the graph $G = (V, P)$ from Figure 1. White (resp. black) vertices of G^* correspond to vertices (resp. edges) of G . We have $r(G^*) = r(G^*, q) = 2(|V| + |P|) - 3 - \text{def}(G)$. Since $\text{def}(G) = 1$, (G^*, q) has exactly four degrees of freedom.

5 Body-and-Pin Frameworks

Let $G = (V, E)$ be a multigraph and (G^*, q) be a body-and-pin realization of G as a bar-and-joint framework. Since, for each $v \in V$, the vertices in $\{v\} \cup E_G(v)$ induce an infinitesimally rigid subframework of (G^*, q) , it is not difficult to see that $r(G^*, q)$ will be uniquely determined by the position of the points $q(e)$, $e \in E$. That is to say $r(G^*, q)$ is independent of the position of the points $q(v)$, $v \in V$, as long as each $q(v)$ is chosen so that the points $q(x)$, $x \in \{v\} \cup E_G(v)$ are not collinear whenever $d_G(v) \geq 2$ holds. This observation leads us to the following definition.

A *body-and-pin framework* (G, q) is a multigraph $G = (V, E)$, together with a map $q : E \rightarrow \mathbb{R}^2$. Let $q(e) = (q_1(e), q_2(e))$ for each $e \in E$. We define an *infinitesimal motion* of the body-and-pin framework (G, q) as a map $S : V \rightarrow \mathbb{R}^3$ satisfying the constraints that, for all $e = uv \in E$, $S(u) - S(v) \in \langle P(e, q) \rangle$, where $P(e, q) = (q_1(e), -q_2(e), 1)$. An infinitesimal motion S is *trivial* if $S(u) = S(v)$ for all $u, v \in V$. The framework (G, q) is *infinitesimally rigid* if all its infinitesimal motions are trivial. Given a body-and-pin realization of a multigraph $G = (V, E)$ as a bar-and-joint framework (G^*, q) , we may define the *body-and-pin framework associated to* (G^*, q) to be the body-and-pin framework (G, \hat{q}) where \hat{q} is the restriction of q to E . We shall see in Lemma 5.1 below that there is a natural correspondence between the infinitesimal motions of (G^*, q) and (G, \hat{q}) .

We first show that the set of infinitesimal motions of a body-and-pin framework (G, q) is the null space of a matrix. For each $e \in E$, let $Q(e, q) = (1, 0, -q_1(e))$ and $R(e, q) = (0, 1, q_2(e))$. Then $\{Q(e, q), R(e, q)\}$ is a basis for the orthogonal complement of $\langle P(e, q) \rangle$ in \mathbb{R}^3 . Thus the constraint that $S(u) - S(v) \in \langle P(e, q) \rangle$ for $e = uv \in E$ is equivalent to the simultaneous constraints $(S(u) - S(v)) \cdot Q(e, q) = 0$ and $(S(u) - S(v)) \cdot R(e, q) = 0$. Combining these constraints for each edge $e \in E$, we obtain a system of $2|E|$ equations in the unknowns $S(v)$, $v \in V$. The matrix of coefficients

of this system is the $2|E| \times 3|V|$ matrix $R_{BP}(G, q)$ with pairs of consecutive rows indexed by E and triples of consecutive columns indexed by V . The entries in the rows corresponding to an edge $e \in E$ and columns corresponding to a vertex $u \in V$ are given by the 2×3 matrix $X_{e,u}$ where $X_{e,u} = \begin{pmatrix} Q(e, q) \\ R(e, q) \end{pmatrix}$ if $e = uv$ is incident to u and $u < v$ in the ordering on V induced by the order of the column labels, $X_{e,u} = -\begin{pmatrix} Q(e, q) \\ R(e, q) \end{pmatrix}$ if $e = uv$ is incident to u and $u > v$, and $X_{e,u}$ is the zero matrix if e is not incident to u . We refer to $R_{BP}(G, q)$ as the *body-and-pin rigidity matrix* of (G, q) . By the above, a map $S : V \rightarrow \mathbb{R}^3$ is an infinitesimal motion of (G, q) if and only if S belongs to the null space, $Z_{BP}(G, q)$ of $R_{BP}(G, q)$. Hence the infinitesimal motions of (G, q) form a vector space. Since every body-and-pin framework (G, q) has three linearly independent (trivial) infinitesimal motions, obtained for example by putting $S(v) = \mathbf{x}$ for all $v \in V$, for each $\mathbf{x} \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, the dimension of $Z_{BP}(G, q)$ is at least three, and (G, q) is infinitesimally rigid if and only if $Z_{BP}(G, q)$ has dimension equal to three. Equivalently, $\text{rank } R_{BP}(G, q) \leq 3|V| - 3$ and (G, q) is infinitesimally rigid if and only if $\text{rank } R_{BP}(G, q) = 3|V| - 3$.

The matrix $R_{BP}(G, q)$ defines a 2-polymatroid $\mathcal{R}_{BP}(G, q)$ on the groundset E , in which the rank of a subset $E' \subseteq E$ is given by the rank of the submatrix of $R_{BP}(G, q)$ indexed by E' and V . We refer to $\mathcal{R}_{BP}(G, q)$ as the *body-and-pin polymatroid* of (G, q) and denote its rank by $r_{BP}(G, q)$.

The proof of the next lemma is similar to a related result for 3-dimensional frameworks due to Whiteley [23].

Lemma 5.1. *Let $G = (V, E)$ be a multigraph, (G^*, q) be a body-and-pin realization of G as a bar-and-joint framework, and (G, \hat{q}) be the body-and-pin framework associated to (G^*, q) . Then the null spaces of $R(G^*, q)$ and $R_{BP}(G, \hat{q})$ are isomorphic vector spaces.*

Proof: Let $Z(G^*, q)$ and $Z_{BP}(G, \hat{q})$ be the null spaces of $R(G^*, q)$ and $R_{BP}(G, \hat{q})$, respectively. Choose $\sigma \in Z(G^*, q)$. For $v \in V$, let G_v^* be the subgraph of G^* induced by $\{v\} \cup E_G(v)$, q_v the restriction of q to $V(G_v^*)$, and σ_v the restriction of σ to $V(G_v^*)$. Then the bar-and-joint framework (G_v^*, q_v) is infinitesimally rigid and $\sigma_v \in Z(G_v^*, q_v)$. By Lemma 2.9(b), the map $f : \mathbb{R}^3 \rightarrow Z(G_v^*, q_v)$ defined by $f(S) = f_S \circ q_v$ is an isomorphism. Thus we may define $S_\sigma : V \rightarrow \mathbb{R}^3$ by choosing $S_\sigma(v)$ to be the unique vector in \mathbb{R}^3 which satisfies $f(S_\sigma(v)) = \sigma_v$.

