Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2006-09. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Matroid parity and jump systems: a solution to a conjecture of Recski

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December 2006

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Abstract

In 1981 András Recski conjectured that if we are given a number $q \in \mathbb{N}$, a linearly represented matroid M, a subpartition $S_1 \dot\cup \ldots \dot\cup S_n$ of its ground set S into classes of size k, and a prescription $A \subseteq \{0, 1, \ldots, k\}$ without two consecutive gaps, then one can find in polynomial time an independent set F of M of size q such that $|F \cap S_i| \in A$ for all $1 \leq i \leq n$, if one exists. In this paper we prove this conjecture. The proof is based on Lovász' result on the polynomial solvability of the matroid parity problem for linearly represented matroids, and on an important technique about jump systems, proved by Sebő. We give an application to rigidity theory, and another one to the unique solvability of linear networks containing memoryless multiports.

1 Introduction

Matroid parity is a problem of high importance in combinatorial optimization. It is a common generalization of the matching problem of graphs and the matroid intersection problem. If M is a matroid on ground set S, and some (not necessarily disjoint) pairs are designated in S, then we call a set of k pairs a **matching** in M, if their union has rank $2k$ (that is, maximum possible). Given a matroid together with some designated pairs of its ground set, the matroid parity problem is to find a maximum matching in it. Jensen and Korte [5], and Lovász [8] have shown that in general the matroid parity problem is of exponential complexity under the independence oracle framework. However, as a groundbreaking result, Lovász [7] gave a polynomial algorithm for finding a maximum matching in a matroid which is linearly represented over $\mathbb{Q}$. The diversity and the importance of solvable special cases of the matroid parity problem makes it a vivid and intriguing field of research. For more references and background, see [10, 16].

Definition 1.1. We denote by $\mathbb{N}$ the set of non-negative integers. If $H \subseteq \mathbb{N}$ then $i \in \mathbb{N} - H$ is called a **gap** of H if $\min H < i < \max H$.

In [14] Recski studied the following slight generalization of the matroid parity problem. For $k \in \mathbb{N}$ and $A \subseteq \{0, \ldots, k\}$ we define the problem $P(A)$ as follows. Given a number $q \in \mathbb{N}$, a matroid $M = (S, r)$, and a subpartition $\pi = \dot\bigcup_{t \in T} S_t$ of S with classes of size k, decide whether M has an independent set F of size q such that

$|F \cap S_t| \in A$ for all $t \in T$. This problem was motivated by engineering applications, see [12, 13]. It is easy to see that if π is a partition into pairs and $A = \{0, 2\}$ then we get back the matroid parity problem. Recski [14] proved that under the independence oracle model, $P(A)$ is polynomial if A is an interval, and is of exponential complexity otherwise. He also proved that if we are given a representation of M over $\mathbb{Q}$ then $P(A)$ is NP-complete if A has two adjacent gaps, and for the prescriptions A without two consecutive gaps he conjectured $P(A)$ to be polynomial [14]. In this paper we verify a slight extension of this conjecture, Theorem 1.3 below.

Definition 1.2. Let S be a finite set, $S = \dot{\bigcup}_{t \in T} S_t$ be a partition and $H(t) \subseteq \mathbb{N}$ be prescriptions for all $t \in T$. We say that a set $F \subseteq S$ is **H-conformal** if $|F \cap S_t| \in H(t)$ for all $t \in T$.

Theorem 1.3. *The following problem is solvable in polynomial time. Given $q \in \mathbb{N}$, a matroid $M = (S, r)$ represented over $\mathbb{Q}$, a partition $S = \dot{\bigcup}_{t \in T} S_t$, and for all $t \in T$ a prescription $H(t) \subseteq \mathbb{N}$ without two consecutive gaps. Find an H-conformal independent set F of M of size q, if one exists.*

We show two applications. First we prove a result on a three-dimensional pinning problem of bar-and-joint frameworks. Then we show that Theorem 1.3 yields a polynomially verifiable sufficient condition to unique solvability of linear networks with memoryless multiports, a question raised by Recski and Takács [13].

2 Proof of the conjecture

Jump systems play in important role in the proof.

