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Abstract

In this paper we consider problems related to Nash-Williams' well-balanced orientation theorem and odd-vertex pairing theorem. These theorems date to 1960 and up to now not much is known about their relationship to other subjects in graph theory. We investigated many approaches to find a more transparent proof for these theorems and possibly generalizations of them. In many cases we found negative answers: counter-examples and NP-completeness results. For example we show that the weighted and the degree-constrained versions of the well-balanced orientation problem are NP-hard. We also show that it is NP-hard to find a minimum cost feasible odd-vertex pairing or to decide whether two graphs with some common edges have simultaneous well-balanced orientations or not.

Nash-Williams' original approach was to define best-balanced orientations with feasible odd-vertex pairings: we show here that not every best-balanced orientation can be obtained this way. However we prove that in the global case this is true: every smooth k-arc-connected orientation can be obtained through a k-feasible odd-vertex pairing.

The aim of this paper is to help to find a transparent proof for the well-balanced orientation theorem. In order to achieve this we propose some other approaches and raise some open questions, too.

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1 Introduction

In 1960 Nash-Williams proved his strong orientation theorem about the existence of *well-balanced* (and *best-balanced*) orientations. He did this proving a stronger result, the so-called *odd-vertex pairing* theorem. These two results give rise to many intriguing questions, some of which are answered in this paper. For example we show that it is NP-hard to find a minimum cost well-balanced orientation (given the cost for the two possible orientations of each edge) or a well-balanced orientation satisfying lower and upper bounds on the out-degrees at each vertex. The same results are proved for best-balanced orientations. For feasible odd-vertex pairings we have similar results: we prove that it is NP-hard to find a minimum cost feasible odd-vertex pairing (where the cost of choosing a pair of odd-degree vertices is given for each pair). We propose several other properties observed for k-arc-connected orientations but in most of the cases we prove with counter-examples that these do not extend to well-balanced orientations. Many of the results of this paper (although not all of them) appeared already in two technical reports of the Egerváry Research Group, in [13] and [2]: in some cases we omit details and refer the reader to these papers. In this work we present our results in a way that starts with the most natural and straightforward questions and goes towards the more involved and sophisticated ones: this method intends to make the paper easier to read.

Let us give a more detailed overview of the results of this paper. Let $G := (V, E)$ be an undirected (or a directed) graph. For two vertices $u, v \in V$ of G the **local edge-connectivity** $\lambda_G(u, v)$ from u to v in G is defined to be the maximum number of pairwise edge (arc resp.) disjoint paths from u to v in G. G is **k-edge-connected** (**k-arc-connected** resp.) if $\lambda_G(u, v) \geq k \ \ \forall (u, v) \in V \times V$. More generally, for $U \subseteq V$, G is **k-edge-connected** (**k-arc-connected** resp.) **in U** if $\lambda_G(u, v) \geq k \ \ \forall (u, v) \in U \times U$.

Nash-Williams' well-balanced orientation theorem [19] states that for any undirected graph G there exists an orientation $\vec{G}$ of G for which $\lambda_{\vec{G}}(u, v) \geq \lfloor \frac{1}{2}\lambda_G(u, v) \rfloor$ $\forall (u, v) \in V \times V$: such an orientation will be called **well-balanced**. For global edge-connectivity this specializes to: G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

Let $G := (V + s, E)$ be an undirected graph. The operation **splitting off** is defined as follows: two edges rs, st incident to s are replaced by a new edge rt. The splitting off theorem of Lovász [15] concerns global edge-connectivity: if G is k-edge-connected in V ($k \geq 2$) and $d(s)$ is even then there exists a pair of edges rs, st incident to s whose splitting off maintains the k-edge-connectivity in V. Lovász [15] also showed that the global case of the well-balanced orientation theorem is an easy consequence of his splitting off theorem. Mader [17] generalized Lovász' result for local edge-connectivity: if $d(s) \geq 4$ and no cut edge of G is incident to s then there exists a pair of edges rs, st incident to s whose splitting off maintains the local edge-connectivities in V. A simple proof for Mader's theorem can be found in [7]. Mader [17] provided a new proof for the well-balanced orientation theorem by applying his splitting off theorem.