We next show that $S_\sigma \in Z_{BP}(G, \hat{q})$. It suffices to verify that, for each $e = uv \in E$, we have $S_\sigma(u) - S_\sigma(v) \in \langle P(e, \hat{q}) \rangle$. Since $e \in V(G_u^*) \cap V(G_v^*)$ we have $\sigma_u(e) = \sigma_v(e)$. Hence $f_{S_\sigma(u)}(q(e)) = f_{S_\sigma(v)}(q(e))$. By Lemma 2.8(a), $f_{S_\sigma(u) - S_\sigma(v)}(q(e)) = \mathbf{0}$. Now Lemma 2.8(b) and the fact that $\hat{q}(e) = q(e)$, imply that $S_\sigma(u) - S_\sigma(v) \in \langle P(e, \hat{q}) \rangle$. Thus $S_\sigma \in Z_{BP}(G, \hat{q})$.

We may now define $h : Z(G^*, q) \rightarrow Z_{BP}(G, \hat{q})$ by putting $h(\sigma) = S_\sigma$ for all $\sigma \in Z(G^*, q)$. It is easy to check that h is a bijective linear map. Hence $Z(G^*, q)$ and $Z_{BP}(G, \hat{q})$ are isomorphic. •

We say that a body-and-pin framework (G, q) is *generic* if the (multi)set containing the coordinates of the vectors $q(e)$, $e \in E$, is algebraically independent over \mathbb{Q} . Since the entries in $R_{BP}(G, q)$ are polynomial functions of the coordinates of the vectors $q(e)$, $e \in E$, with rational coefficients, all generic body-and-pin frameworks for G give rise to the same 2-polymatroid. We refer to this as the *body-and-pin polymatroid* of G . We denote it by $\mathcal{R}_{BP}(G)$, and its rank by $r_{BP}(G)$. We have $r_{BP}(G, q) \leq r_{BP}(G)$ for all body-and-pin frameworks (G, q) , with equality whenever (G, q) is generic.

Now suppose that (G^*, q) is a non-degenerate pin-collinear body-and-pin realization of a graph G as a bar-and-joint framework. The fact that, for each $uv \in E$, the point $q(uv)$ is uniquely determined as the point of intersection of $L_{(G^*, q)}(u)$ and $L_{(G^*, q)}(v)$ indicates that $r(G^*, q)$ will be uniquely determined by the pin-lines of (G^*, q) . We formalize this observation by defining yet another kind of framework, and use it to show that $r(G^*, q)$ is maximized when its pin-lines are ‘generic’.

A *rod-and-pin framework* (G, p) is a graph $G = (V, E)$, together with a map $p : V \rightarrow \mathbb{R}^2$ such that $p(u)$ and $p(v)$ are linearly independent for all $uv \in E$. Let $p(v) = (p_1(v), p_2(v))$ for each $v \in V$. We define the *pin-line* of v to be $L_v = \{(x, y) \in \mathbb{R}^2 : p_1(v)x + p_2(v)y = 1\}$. The map p induces a map $\tilde{p} : E \rightarrow \mathbb{R}^2$ by $\tilde{p}(uv) = L_u \cap L_v$ for all $uv \in E$. Thus

$$\tilde{p}(uv) = d(u, v)^{-1}(p_2(v) - p_2(u), p_1(u) - p_1(v)),$$

where

$$d(u, v) = \det \begin{pmatrix} p_1(u) & p_2(u) \\ p_1(v) & p_2(v) \end{pmatrix}.$$

The map \tilde{p} gives rise to a body-and-pin framework (G, \tilde{p}) which we refer to as the *body-and-pin framework associated to* (G, p) .⁵ We define the *rod-and-pin rigidity matrix* $R_{RP}(G, p)$ and 2-polymatroid $\mathcal{R}_{RP}(G, p)$ of (G, p) by putting $R_{RP}(G, p) = R_{BP}(G, \tilde{p})$ and $\mathcal{R}_{RP}(G, p) = \mathcal{R}_{BP}(G, \tilde{p})$. Let $r_{RP}(G, p) = \text{rank } R_{RP}(G, p)$.

We say that a rod-and-pin framework (G, p) is *generic* if the (multi)set containing the coordinates of the vectors $p(v)$, $v \in V$, is algebraically independent over \mathbb{Q} . Since the entries in $R_{RP}(G, p)$ are polynomial functions of the coordinates of the vectors $p(v)$, $v \in V$, with rational coefficients, all generic rod-and-pin frameworks for G give rise to the same 2-polymatroid. We refer to this as the *rod-and-pin polymatroid* of G . We denote it by $\mathcal{R}_{RP}(G)$, and its rank by $r_{RP}(G)$. We have $r_{RP}(G, p) \leq r_{RP}(G)$ for all rod-and-pin frameworks (G, p) , with equality whenever (G, p) is generic. Note also that even if (G, p) is a generic rod-and-pin framework, its associated body-and-pin framework (G, \tilde{p}) will not in general be generic. Thus $r_{RP}(G) = r_{RP}(G, p) = r_{BP}(G, \tilde{p}) \leq r_{BP}(G)$. We will see later that, by Theorems 7.1 and 7.2, equality must hold.

Let (G^*, q) be a non-degenerate pin-collinear body-and-pin realization of a graph $G = (V, E)$ in which no pin-line passes through the origin. Then the equation of a pin-line L_v can be uniquely written as $p_1(v)x + p_2(v)y = 1$ for each $v \in V$. We define the

⁵For each $v \in V$, the points $\tilde{p}(e)$, $e \in E(v)$, all lie on the line L_v . Thus (G, \tilde{p}) is a 2-dimensional body-and-hinge framework in which the 0-dimensional hinges, i.e. pins, incident with each body all lie on a 1-dimensional hyperplane, i.e. line. We may imagine each body as a rigid segment of this pin-line, i.e. rod, containing the pins it is incident with, see Figure 3.

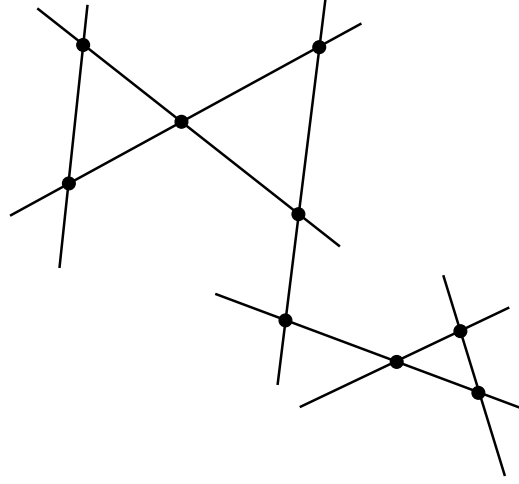


Figure 3: The rod-and-pin framework associated to the non-degenerate pin-collinear body-and-pin realization (G^*, q) of Figure 2.

rod-and-pin framework associated to (G^, q)* to be the rod-and-pin framework (G, p) where $p : V \rightarrow \mathbb{R}^2$ by $p(v) = (p_1(v), p_2(v))$ for all $v \in V$, see Figure 3.