Definition 2.1. For $F \subseteq T$ we denote by $\chi_F \in \{0, 1\}^T$ the characteristic vector of F. For $a, b \in \mathbb{Z}^T$ we say that a' is a **step from** a **to** b, if either $a' = a + \chi_t$ and $a(t) < b(t)$, or $a' = a - \chi_t$ and $a(t) > b(t)$ for some $t \in T$. We call $J \subseteq \mathbb{Z}^T$ a **jump system** if for all $a, b \in J$ and step a' from a to b, either $a' \in J$ or some step from a' to b is contained in J.

Let $J \subseteq \mathbb{Z}^S$ be a jump system and $\varphi : S \to T$ a function. For $x \in \mathbb{Z}^S$ let $\varphi x \in \mathbb{Z}^T$ be the vector with components $\varphi x(t) = \sum\{x(s) : \varphi(s) = t\}$ for $t \in T$. Finally, let $\varphi(J) = \{\varphi x : x \in J\}$, the **homomorphic image** of J.

Jump systems were introduced by Bouchet and Cunningham [1], who justified their importance by claiming that the following sets are jump systems:

- the set of degree sequences of all subgraphs of an undirected graph $G = (V, E)$, denoted by $J_G \subseteq \mathbb{N}^V$,
- the set of characteristic vectors of bases of a matroid $M = (S, r)$, denoted by $J_M \subseteq \mathbb{N}^S$.

The notion of homomorphic image was introduced by Cunningham [3], who proved that the homomorphic image of a jump system is a jump system again.

Definition 2.2. Let T be a finite set, $J \subseteq \mathbb{Z}^T$ be a jump system and $H(t) \subseteq \mathbb{N}$ be a prescription for $t \in T$. For $x \in \mathbb{Z}^T$ and $t \in T$ define

$$\delta_H^x(t) = \min\{|x(t) - i| : i \in H(t)\},$$

and let $\delta_H^x = \sum\{\delta_H^x(t) : t \in T\}$. Finally, $\delta_H = \min\{\delta_H^x : x \in J\}$. A vector $x \in J$ minimizing δ_H^x is called **H-optimal**.

A set $H \subseteq \mathbb{N}$ of the form $\{l, l+2, \ldots, u-2, u\}$ is called a **parity interval**.

The membership problem of jump systems is to determine δ_H, under the assumption that no $H(t)$ contains two consecutive gaps. An involved formula to the membership problem was worked out by Lovász [9]. We do not need this deep result here, only a simpler but very important tool, Lemma 2.4, which was developed by Sebő. This lemma appeared in Cornuéjols [2], where it was originally used in constructing the first polynomial algorithm to the so-called *degree prescribed subgraph problem*, that is the problem of determining δ_H for a jump system J_G, if $G = (V, E)$ is a graph. Building on this lemma, it was possible to reduce the improvement of δ_H^x to the much easier problem of determining $\delta_{H'}$, where $H'(v)$ is a parity interval for all $v \in V$. This is exactly what we shall do, but now for a jump system $\varphi(J_M)$, where M is a matroid.

Definition 2.3 (Sebő [2]). Let $J \subseteq \mathbb{Z}^T$ be a jump system, $H(t) \subseteq \mathbb{Z}$, for $t \in T$, be a prescription without two consecutive gaps, and let $x \in J$. Define $I(t) \subseteq \mathbb{Z}$ to be the maximal interval such that $x(t) \in I(t)$ and the elements of

$$H^x(t) := I(t) \cap H(t)$$

have the same parity. (Observe that either $I(t)$ is unbounded from above or $\max I(t)$ and $\max I(t) + 1$ belong to $H(t)$. Similarly from below.) Now we define at most $2|T| + 1$ collections of prescriptions:

- H^x.
- ${}^wH^x$ for $w \in T$ with $I(w)$ bounded from above: ${}^wH^x(t) = H^x(t)$ for $t \neq w$ and ${}^wH^x(w) = \max I(w) + 1$.
- ${}_wH^x$ for $w \in T$ with $I(w)$ bounded from below: ${}_wH^x(t) = H^x(t)$ for $t \neq w$ and ${}_wH^x(w) = \min I(w) - 1$.