Let $\vec{G} := (V + s, E)$ be a directed graph. Splitting off can be naturally reformulated for directed graphs: two arcs rs, st are replaced by rt. Mader [18] proved a splitting off theorem preserving global arc-connectivity in directed graphs: if G is k-arc-connected in V and $\varrho(s) = \delta(s)$ then there exists a pair of arcs rs, st incident to s whose splitting off maintains the k-arc-connectivity in V. An example of Enni [4] shows that there is no splitting off theorem preserving local arc-connectivities in directed graphs. In Question 3 we provide a smaller example showing that even if $\vec{G}$ is a well-balanced orientation of G there is no splitting off that preserves local arc-connectivities in V.

Nash-Williams' odd vertex pairing theorem [19] states that every undirected graph G has a **pairing** M (a set of new edges on the set T_G of odd degree vertices of G such that $d_M(v) = 1\ \forall v \in T_G$) that is **feasible** ($d_M(X) \leq b_G(X)\ \forall X \subset V$, where $b_G(X) := d_G(X) - 2\lfloor \frac{R_G(X)}{2} \rfloor$ and $R_G(X) := \max\{\lambda_G(x,y) : x \in X,\ y \notin X\}$). A simpler proof of the odd vertex pairing theorem can be found in [8]. For the global case, let us call a pairing M to be k**-feasible** (where k is a nonnegative integer) if $d_M(X) \leq d_G(X) - 2k\ \forall \emptyset \neq X \subsetneq V$: in this case the odd vertex pairing theorem (i.e. the existence of a k-feasible pairing in a $2k$-edge-connected graph) can be proved easily by the global splitting off theorem as it was shown in [14].

The well-balanced orientation theorem is trivial for Eulerian graphs (any Eulerian orientation will do) but this special case plays an important role in the theory. It was shown in [14] that for Eulerian graphs, an orientation is well-balanced if and only if it is Eulerian.

Nash-Williams [19] showed that if M is a feasible pairing of G then for every Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$, $\vec{G}$ is well-balanced and furthermore it is **smooth**, that is the in-degree and the out-degree of every vertex differ by at most one. A smooth well-balanced orientation is called **best-balanced**. We show (Question 8) that not every best-balanced orientation can be defined by a feasible pairing. On the other hand we prove in Theorem 8.1 that for the global case it can be: every smooth k-arc-connected orientation can be defined with a k-feasible pairing.

The above mentioned two proofs of the odd-vertex pairing theorem (the original due to Nash-Williams and that of András Frank) both imply a polynomial algorithm to find a feasible odd-vertex pairing, though it is not explicitly stated in either of them. An explicit algorithm for this problem is sketched in [10], it states that an odd-vertex pairing (and consequently a best-balanced orientation) can be found in $O(nm^2)$ time in a graph and in $O(n^6)$ time in a multigraph. It is a natural question to look for a feasible odd-vertex pairing of minimum cost where the cost for any pair of odd-degree vertices is given. However we show (Corollary 9.2) that this problem is NP-complete, even for the global case. Another natural question is whether one can find a well-balanced orientation of minimum cost (with costs given for the two orientations of every edge) or whether one can find a well-balanced orientation satisfying some other constraints, for example lower and upper bounds on the out-degrees at each vertex. In his survey paper [8] András Frank mentions these questions when he writes the following about his proof of the odd-vertex pairing theorem: *I keep feeling that there must be an even more illuminating proof which finally will lead to methods to solve the minimum cost and/or degree-constrained well-balanced orientation problem.* Here we present negative answers in this direction: we prove the NP-completeness of these problems (see Theorem 4.3). We have similar results for best-balanced orientations.

Nash-Williams [20] formulated the following extension of the well-balanced orientation theorem for a subgraph chain of length two: if H is a subgraph of G, then there exists an orientation of H that can be extended to an orientation of G both being best-balanced. A simple proof is given in [14]: it is shown there that the odd vertex pairing theorem easily implies this. It was also shown that the global case of this extension has a simple proof. We show that the *general subgraph chain property* is not valid, that is this extension cannot be generalized for subgraph chain of length three, neither for the global case (see Question 6).