We say (G^*, q) is *pin-line-generic* if (G^*, q) is non-degenerate and its associated rod-and-pin framework is generic. It is easy to see that every graph $G = (V, E)$ has a pin-line-generic pin-collinear body-and-pin realization (G^*, q) : we first choose an algebraically independent set $\{p_1(v), p_2(v) : v \in V\}$ to define the pin-lines L_v , then define $q : V \rightarrow \mathbb{R}^2$ by putting $q(uv)$ equal to the point of intersection of L_u and L_v for each $uv \in E$, and then choosing the points $q(v)$, $v \in V$, such that the subframework induced by $\{v\} \cup E(v)$ is infinitesimally rigid.

Our final result of this section verifies the intuitively obvious fact that the rank of a pin-collinear body-and-pin realization of a graph as a bar-and-joint framework will be maximized when it is pin-line-generic.

Lemma 5.2. *Let G be a graph with no isolated vertices and (G^*, q_0) be a pin-line-generic pin-collinear body-and-pin realization of G . Then $r(G^*, q_0) = \max\{r(G^*, q)\}$ over all pin-collinear body-and-pin realizations (G^*, q) of G .*

Proof: Let (G^*, q_1) be a pin-collinear body-and-pin realization of G . By Lemma 4.2 there exists a non-degenerate pin-collinear body-and-pin realization (G^*, q_2) of G such that $r(G^*, q_2) \geq r(G^*, q_1)$. Let (G, p_0) and (G, p_2) be the rod-and-pin frameworks associated to (G^*, q_0) and (G^*, q_2) , respectively. Then (G, p_0) is generic so $\text{rank } R_{RP}(G, p_0) \geq \text{rank } R_{RP}(G, p_2)$. Let (G, \hat{q}_0) and (G, \hat{q}_2) be the body-and-pin frameworks associated to (G^*, q_0) and (G^*, q_2) , respectively. Then $R_{RP}(G, p_0) = R_{BP}(G, \hat{q}_0)$ and $R_{RP}(G, p_2) = R_{BP}(G, \hat{q}_2)$. Hence $\dim Z_{BP}(G, \hat{q}_0) \leq \dim Z_{BP}(G, \hat{q}_2)$. Lemma 5.1 now implies that $\dim Z(G^*, q_0) \leq \dim Z(G^*, q_2)$ and hence $r(G^*, q_0) \geq r(G^*, q_2) \geq r(G^*, q_1)$.

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6 Maximum Rank of Pin-Collinear Body-and-Pin Realizations

Our first result determines the maximum rank of a pin-collinear body-and-pin realization of a graph as a bar-and-joint framework. It will be extended to multigraphs at the end of this section.

Theorem 6.1. *Let $G = (V, P)$ be a graph with no isolated vertices. Then the maximum rank of a pin-collinear body-and-pin realization of G as a bar-and-joint framework is $2(|V| + |P|) - 3 - \text{def}(G)$.*

Proof: By Lemmas 2.2(a) and 4.1 it will suffice to show that there exists a pin-collinear body-and-pin realization (G^*, q) of G such that $r(G^*, q) = 2(|V| + |P|) - 3 - \text{def}(G)$. We proceed by contradiction. Suppose there exists a graph $G = (V, P)$ such that, for all pin-collinear body-and-pin realizations (G^*, q) of G , we have $r(G^*, q) < 2(|V| + |P|) - 3 - \text{def}(G)$. We may suppose that G has been chosen such that $|V| + |E|$ is as small as possible. It can easily be seen that G has at least four vertices. We denote the number of vertices and edges of G by n and m , respectively. We will extend this notation using subscripts, so that for example, the number of vertices and edges in a graph G_1 will be denoted by n_1 and m_1 , respectively. We will frequently use the fact that if $n_1 < n$ then, by induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_1^*, q) of G_1 with $r(G_1^*, q) = 2(n_1 + m_1) - 3 - \text{def}(G_1)$. Since (G_1^*, q) is pin-line-generic, no two pin-lines of (G_1^*, q) are parallel and every point $q(p)$, $p \in P_1$, belongs to exactly two pin-lines in (G_1^*, q) .

Claim 6.2. *G is connected.*

Proof: Suppose the claim is false. Then there exist disjoint subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$. Clearly $\text{def}(G) = \text{def}(G_1) + \text{def}(G_2) + 3$. By induction, there exists a pin-collinear body-and-pin realization (G_i^*, q_i) of G_i such that $r(G_i^*, q_i) = 2(n_i + m_i) - 3 - \text{def}(G_i)$, for each $i \in \{1, 2\}$. Taking the union of (G_1^*, q_1) and (G_2^*, q_2) we obtain a pin-collinear body-and-pin realization (G^*, q) of G . By Lemma 2.6 satisfies

$$\begin{aligned} r(G^*, q) &= r(G_1^*, q_1) + r(G_2^*, q_2) \\ &= 2(n_1 + m_1) - 3 - \text{def}(G_1) + 2(n_2 + m_2) - 3 - \text{def}(G_2) \\ &= 2(n + m) - 3 - \text{def}(G). \end{aligned}$$

This contradicts the choice of G . •

Claim 6.3. *For each $v \in V$, $d_G(v) \geq 2$.*

Proof: Suppose there exists $v_1 \in V$ with $d_G(v_1) = 1$. Let $p_1 = u_1v_1$ be the edge of G incident to v_1 and $G_1 = G - v_1$. Let \mathcal{B}_1 be a tight partition of G_1 . Put $B = \{v_1\}$ and let $\mathcal{Q} = \mathcal{B}_1 \cup \{B\}$. Then $\text{def}(G) \geq \text{def}_G(\mathcal{Q}) = \text{def}_{G_1}(\mathcal{B}_1) + 1 = \text{def}(G_1) + 1$. By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_1^*, q_1) of G_1 such that $r(G_1^*, q_1) = 2(n_1 + m_1) - 3 - \text{def}(G_1)$. Choose $p_2 \in E_{G_1}(u_1)$.

Let Q_1 be a point on $L_{(G_1^*, q_1)}(u_1)$ such that $Q_1 \neq q_1(p)$ for all $p \in E_{G_1}(u_1)$. Choose another point $Q_2 \neq Q_1$. We may now extend (G_1^*, q_1) to a pin-collinear body-and-pin realization (G^*, q) of G by putting $q(p_1) = Q_1$ and $q(v_1) = Q_2$. Since G^* can be obtained from G_1^* by performing a 0-extension (adding the vertex p_1 and edges p_1u_1, p_1p_2), then adding the vertex v_1 and edge v_1p_1 , and finally adding the remaining edges of $E(G^*)$, Lemma 2.3 implies that

$$r(G^*, q) \geq r(G_1^*, q_1) + 3 = 2(n_1 + m_1) - 3 - \text{def}(G_1) + 3 \geq 2(n + m) - 3 - \text{def}(G).$$

This contradicts the choice of G . •

Claim 6.4. G is 2-edge-connected.