Lemma 2.4 (Sebő [2]). *Let $J \subseteq \mathbb{Z}^T$ be a jump system and $H(t) \subseteq \mathbb{Z}$, for $t \in T$, be a prescription without two consecutive gaps. Let $x \in J$ be a vector which is not H-optimal. Then there exists an $x_0 \in J$ such that $\delta_{H_0}^{x_0} < \delta_H^x$ for either $H_0 = H^x$ or $H_0 = {}^wH^x$ or $H_0 = {}_wH^x$ for some $w \in T$.*

Proof. Let $x^* \in J$ with $\delta_H^{x^*} < \delta_H^x$. Choose x' minimizing $\sum\{|x'(t) - x^*(t)| : t \in T\}$ among all vectors $x' \in J$ for which $\delta_H^{x'} \leq \delta_H^x$ and $x'(t) \in I(t)$ for all $t \in T$. If $\delta_H^{x'} < \delta_H^x$ then we are done with $x_0 = x'$, $H_0 = H^x$. Otherwise there exists a coordinate $z \in T$ with $\delta_H^{x'}(z) > \delta_H^{x^*}(z)$. Assume that $x'(z) > x^*(z)$ (one can argue analogously if $x'(z) < x^*(z)$). Let $x'' = x' - \chi_z$ be a step from x' to x^*. It is easy to see that

$\delta_H^{x''} < \delta_H^{x'}$ and $x''(t) \in I(t)$ for all $t \in T$. Thus $x'' \in J$ is impossible by the choice of x'. So we assume that there exists a step $x''' = x'' + \chi_w \in J$ from x'' to x^* (the case $x''' = x'' - \chi_w$ can be treated analogously). Clearly, $\delta_H^{x'''} \leq \delta_H^{x''} + 1 \leq \delta_H^{x'}$, hence by the choice of x' we get that $x'''(w) \notin I(w)$. This can happen only if $x''(w) \in H(w)$ and $x''(w) + 1 = x'''(w) \in H(w)$, and thus $\delta_H^{x'''} = \delta_H^{x''} < \delta_H^{x'}$. This means that we are done with $x_0 = x'''$ and $H_0 = {}^wH^x$. □

Proof of Theorem 1.3 We denote $r(M)$ simply by r, and let $l_t = \min H(t)$, $u_t = \max H(t)$ for $t \in T$. We can clearly assume that $q \leq r$ and that $u_t \leq |S_t|$ for all $t \in T$. We already mentioned that the set of characteristic vectors of bases of a matroid is a jump system. As the q-truncation of M is a matroid, we have that

$$J' = \{\chi_F : F \subseteq S \text{ is independent in } M \text{ of size } q\} \subseteq \{0,1\}^S$$

is a jump system. We define a function $\varphi : S \to T$, which maps $s \in S_t$ to $t \in T$ for all $s \in S$, and let $J = \varphi(J') \subseteq \mathbb{N}^T$. For a set $F \subseteq S$ we denote $\varphi\chi_F \in \mathbb{N}^T$ by $w(F)$. Choose a vector $x \in J$. By Lemma 2.4 it is enough to construct a polynomial algorithm which finds an H_0-optimal vector in J, where H_0 runs over the prescription-sets listed in Definition 2.3. Fix such a prescription-set H_0, and recall that $H_0(t)$ is a parity interval for all $t \in T$.

Next we construct a representation over $\mathbb{Q}$ of an auxiliary linear matroid M^*, see Figure 1. First, for all $t \in T$ add a set C_t of u_t coloops to M, then add a set C of $r - q$ coloops to M. The representation of this new linear matroid M^* is trivial to construct, its rank is $r^* = r + \sum u_t + r - q$ and its ground set S^* has size $|S| + \sum u_t + r - q$. We designate some pairs in M^* as follows: for all $t \in T$ take all $|C_t||S_t|$ pairs between C_t and S_t, together with $(u_t - l_t)/2$ disjoint pairs inside C_t, then take all pairs between C and S. Using the matroid parity algorithm of Lovász [7] we can find a maximum matching of M^* in polynomial time. Let us denote the size of this matching by ν, and the union of pairs in it by $K \subseteq S^*$. Recall that K is independent in M^* and $|K| = 2\nu$.