The authors of [14] generalized further the above extension by showing that the following *edge disjoint subgraphs property* is valid: if $\{G_1, G_2, ..., G_k\}$ is a partition of G into edge disjoint subgraphs then there is an orientation $\vec{G}$ of G such that each $\vec{G_i}$ and $\vec{G}$ are best-balanced orientations of G_i and of G. We show that deciding for two non-edge-disjoint graphs whether they have simultaneous best-balanced orientations is NP-complete, even for two Eulerian graphs (see Question 7).

Király and Szigeti [14] also showed that for every pairing M of G there exists an Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$ so that $\vec{G}$ is best-balanced. We mention that, for an Eulerian subgraph H of G, any Eulerian orientation of H can be extended to a best-balanced orientation of G.

Frank [6] proved the following *reorientation property* for the k-arc-connected orientations: given two k-arc-connected orientations of G, there exists a series of k-arc-connected orientations of G (leading from the first to the second given orientation), such that in each step we reverse a directed path or a circuit. For well-balanced (or best-balanced) orientations it is not known whether the reorientation property is valid.

Frank [5] also proved that the *linkage property* is valid for the k-arc-connected orientation problem, namely there exists a k-arc-connected orientation whose in-degree function satisfies lower and upper bounds if and only if there is one satisfying the lower bound and one satisfying the upper bound. É. Tardos [21] showed that the linkage property is not valid for the well-balanced orientation problem. Here we present another example (see Question 16).

The original proof of the odd vertex pairing theorem in [19] and Frank's proof [8] as well relies heavily on the skew-submodularity of the function b_G. We show (Question 10) that the existence of a feasible pairing cannot be generalized to arbitrary skew-submodular functions. Skew-submodular functions correspond to local edge-connectivity, while crossing submodular functions can be considered as generalizations of global edge-connectivity. For such a function it is an open problem whether there exists a feasible pairing. However the corresponding orientation theorem can be proved easily (see Theorem 10.1).

By the proof of Frank [6] it is easy to see that the following *matroid property* is valid for smooth k-arc-connected orientations: the family of sets, over smooth k-arc-connected orientations, consisting of vertices whose in-degree is larger than the out-degree, forms the basis of a matroid. We show that this is not true for best-balanced orientations (see Question 14).

The aim of this paper is to help to find a transparent proof for the well-balanced orientation theorem. A possible way could be to find a convenient generalization that has a simple inductive proof. Here we think of results like Theorems 3.7, 3.8 and 3.9. Unfortunately we do not have direct proofs for them, they follow easily from the odd vertex pairing theorem. This result (Theorem 3.4) is a miracle, it has no generalization, no application (except the well-balanced orientation theorem), no relation to any other result in graph theory.

The rest of this paper is organized as follows. In Section 2 we introduce some further notations. In Section 3 we summarize known results on well-balanced orientations and odd-vertex pairings. In Section 4 we consider well-balanced orientations with extra requirements: we prove the NP-completeness of questions such as finding a well-balanced orientation of minimum cost or one satisfying lower and upper bounds on the out-degrees. In Section 5 we consider mixed graphs and their well-balanced orientations. In Section 6 we look at the splitting-off operation. In Section 7 we consider the question of orienting two graphs with possibly some common edges resulting in an orientation that is simultaneously well-balanced. In Section 8 we show that not every best-balanced orientation can be defined with a feasible odd-vertex pairing, however this is true in the global case. In the next section we investigate the structure of feasible pairings. In Section 10 we deal with a more general setting and investigate feasible pairings for connectivity functions. In the last section we show that the matroid property that is valid for k-arc-connected orientations does not extend to well-balanced orientations.