Proof: Suppose the claim is false. Then there exists $p_0 \in P$ and disjoint subgraphs G_1, G_2 of $G_0 = G - p_0$ such that $G_0 = G_1 \cup G_2$. By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_0^*, q_0) of G_0 which satisfies $r(G_0^*, q_0) = 2(n_0 + m_0) - 3 - \text{def}(G_0)$. Clearly $\text{def}(G) = \text{def}(G_0) - 2$. Let u_i be the vertex of G_i incident to p_0 , for each $i \in \{1, 2\}$ and Q be the point of intersection of the lines $L_{(G_0^*, q_0)}(u_1)$ and $L_{(G_0^*, q_0)}(u_2)$. We may extend (G_0^*, q_0) to a pin-collinear body-and-pin realization (G^*, q) of G by putting $q(p_0) = Q$. Let H_i be the subgraph of G^* induced by $V(G_i^*) \cup \{p_0\}$, q_i the restriction of q to $V(G_i^*)$ and q'_i the restriction of q to $V(H_i)$, for $i \in \{1, 2\}$. Lemmas 2.3 and 2.7 imply that

$$\begin{aligned} r(G^*, q) &= r(H_1, q'_1) + r(H_2, q'_2) \geq r(G_1^*, q_1) + 2 + r(G_2^*, q_2) + 2 \geq r(G_0^*, q_0) + 4 \\ &= 2(n_0 + m_0) - 3 - \text{def}(G_0) + 4 = 2(n + m) - 3 - \text{def}(G). \end{aligned}$$

This contradicts the choice of G . •

Claim 6.5. For each $v \in V$, $d_G(v) \geq 3$.

Proof: Suppose there exists $v_1 \in V$ with $d_G(v_1) = 2$. Let $p_1 = u_1v_1$ and $p_2 = u_2v_1$ be the edges of G incident to v_1 and $G_1 = G - v_1$. Let \mathcal{B}_1 be the brick partition of G_1 .

We first consider the case when u_1 and u_2 both belong to the same brick B_1 of G_1 . Let $B = B_1 + v_1$ and $\mathcal{Q} = \mathcal{B}_1 - \{B_1\} \cup \{B\}$. Then

$$\text{def}(G) \geq \text{def}_G(\mathcal{Q}) = \text{def}_{G_1}(\mathcal{B}_1) = \text{def}(G_1).$$

By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_1^*, q_1) of G_1 such that

$$r(G_1^*, q_1) = 2(n_1 + m_1) - 3 - \text{def}(G_1) \geq 2(n + m) - 9 - \text{def}(G).$$

For $i \in \{1, 2\}$, let Q_i be a point on $L_{(G_1^*, q_1)}(u_i)$ such that $Q_i \neq q_1(p)$, $p \in E_{G_1}(u_i)$, and $Q_1 \neq Q_2$. Choose a point Q which does not lie on the line through Q_1, Q_2 . We may now extend (G_1^*, q_1) to a pin-collinear body-and-pin realization (G^*, q) of

G by putting $q(p_i) = Q_i$ for $i \in \{1, 2\}$ and $q(v_1) = Q$. Lemma 2.3 implies that $r(G^*, q) \geq r(G_1^*, q_1) + 6 \geq 2(n + m) - 3 - \text{def}(G)$. This contradicts the choice of G .

Thus u_1 and u_2 must belong to distinct bricks of \mathcal{B}_1 . Let $B = \{v_1\}$ and $\mathcal{Q} = \mathcal{B}_1 \cup \{B\}$. Then

$$\text{def}(G) \geq \text{def}_G(\mathcal{Q}) = \text{def}_{G_1}(\mathcal{B}_1) - 1 = \text{def}(G_1) - 1.$$

Consider the following three cases.

Case 1. $u_1 u_2 \notin E$.

Let $p_0 = u_1 u_2$ and $G_2 = G_1 + p_0$. Since u_1, u_2 belong to distinct bricks of G_1 , $\text{def}(G_2) \leq \text{def}(G_1) - 1$. By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_2^*, q_2) of G_2 such that

$$r(G_2^*, q_2) = 2(n_2 + m_2) - 3 - \text{def}(G_2) \geq 2(n_1 + m_1) - 3 - \text{def}(G_1) + 3.$$

Let q_1 be the restriction of q_2 to $V(G_1^*)$. Then (G_1^*, q_1) is a pin-line-generic pin-collinear body-and-pin realization of G_1 so, again by induction and Lemma 5.2, $r(G_1^*, q_1) = 2(n_1 + m_1) - 3 - \text{def}(G_1)$. Thus $r(G_1^* + p_0 u_1 + p_0 u_2 + p_0 p_4, q_2) = 2(n_1 + m_1) - 3 - \text{def}(G_1) + 3$ for some $p_4 \in E_{G_1}(u_1) \cup E_{G_1}(u_2)$. By symmetry we may assume that $p_4 \in E_{G_1}(u_1)$. Let $H_2 = G_1^* + p_0 u_1 + p_0 u_2 + p_0 p_4$. By Lemma 2.2, $r(H_2, q_2)$ does not decrease if we move $q_2(p_0)$ in a small enough neighbourhood to a new position Q_1 , in such a way that it remains on $L_{(G_2^*, q_2)}(u_1)$, but no longer lies on $L_{(G_2^*, q_2)}(u_2)$, and such that the line L_0 through Q_1 and $q_2(u_2)$ intersects $L_{(G_2^*, q_2)}(u_2)$ at a point $Q_2 \neq q_2(p)$ for all $p \in E_{G_2}(u_2)$. Choose a point Q_0 such that Q_0, Q_1, Q_2 are not collinear. Define $q : V(G^*) \rightarrow \mathbb{R}^2$ by putting $q(x) = q_2(x)$ for $x \in V(G_2^*) - p_0$, $q(p_1) = Q_1$, $q(p_2) = Q_2$, and $q(v_1) = Q_0$. Then (G^*, q) is a pin-collinear body-and-pin realization of G . Since G^* can be obtained from H_2 by first relabelling p_0 as p_1 , then performing a 1-extension (deleting $p_1 u_2$ and adding $p_2, p_2 p_1, p_2 u_2, p_2 p_3$ for some $p_3 \in E_{G_1}(u_2)$), then performing a 0-extension (adding $v_1, v_1 p_1, v_1 p_2$), and finally adding the remaining edges of $E(G^*)$, Lemmas 2.3 and 2.4 imply that

$$r(G^*, q) \geq r(H_2, q_2) + 4 \geq 2(n_1 + m_1) - 3 - \text{def}(G_1) + 7 \geq 2(n + m) - 3 - \text{def}(G).$$

This contradicts the choice of G .