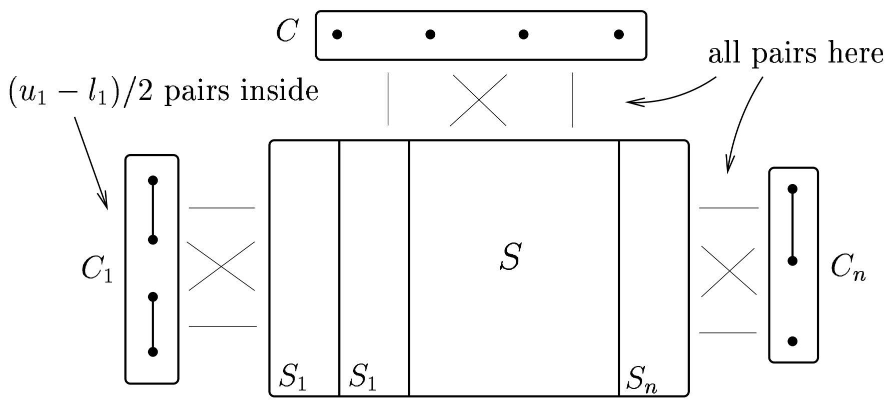

Figure 1: The auxiliary represented matroid M^*

Denote by $K' \subseteq K$ the union of those pairs which are disjoint from C, and let $F' = K' \cap S$. We will be done if we prove that if $F_0 \subseteq S$ is an independent set of M of size q, minimizing $|F_0 \triangle F'|$ (that is, if $|F_0 \triangle F'| = ||F'| - q|$), then $w(F_0)$ is H_0-optimal in J. First, it is not hard to see that

$$\delta_{H_0}^{w(F')}(t) \leq |C_t \setminus K| = u_t - |C_t \cap K| \quad \text{for all } t \in T.$$

Trivially, $r \geq |S \cap K|$ and $r - q \geq |C \cap K|$. It is easy to see that

$$q - |F'| \leq r - |S \cap K| \quad \text{and} \quad |F'| \leq r - |C \cap K|.$$

Thus

$$||F'| - q| \leq (r - |S \cap K|) + (r - q - |C \cap K|).$$

Summarizing,

$$\delta_{H_0}^{w(F_0)} \leq \sum_{t \in T} \delta_{H_0}^{w(F')}(t) + ||F'| - q| \leq \sum_{t \in T} u_t - \sum_{t \in T} |C_t \cap K| + 2r - q - |(S \cup C) \cap K| = r^* - |K|.$$

On the other hand, if F is an independent set of M of size q such that $w(F)$ is H_0-optimal in J then it is straightforward to construct a matching of M^* with union of size $r^* - \delta_{H_0}^{w(F)}$. Thus $\delta_{H_0} = r^* - |K|$, and $w(F)$ is indeed H_0-optimal in J. □

3 Applications to rigidity and to unique solvability of linear networks

For a thorough background on rigidity we refer to Jackson and Jordán [4]. If $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is an embedding into the Euclidean d-space then (G, p) is said to be a **framework**. We think of the edges of G as rigid bars with flexible joints at the vertices. An **infinitesimal motion** means an assignment of velocity $x(v) \in \mathbb{R}^d$ to each vertex $v \in V$ such that the bar lengths are preserved, that is $(p(u) - p(v)) \perp (x(u) - x(v))$ for all $uv \in E$. Given a subspace $U_v \leq \mathbb{R}^d$ for all $v \in V$ we say that (G, p) is **fixed** by the collection $(U_v : v \in V)$ if 0 is the only infinitesimal motion under the constraint $x(v) \in U_v$ for all $v \in V$. We say that $Z \subseteq V$ is a **pinning set** if (G, p) is fixed by the collection

$$U_v = \begin{cases} \{0\} & \text{if } v \in Z, \\ \mathbb{R}^d & \text{if } v \notin Z. \end{cases} \tag{1}$$

We recollect the notion of the **rigidity matrix** $A \in \mathbb{R}^{E \times dV}$ of the framework (G, p), where $dV = \{(v, i) : v \in V, 1 \leq i \leq d\}$: for each edge $uv \in E$, in the row indexed by uv, the entries in the d columns of u and v contain the d coordinates of $p(v) - p(u)$ and $p(u) - p(v)$, respectively, and the other entries are 0. The rigidity matrix has the important and well-known property that the infinitesimal motions of (G, p) are exactly the vectors $x \in \mathbb{R}^{dV}$ with $Ax = 0$. Let $Z \subseteq V$ and let A_Z be the submatrix of A we get when deleting the $d|Z|$ columns indexed by Z. Observe that $Z \subseteq V$ is a pinning set if and only if $x = 0$ is the only solution of $A_Z x = 0$. This is tantamount to that the columns of A_Z are linearly independent, or in other words, that they form an independent set in the column matroid M of A. Thus in dimension $d = 2$, the problem of finding a minimum pinning set leads to the parity problem of a linearly represented matroid, which can be solved by the algorithm of Lovász [7]. A combinatorial formula to this problem was subsequently given by Lovász in [6]. For $d = 3$, finding a minimum pinning set is NP-complete by Mansfield [11].