Case 2. $u_1 u_2 \in E$ and $d_G(u_i) \geq 3$ for all $i \in \{1, 2\}$.

Let $p_3 = u_1 u_2$, $E_{G_1}(u_1) = \{p_3, p_4, \dots, p_j\}$ and $E_{G_1}(u_2) = \{p_3, p_{j+1}, p_{j+2}, \dots, p_k\}$. Since $d_G(u_1), d_G(u_2) \geq 3$, $j \geq 4$ and $k \geq 5$. Since u_1, u_2 belong to distinct bricks of G_1 and $u_1 u_2 \in E$, no vertex of G_1 is adjacent to both u_1 , and u_2 . Thus the (multi)graph G_3 obtained from G_1 by contracting the edge p_3 onto a new vertex z contains no parallel edges. Furthermore, the fact that u_1, u_2 belong to distinct bricks of G_1 also implies that $\text{def}(G_3) \leq \text{def}(G_1) - 1$. Since $d_G(v_1) = 2$ and G is a 2-edge-connected graph on at least four vertices, G_3 contains no isolated vertices. By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_3^*, q_3) of G_3 such that

$$r(G_3^*, q_3) = 2(n_3 + m_3) - 3 - \text{def}(G_3).$$

Each vertex p_i , $4 \leq i \leq k$, lies on exactly two pin-lines of (G_3^*, q_3) . Let L_i be the pin line of (G_3^*, q_3) which contains p_i and is distinct from $L_{(G_3^*, q_3)}(z)$.

Let $S = \{p_s p_t : 4 \leq s \leq j, j+1 \leq t \leq k, (s, t) \neq (4, j+1)\}$, $H_0 = G_3^* - S$, F be the subgraph of H_0 induced by $\{z\} \cup \{p_i : 4 \leq i \leq k\}$, and q_0 be the restriction of q_3 to F . Since (F, q_0) is infinitesimally rigid, $r(H_0, q_3) = r(G_3^*, q_3)$. Let H_1 be the graph obtained from $H_0 - p_4 p_{j+1}$ by adding two new vertices, p_1, p_3 , and edges $p_1 p_3, p_1 p_4, p_1 z, p_3 p_{j+1}, p_3 z$. Choose two distinct points Q_1, Q_3 on $L_{(G_3^*, q_3)}(z)$ such that $Q_i \neq q_3(p)$ for all $i \in \{1, 3\}$ and $p \in E_{G_3}(z)$, and define $q_4 : V(H_1) \rightarrow \mathbb{R}^2$ by $q_4(x) = q_3(x)$ for $x \in V(G_3^*)$, $q_4(p_1) = Q_1$ and $q_4(p_3) = Q_3$. Since (H_1, q_4) can be obtained from (H_0, q_3) by applying two 1-extensions, Lemma 2.4 implies that $r(H_1, q_4) = r(H_0, q_3) + 4 = r(G_3^*, q_3) + 4$.

Let H_2 be the graph obtained from $H_1 - z$ by adding two new vertices, u_1, u_2 , edges $u_1 p_i$ for $i \in \{1, 3, 4, \dots, j\}$, and edges $u_2 p_i$ for $i \in \{1, 3, j+1, j+2, \dots, k\}$. Define $q_5 : V(H_2) \rightarrow \mathbb{R}^2$ by $q_5(x) = q_4(x)$ for $x \in V(H_1) - z$, and $q_5(u_1) = q_5(u_2) = q_4(z)$. Since (H_2, q_5) can be obtained from (H_1, q_4) by a vertex-split, Lemma 2.5 implies that $r(H_2, q_5) \geq r(H_1, q_4) + 2$. By Lemma 2.2, there exists a neighborhood S_i around each point $q_5(p_i)$, $i \in \{1, 4, \dots, j\}$, such that $r(H_2, q_5)$ does not decrease if we move $q_5(p_i)$ within S_i . Thus we may modify (H_2, q_5) by moving each point $q_5(p_i)$, $i \in \{1, 4, \dots, j\}$, slightly, in such a way that it continues to lie on L_i and belong to S_i , and also such that $p_1, p_3, p_4, \dots, p_j$ all lie on a line L_0 which is not parallel to $L_{(G_3^*, q_3)}(z)$. (We may imagine L_0 is obtained by a small rotation of $L_{(G_3^*, q_3)}(z)$ about the point $q_5(p_3)$.)

Let L_1 be the line through $q_5(p_1)$ and $q_5(u_2)$. By using Lemma 2.2(b) to move $q_5(u_2)$ if necessary, we may suppose that Q_2 , the point of intersection of L_1 and $L_{(G_3^*, q_3)}(z)$, is distinct from $q_5(p)$ for all $p \in E_{G_1}(u_2)$. Choose a point Q_0 which does not lie on L_1 . Define $q : V(G^*) \rightarrow \mathbb{R}^2$ by $q(x) = q_5(x)$ for $x \in V(G^*) - \{p_2, v_1\}$, $q(p_2) = Q_2$ and $q(v_1) = Q_0$. Then (G^*, q) is a pin-collinear body-and-pin realization of G . Since (G^*, q) can be obtained from (H_2, q_5) by a 1-extension, (which deletes $u_2 p_1$, and adds a new vertex p_2 and edges $p_2 p_1, u_2 p_2, p_2 p_3$), a 0-extension (which adds $v_1, v_1 p_1, v_1 p_2$) and by adding other edges, we have

$$\begin{aligned} r(G^*, q) &\geq r(H_2, q_5) + 4 \geq r(H_1, q_4) + 6 = r(G_3^*, q_3) + 10 \\ &= 2(n_3 + m_3) - 3 - \text{def}(G_3) + 10 \geq 2(n + m) - 3 - \text{def}(G), \end{aligned}$$

since $\text{def}(G) \geq \text{def}(G_1) - 1 \geq \text{def}(G_3)$. This contradicts the choice of G .

Case 3. $u_1 u_2 \in E$ and $d_G(u_i) = 2$ for some $i \in \{1, 2\}$.

Let $p_3 = u_1 u_2$. Suppose, without loss of generality, that $E_G(u_2) = \{p_2, p_3\}$. Let $G_4 = G - \{v_1, u_2\}$. Note that G is obtained from G_4 by attaching a complete graph on three vertices (which is strong) at vertex u_1 , and hence $\text{def}(G_4) = \text{def}(G)$. By Claim 6.4 we have $|E_{G_4}(u_1)| \geq 2$.

By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_4^*, q_6) of G_4 such that

$$r(G_4^*, q_6) = 2(n_4 + m_4) - 3 - \text{def}(G_4).$$

Let $p_4 \in E_{G_4}(u_1)$. Let H_4 be the graph obtained from G_4^* by adding new vertices p_1, p_2, p_3, v_1, u_2 and edges $p_1 u_1, p_1 p_4, p_3 u_1, p_3 p_4, p_2 p_1, p_2 p_3, v_1 p_1, v_1 p_2, u_2 p_3, u_2 p_2$.

Choose distinct points Q_1, Q_3 on $L_{(G_4^*, q_6)}(u_1)$ such that $Q_i \neq q_6(p)$ for all $i \in \{1, 3\}$ and $p \in E_{G_4}(u_1)$. Choose a point Q_2 which is not on $L_{(G_4^*, q_6)}(u_1)$, and choose points Q_5, Q_6 such that Q_5 is not on the line through Q_1, Q_2 , and Q_6 is not on the line through Q_3, Q_2 . Define $q_7 : V(H_4) \rightarrow \mathbb{R}^2$ by $q_7(x) = q_6(x)$ for $x \in V(G_4^*)$, $q_7(p_1) = Q_1$, $q_7(p_3) = Q_3$, $q_7(p_2) = Q_2$, $q_7(v_1) = Q_5$, and $q_7(u_2) = Q_6$. Since (H_4, q_7) can be obtained from (G_4^*, q_6) by five 0-extensions, Lemma 2.3 implies that $r(H_4, q_7) = r(G_4^*, q_6) + 10$. Since $\text{def}(G_4) = \text{def}(G)$ and $|V(G^*)| = |V(G_4^*)| + 5$, we have

$$r(G^*, q_7) \geq r(H_4, q_7) = r(G_4^*, q_6) + 10 = 2(n + m) - 3 - \text{def}(G).$$

This contradicts the choice of G . •

Claim 6.6. *For each $p \in P$, $\text{def}(G - p) \geq \text{def}(G) + 1$.*

Proof: Clearly $\text{def}(G - p) \geq \text{def}(G)$. Suppose $\text{def}(G - p_1) = \text{def}(G)$ for some $p_1 \in P$. Let $p_1 = v_1v_2$ and $G_1 = G - p_1$. By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_1^*, q_1) of G_1 such that $r(G_1^*, q_1) = 2(n_1 + m_1) - 3 - \text{def}(G_1)$. Let Q be the point of intersection of $L_{(G_1^*, q_1)}(v_1)$ and $L_{(G_1^*, q_1)}(v_2)$. We may extend (G_1^*, q_1) to a pin-collinear body-and-pin realization (G^*, q) of G by putting $q(p_1) = Q$. Lemma 2.3 implies that

$$r(G^*, q) \geq r(G_1^*, q_1) + 2 = 2(n_1 + m_1) - 3 - \text{def}(G_1) + 2 = 2(n + m) - 3 - \text{def}(G).$$

This contradicts the choice of G . •

Claim 6.7. *G is not 3-edge-connected.*

Proof: Choose $p_0 \in P$ and let $G_0 = G - p_0$. By Claim 6.6, $\text{def}(G_0) \geq \text{def}(G) + 1 \geq 1$. Thus G_0 is not a brick. Let \mathcal{B}_0 be a tight partition of G_0 . We have $\text{def}(G_0) = \text{def}_{G_0}(\mathcal{B}_0) = 3(|\mathcal{B}_0| - 1) - 2e_{G_0}(\mathcal{B}_0) \geq 1$. Thus $2e_{G_0}(\mathcal{B}_0) \leq 3|\mathcal{B}_0| - 4$ and $2e_G(\mathcal{B}_0) \leq 3|\mathcal{B}_0| - 2$. Hence, there exists $B \in \mathcal{B}_0$ such that $d_G(B) \leq 2$. •

Claim 6.8. *If $S = \{p_1, p_2\}$ is a 2-edge-cut of G then $\text{def}(G - p_1) = \text{def}(G) + 2$.*

Proof: Let $p_1 = uv$ and $G_1 = G - p_1$. It follows from the definition of $\text{def}(G)$ and Claim 6.6 that $\text{def}(G) + 1 \leq \text{def}(G_1) \leq \text{def}(G) + 2$. Suppose that $\text{def}(G_1) = \text{def}(G) + 1$. By induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_1^*, q_1) of G_1 such that $r(G_1^*, q_1) = 2(n_1 + m_1) - 3 - \text{def}(G_1)$. Let Q be the point of intersection of $L_{(G_1^*, q_1)}(u)$ and $L_{(G_1^*, q_1)}(v)$. We may assume that Q does not lie on the line through $q_1(p_2)$ and $q_1(v)$, since if it does, then we can use Lemma 2.2(b) to move $q_1(v)$ in a small neighbourhood without decreasing $r(G_1^*, q_1)$. Choose $p_3 \in E_{G_1}(u)$ and let H_1 be the graph obtained from G_1^* by adding the vertex p_1 and edges p_1u, p_1p_3 . We may extend (G_1^*, q_1) to a bar-and-joint realization (H_1, q) of H_1 by putting $q(p_1) = Q$. Lemma 2.3 implies that $r(H_1, q) = r(G_1^*, q_1) + 2$. Let H_2 be obtained from H_1 by adding the edge p_1v . Since Q is not on the line through $q_1(p_2)$

and $q_1(v)$, the infinitesimal motion of (H_1, q) which keeps both p_2 and the component of $H_1 - p_2$ containing u fixed, and rotates the other component about p_2 , is not an infinitesimal motion of (H_2, q) (since it changes $\|q(p_1) - q(v)\|$). Thus (H_2, q) has fewer degrees of freedom than (H_1, q) and $r(H_2, q) = r(H_1, q) + 1$. Hence (G^*, q) is a pin-collinear body-and-pin realization of G for which

$$\begin{aligned} r(G^*, q) &\geq r(H_2, q) \geq r(G_1^*, q_1) + 3 \\ &= 2(n_1 + m_1) - 3 - \text{def}(G_1) + 3 = 2(n + m) - 3 - \text{def}(G). \end{aligned}$$

This contradicts the choice of G . •

Claim 6.9. G is not a superbrick.

Proof: Suppose G is a superbrick. Then $2e_G(\mathcal{P}) \geq 3(|\mathcal{P}| - 1) + 1$ for all partitions \mathcal{P} of G with $|\mathcal{P}| \geq 2$. Hence $\text{def}(G - p) \leq \text{def}(G) + 1$ for all $p \in P$. This contradicts Claims 6.7, 6.8. •

We now continue the proof of the theorem. Let \mathcal{B} be the superbrick partition of G . By Claim 6.9, $|\mathcal{B}| \geq 2$. Since $\text{def}(G) = 3(|\mathcal{B}| - 1) - 2e_G(\mathcal{B}) \geq 0$, we may use a similar argument to that given in the proof of Claim 6.7 to deduce that there exists a superbrick $B_1 \in \mathcal{B}$ with $d_G(B_1) = 2$. By Claim 6.5, we have $2 \leq |B_1| \leq n - 2$. Let $p_1 = u_1v_1$ and $p_2 = u_2v_2$ be the edges in G from B_1 to $V - B_1$, where $u_1, u_2 \in B_1$. Let H_1, H_2 be the components of $G - \{p_1, p_2\}$ where $V(H_1) = B_1$. Let G_1 be the graph obtained from H_1 by adding two new vertices w_0, w_1 and edges $p_3 = u_1w_0, p_4 = w_0w_1, p_5 = w_1u_2$. Let G_2 be the graph obtained from H_2 by adding a new vertex w_2 and edges $p_6 = v_1w_2, p_7 = v_2w_2$.