However, with a certain modification we get a tractable problem for $d = 3$. Take a partition of V into two subsets V_1 and V_2, and define $H(v) = \{0, i, 3\}$ for $v \in V_i$. $H(v)$ has no two consecutive gaps for all $v \in V$, so Theorem 1.3 implies that we can find in polynomial time the maximum q for which there exists an H-conformal independent set of M of size q. If $J \subseteq 3V$ is such an independent set and $U_v \leq \mathbb{R}^3$ is the subspace spanned by $\{e_i : (v,i) \in 3V \setminus J\}$ ($e_i \in \mathbb{R}^3$ is the i^{th} unit vector) then by the above considerations, (G,p) is fixed by $(U_v : v \in V)$. We obtain:

Theorem 3.1. *The following problem is polynomial time solvable. Given a 3-dimensional framework (G,p) and a partition $V = V_1 \dot\cup V_2$. Find a collection $(U_v \leq \mathbb{R}^3 : v \in V)$ minimizing $\sum_{v\in V} \dim U_v$ such that (G,p) is fixed by $(U_v : v \in V)$ and*

- *for $v \in V_1$ either $U_v = \{0\}$, or $U_v = \mathbb{R}^3$, or U_v is a coordinate-line,*
- *for $v \in V_2$ either $U_v = \{0\}$, or $U_v = \mathbb{R}^3$, or U_v is a coordinate-plane.*

More generally, assign to every vertex $v \in V$ a set of vectors $\{a_i^v \in \mathbb{R}^d : i \in S_v\}$, where the S_v's are disjoint finite sets, and $S = \dot\bigcup_{v\in V} S_v$. Then define a matrix $A \in \mathbb{R}^{E\times S}$ as follows. For all $uv = e \in E$ and $i \in S_v$ let

$$A_{e,i} = (p(u) - p(v))^\mathsf{T} a_i^v,$$

and let the other entries be 0. We point out that if a_i^v is the i^{th} unit vector for all $v \in V$ and $1 \leq i \leq d$ then A is simply the rigidity matrix. Again, if $y \in \mathbb{R}^S$ with $Ay = 0$ then $x \in \mathbb{R}^{dV}$ with $x(v) = \sum\{a_i^v y_i : i \in S_v\}$ is an infinitesimal motion of (G,p). Thus given a framework (G,p), and for each $v \in V$ a set of vectors $\{a_i^v \in \mathbb{R}^d : i \in S_v\}$ and a prescription $H(v) \subseteq \mathbb{N}$ without two consecutive gaps, we can find in polynomial time a collection $(U_v \leq \mathbb{R}^d : v \in V)$ minimizing $\sum_{v\in V} \dim U_v$ with the following properties

- (G,p) is fixed by $(U_v : v \in V)$,
- U_v is generated by some vectors a_i^v for $v \in V$,
- $\dim U_v \in H(v)$ for $v \in V$.

Another application of Theorem 1.3 is that we can check in polynomial time a weaker version of a sufficient condition obtained by Recski and Takács [12, 13] to the unique solvability of linear networks with memoryless multiports. It is well-known how to assign a **network graph** $G = (V,F)$ to an electric network, containing voltage and current sources, and linear memoryless multiports (e.g. resistors, gyrators), in such a way that V corresponds to the wires, disjoint n-tuples of F correspond to the n terminal-pairs of the n-ports, while the remaining edges of F correspond to the voltage and current sources (for a survey on electric networks, see Recski [15]). Recski and Takács [12, 13] introduced the notion of **potentially singular** multiports, which are the only possible devices in the network which can contribute to singularities. If D is a potentially singular n-port and $F_D \subseteq F$ is the set of n corresponding edges in G, then they introduce a collection of k_D partitions of F_D: $\pi_D^j = \{L^j, C^j, N_1^j, \ldots, N_{l_j}^j\}$ for $1 \leq j \leq k_D$. Extending the well-known notion of a normal tree, they introduce the following definition.