Claim 6.10. (a) G_1 is a brick.
(b) $\text{def}(G) \geq \text{def}(G_2)$.

Proof: (a) Suppose G_1 is not a brick. Let \mathcal{B}_1 be the brick partition of G_1 . Then $|\mathcal{B}_1| \geq 2$. Since $G[B_1]$ is strong and since strong graphs with at least two vertices have minimum degree at least two, we must have $\mathcal{B}_1 = \{B_1, \{w_0\}, \{w_1\}\}$. But then $\text{def}(G_1) = \text{def}_{G_1}(\mathcal{B}_1) = 0$, contradicting the assumption that G_1 is not a brick.

(b) Consider the brick partition \mathcal{B}_2 of G_2 . Let B_2 be the brick of G_2 which contains w_2 and $\mathcal{Q} = \mathcal{B}_2 - \{B_2\} \cup \{B_2 \cup B_1\}$. Then

$$\text{def}(G) \geq \text{def}_G(\mathcal{Q}) = \text{def}_{G_2}(\mathcal{B}_2) = \text{def}(G_2).$$
•

It follows from Claims 6.5 and 6.9 that $|V(G_i)| < |V(G)|$ for $i \in \{1, 2\}$. Thus, by induction and Lemma 5.2, there exists a pin-line-generic pin-collinear body-and-pin realization (G_i^*, q_i) of G_i such that $r(G_i^*, q_i) = 2(n_i + m_i) - 3 - \text{def}(G_i)$, for each $i \in \{1, 2\}$. Since $r(G_2^*, q_2)$ is preserved by a translation, rotation, and dilation of \mathbb{R}^2 , we may assume that $q_1(p_3) = q_2(p_6)$ and $q_1(p_5) = q_2(p_7)$. Define $q : V(G^*) \rightarrow \mathbb{R}^2$

by putting $q(x) = q_1(x)$ for $x \in V(G_1^*) - \{w_0, w_1, p_3, p_4, p_5\}$, $q(x) = q_2(x)$ for $x \in V(G_2^*) - \{w_2, p_6, p_7\}$, $q(p_1) = q_1(p_3) = q_2(p_6)$, and $q(p_2) = q_1(p_5) = q_2(p_7)$. Then (G^*, q) is a pin-collinear body-and-pin realization of G .

By Claim 6.10(a), $\text{def}(G_1) = 0$. Thus $r(G_1^*, q_1) = 2(n_1 + m_1) - 3$ and (G_1^*, q_1) is infinitesimally rigid. Hence $r(G_1^* + p_3p_5, q_1) = r(G_1^*, q_1)$. Let $F_1 = G_1^* - \{w_0, w_1, p_4\}$ and $F_2 = G_2^* - \{w_2\}$. For $i = 1, 2$ let t_i be the restriction of q_i to $V(F_i)$. Using Lemma 2.3 we may deduce that $r(F_1, t_1) = 2(n_1 + m_1) - 3 - 6 = r(F_1 + p_3p_5, t_1)$ and $r(F_2, t_2) = 2(n_2 + m_2) - 3 - \text{def}(G_2) - 2$. Now Lemma 2.7 implies that

$$\begin{aligned} r(G^* + p_1p_2, q) &= r(F_1 + p_3p_5, t_1) + r(F_2, t_2) - 1 \\ &= 2(n_1 + m_1) + 2(n_2 + m_2) - \text{def}(G_2) - 15. \end{aligned}$$

Since $r(F_1, t_1) = r(F_1 + p_3p_5, t_1)$ we have $r(G^*, q) = r(G^* + p_1p_2, q)$. Since $n = n_1 + n_2 - 3$ and $m = m_1 + m_2 - 3$ this gives

$$r(G^*, q) = 2(n + m) - 3 - \text{def}(G_2) \geq 2(n + m) - 3 - \text{def}(G),$$

by Claim 6.10(b). This contradicts the choice of G and completes the proof of the theorem. \bullet

We close this section by extending Theorem 6.1 to multigraphs.

Theorem 6.11. *Let $G = (V, P)$ be a multigraph without isolated vertices. Then there exists a pin-collinear body-and-pin realization (G^*, q) of G such that $r(G^*, q) = 2(|V| + |P|) - 3 - \text{def}(G)$.*

Proof: Suppose the theorem is false and choose a counterexample G with as few multiple edges as possible. If G has no multiple edges then the theorem follows from Theorem 6.1. Hence G has a pair u_1, u_2 of vertices joined by $t \geq 2$ parallel edges. We may show that G is connected as in the proof of Claim 6.2. If $t \geq 3$ then we may proceed as in the proof of Claim 6.6, since deleting one of the parallel edges does not increase the deficiency in this case. Thus we may suppose that $t = 2$ and proceed as in Case 2 of the proof of Claim 6.5 when both u_1 and u_2 have a neighbour outside u_1, u_2 , and as in Case 3 of the proof of Claim 6.5 when one of u_1, u_2 has its only neighbour in $\{u_1, u_2\}$. (Note that the graph G_1 obtained from G by contracting the multiple edge u_1u_2 to a single vertex satisfies $\text{def}(G_1) = \text{def}(G)$.) \bullet

Corollary 6.12. *Let $G = (V, P)$ be a multigraph without isolated vertices. Then $r(G^*) = 2(|V| + |P|) - 3 - \text{def}(G)$.*

Proof: This follows immediately from Theorem 6.11 and Lemmas 4.1 and 2.2(a). \bullet

Corollary 6.13. *Let $G = (V, P)$ be a multigraph. Then G has an infinitesimally rigid pin-collinear body-and-pin realization if and only if G has an infinitesimally rigid body-and-pin realization.*

Proof: This follows by putting $\text{def}(G) = 0$ in Theorem 6.11 and Corollary 6.12. \bullet

7 The Molecular Conjecture in 2-Dimensional Space

In this section we use our results on body-and-pin realizations of graphs as bar-and-joint frameworks to show that the rank functions for the body-and-pin and rod-and-pin 2-polymatroids of a graph are identical. We also deduce that the Molecular Conjecture holds in 2-dimensional space.

Theorem 7.1. *Let $G = (V, E)$ be a graph with no isolated vertices. Then $r_{RP}(G) = 3|V| - 3 - \text{def}(G)$.*

Proof: Let (G^*, q) be a pin-line-generic pin-collinear body-and-pin realization of G as a bar-and-joint framework. By Theorem 6.1 and Lemma 5.2, $r(G^*, q) = 2(|V| + |E|) - 3 - \text{def}(G)$ and hence $\dim Z(G^*, q) = 3 + \text{def}(G)$. Let (G, p) be the rod-and-pin framework associated to (G^*, q) and (G, \hat{q}) be the body-and-pin framework associated to (G^*, q) . By Lemma 5.1, $\dim Z(G^*, q) = \dim Z_{BP}(G, \hat{q})$. Thus $\dim Z_{RP}(G, p) = \dim Z_{BP}(G, \hat{q}) = 3 + \text{def}(G)$. Hence $r_{RP}(G) = r_{RP}(G, p) = 3|V| - 3 - \text{def}(G)$. •

Theorem 7.2. *Let $G = (V, E)$ be a multigraph with no isolated vertices.*

(a) $r_{BP}(G) = 3|V| - 3 - \text{def}(G)$.

(b) *There exists a body-and-pin framework (G, q) such that $r_{BP}(G) = 3|V| - 3 - \text{def}(G)$ and, for each $v \in V$, the sets of points $\{q(e) : e \in E_G(v)\}$ are collinear.*

Proof: We proceed as in the proof of Theorem 7.1. For (a), we choose a generic body-and-pin realization (G^*, q_1) of G as a bar-and-joint framework, and use Corollary 6.12 rather than Theorem 6.1. For (b), we again choose a pin-line-generic pin-collinear body-and-pin realization (G^*, q_1) of G as a bar-and-joint framework and consider the body-and-pin framework (G, \hat{q}) associated to (G^*, q_1) . •

Theorems 7.1 and 7.2 imply that, if G is a graph, then its rod-and-pin 2-polymatroid $\mathcal{R}_{RP}(G)$ is identical to its body-and-pin 2-polymatroid $\mathcal{R}_{BP}(G)$. Theorems 7.2 and 3.1 also imply the following characterization of infinitesimal rigidity (by taking $\text{def}(G) = 0$), and hence solve the 2-dimensional version of the Molecular Conjecture.

Theorem 7.3. *Let $G = (V, E)$ be a multigraph. Then the following statements are equivalent.*

(a) G has a realization as an infinitesimally rigid body-and-hinge framework in \mathbb{R}^2 .

(b) G has a realization as an infinitesimally rigid body-and-hinge framework (G, q) in \mathbb{R}^2 with each of the sets of points $\{q(e) : e \in E_G(v)\}$, $v \in V$, collinear.

(c) $2G$ contains three edge-disjoint spanning trees.

A body-and-pin framework (G, q) is *independent* if the rows of $R_{BP}(G, q)$ are linearly independent i.e. $r_{BP}(G, q) = 2|E|$. We close this section by using Theorems 7.2 and 3.1 to derive a result of Whiteley which characterizes when a graph G can be realized as an independent ‘pin-collinear’ body-and-pin framework.

Theorem 7.4. [19, Theorem 5.4] Let $G = (V, E)$ be a multigraph. Then the following statements are equivalent.

- (a) G has a realization as an independent body-and-hinge framework in \mathbb{R}^2 .
- (b) G has a realization as an independent body-and-hinge framework (G, q) in \mathbb{R}^2 with each of the sets of points $\{q(e) : e \in E_G(v)\}$, $v \in V$, collinear.
- (c) $2G$ can be covered by three forests.

Proof: We may suppose that G has no isolated vertices. By Theorem 7.2, it will suffice to show that $2|E| = 3|V| - 3 - \text{def}(G)$ if and only if $2G$ can be covered by three forests. This follows easily from Theorem 3.1(a). •

Note that an infinitesimally rigid body-and-pin framework (G, q) need not have an independent infinitesimally rigid spanning subframework. For example, when $|V|$ is even, we have $r_{BP}(G, q) = 3|V| - 3$ is odd, whereas all independent body-and-pin frameworks have even rank. On the other hand, the body-and-pin 2-polymatroid of an arbitrary body-and-pin framework (G, q) is linear, so we can determine the maximum rank of an independent subframework of (G, q) using the results of Lovász [8].

8 Concluding Remarks

In the body-and-hinge frameworks investigated so far in this paper, each hinge is shared by exactly two bodies. We can obtain more general structures by relaxing this condition.

An *identified body-and-hinge framework* in \mathbb{R}^d is an ordered pair (H, q) where $H = (V \cup P, I)$ is a bipartite graph and q is a map which associates a $(d - 2)$ -dimensional affine subspace $q(p)$ with each $p \in P$. Infinitesimal motions and infinitesimal rigidity of (H, q) are defined in an analogous way as for body-and-hinge frameworks. Tay and Whiteley [16, Conjecture 2, Theorem 3] give a conjectured characterization of when a bipartite graph has an infinitesimally rigid realization as a d -dimensional identified body-and-hinge framework and point out that their conjecture holds when $d = 2$. Indeed we may use the rank formula for the 2-dimensional generic (bar-and-joint) rigidity matroid given by Lovász and Yemini in [9] to determine the maximum rank of a realization of a bipartite graph as a 2-dimensional identified body-and-hinge framework. To see this we extend the definition of a body-and-pin graph.

Let $H = (V \cup P, I)$ be a bipartite graph without isolated vertices. The *identified body-and-pin graph* of H is the graph H^{BP} with $V(H^{BP}) = V \cup P$ and

$$E(H^{BP}) = \{vp : v \in V, p \in P, vp \in I\} \cup \{p_1p_2 : v \in V \text{ and } p_1, p_2 \in E_H(v)\}.$$

(This definition extends the earlier definition for a graph G by taking H to be the bipartite graph obtained by subdividing each edge of G . We then have $G^* = H^{BP}$.) Let \mathcal{F} be a partition of V . For each $p \in P$ let $w_{\mathcal{F}}(p)$ be the number of sets $F \in \mathcal{F}$ for which $N_H(p) \cap F \neq \emptyset$. Put $\text{def}_H(\mathcal{F}) = 3(|\mathcal{F}| - 1) - 2(\sum_{p \in P} (w_{\mathcal{F}}(p) - 1))$ and let $\text{def}(H) = \max_{\mathcal{F}} \{\text{def}_H(\mathcal{F})\}$. By using the above mentioned rank formula for the 2-dimensional generic rigidity matroid it is not difficult to show that $r(H^{BP}) = 2(|V| + |P|) - 3 - \text{def}(H)$.

We believe that Theorem 6.11 can be extended to identified body-and-pin graphs.

Conjecture 8.1. *Let H be a bipartite graph. Then there exists a pin-collinear realization (H^{BP}, q) of H^{BP} such that $r(H^{BP}, q) = r(H^{BP})$.*

An affirmative answer to Conjecture 8.1 for the special case when H has a realization as an independent body-and-pin framework follows from the above mentioned result of Whiteley [19, Theorem 5.4]. Whiteley also formulated a similar conjecture to Conjecture 8.1 for 3-dimensional frameworks in [19, Page 93].

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