Definition 3.2 ([13]). A **normal tree** of the network graph $G = (V, F)$ is a spanning tree T of G satisfying the following conditions.

1. T contains all edges corresponding to voltage sources.
2. T contains no edges corresponding to current sources.
3. If D is a potentially singular n-port then for all $1 \leq j \leq k_D$ at least one of the following statements is met:
   (a) $E(T) \cap L^j \neq \emptyset$,
   (b) $C^j \not\subseteq E(T)$,
   (c) $|E(T) \cap N_i^j| \neq 1$ for at least one index i with $1 \leq i \leq l_j$.

They proved that

Theorem 3.3 (Recski, Takács [13]). *If the network graph has a normal tree then the network has a unique solution.*

We show how to decide the existence of a normal tree, *under the assumption that $k_D = 1$ for all potentially singular multiports D*. First recall that the edge sets of the spanning trees of a graph form the bases of the cycle matroid M_G of $G = (V, F)$. We define a hypergraph $H = (F, \mathcal{E})$, and a collection of prescriptions $(H(e) \subseteq \mathbb{N} : e \in \mathcal{E})$ as follows. For every voltage source (resp. current source) and corresponding edge $f \in F$ add a hyperedge $e = \{f\}$ to $\mathcal{E}$ with prescription $H(e) = \{1\}$ (resp. $H(e) = \{0\}$). Then do the following for each potentially singular n-port D with partition $\pi_D = \{L, C, N_1, \ldots, N_l\}$ of F_D. First add a set $X = \{x_D, y_D\}$ consisting of two coloops to M_G, and let $\hat{L} = L \cup \{x_D\}$. In addition, add the following hyperedges to $\mathcal{E}$ with prescriptions as indicated:

1. the classes of $\pi_D - L + \hat{L}$, with $H(\hat{L}) = \{1, 2, \ldots, |\hat{L}|\}$, $H(C) = \{0, 1, \ldots, |C| - 1\}$ and $H(N_i) = \{0, 2, 3, \ldots, |N_i|\}$ for $1 \leq i \leq l$,
2. $U := \dot{\bigcup}_{1 \leq i \leq l} N_i$ with $\{0, 1, \ldots, l - 1\}$,
3. X with $\{0, 2\}$,
4. and $W := F_D \cup \{y_D\}$ with $\{0, 1, 2\}$.

Denote the resulting matroid, which is just M_G plus some coloops, by M. Finally, construct a matroid M' from M as follows: for every element f of M which is incident to the hyperedges $e_1, \dots, e_d \in \mathcal{E}$ for $d = \deg_H(f)$, replace f by a parallel class $\{f^{e_1}, \dots, f^{e_d}\}$. For $e \in \mathcal{E}$ let $e' = \{f^e : f \in e\}$. In this way $\mathcal{E}' := \{e' : e \in \mathcal{E}\}$ is a partition of the new ground set F'.

Claim 3.4. *If B' is an H-conformal base of M' and B is the corresponding base of M then $B \cap F$ is the edge set of a normal tree in G.*

Proof. If $f \in F$ corresponds to a voltage source (resp. current source) then, as $|B \cap \{f\}| \in H(\{f\})$, we have $f \in B$ (resp. $f \notin B$). Now let D be a potentially singular n-port.

(a) If $B \cap L = \emptyset$ then necessarily $B' \cap \hat{L}' = \{x_D^{\hat{L}}\}$, thus $B' \cap X' = \emptyset$ and so $y_D^W \in B' \cap W'$.

(b) If $C \subseteq B$ then $B' \cap W'$ contains an element s^W with $s \in C$.

(c) If $|B \cap N_i| = 1$ for all $1 \le i \le l$ then $B' \cap N_i' = \emptyset$. As $|B' \cap U'| \le l - 1$ we can conclude that $B' \cap W'$ contains an element s^W with $s \in U$.

Thus if all of (a), (b) and (c) are violated in Definition 3.2, 3., then $|B' \cap W'| \ge 3$, which is impossible by 4. □

Using similar arguments, from a normal tree of G it is not hard to construct an H-conformal base of M'. As a representation of M' over $\mathbb{Q}$ is trivial to construct using the well-known representation of M_G, we can decide in polynomial time whether M' has an H-conformal base, and we are done.

Acknowledgements

I would like to thank Tibor Jordán for his suggestions and for calling my attention to the application in rigidity, and to András Recski for helpful discussions.

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