2 Notation

A directed graph is denoted by $\vec{G} = (V, A)$ and an undirected graph by $G = (V, E)$. For a directed graph $\vec{G}$, a set $X \subseteq V$, a vector $z : A \to \mathbb{R}$ and $u, v \in V$, let $\delta_{\vec{G}}(X) := |\{uv \in A : u \in X,\ v \notin X\}|$, $\varrho_{\vec{G}}(X) := \delta_{\vec{G}}(V - X)$, $f_{\vec{G}}(X) := \varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)$, $\delta^z_{\vec{G}}(X) := \sum_{\{uv \in A: u \in X,\ v \notin X\}} z(uv)$, $\varrho^z_{\vec{G}}(X) := \delta^z_{\vec{G}}(V - X)$, $\lambda_{\vec{G}}(u, v) := \min\{\delta_{\vec{G}}(Y) : u \in Y,\ v \notin Y\}$, and $\overleftarrow{G} := (V, \{vu : uv \in A\})$. For an undirected graph G, a set $X \subseteq V$ and $u, v \in V$, let $\Delta_G(X) := \{uv \in E : u \in X,\ v \notin X\}$, $d_G(X) := |\Delta_G(X)|$, $d_G(X, Y) := |\{uv \in E(G) : u \in X - Y, v \in Y - X\}|$, $\lambda_G(u, v) := \min\{d_G(X) : u \in X,\ v \notin X\}$, $R_G(X) := \max\{\lambda_G(x, y) : x \in X,\ y \notin X\}$, $\hat{R}_G(X) := 2\lfloor R_G(X)/2\rfloor$, $b_G(X) := d_G(X) - \hat{R}_G(X)$ and $T_G := \{v \in V : d_G(v) \text{ is odd}\}$. Observe that $\forall X \subseteq V$,

$$f_{\vec{G}}(X) \;=\; \sum_{v \in X} f_{\vec{G}}(v). \tag{1}$$

Let $G = (V, E)$ be an undirected graph. G is **connected** if for every pair of vertices u, v there is a (u, v)-path in G. G is called **k-edge-connected** if $G - F$ is connected for $\forall F \subseteq E$ with $|F| \leq k - 1$. For a function $\mathbf{r} : V \times V \to \mathbb{Z}^+_0$, we say that G is **r-edge-connected** if $\lambda_G(u, v) \geq \mathbf{r}(u, v)$ for every pair u, v of vertices.

Let $D = (V, A)$ be a directed graph. D is **strongly connected** if for every ordered pair $(u, v) \in V \times V$ of vertices there is a directed (u, v)-path in D. D is called **k-arc-connected** if $D - F$ is strongly connected for $\forall F \subseteq A$ with $|F| \leq k - 1$. For a function $\mathbf{r} : V \times V \to \mathbb{Z}_0^+$, we say that D is **r-arc-connected** if $\lambda_D(u,v) \geq \mathbf{r}(u,v)$ for every ordered pair u, v of vertices.

An orientation $\vec{G}$ of G is called **well-balanced** if $\vec{G}$ satisfies (2), **smooth** if $\vec{G}$ satisfies (3) and **best-balanced** if it is smooth and well-balanced. Let us denote by $\mathcal{O}_w(G)$ and $\mathcal{O}_b(G)$ the set of well-balanced and best-balanced orientations of G. Note that if $\vec{G}$ is best-balanced then so is $\overleftarrow{G}$.

$$\lambda_{\vec{G}}(x,y) \geq \lfloor \lambda_G(x,y)/2 \rfloor \quad \forall\, (x,y) \in V \times V, \tag{2}$$
$$|f_{\vec{G}}(v)| \leq 1 \quad \forall\, v \in V. \tag{3}$$

A **pairing** M of G is a new graph on vertex set T_G in which each vertex has degree one. Let M be a pairing of G. An orientation $\vec{M}$ of M that satisfies (4) is called **good**. Note that by Claim 3.5 if $\vec{M}$ is good then every Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$ that extends $\vec{M}$ defines a best-balanced orientation of G. Pairing M is **well-orientable** if there *exists* a good orientation of M, M is **strong** if *every* orientation of M is good and M is **feasible** if (5) is satisfied. Clearly an oriented pairing $\vec{M}$ is good iff $\overleftarrow{M}$ is good. Let us denote by $\mathcal{P}_f(G)$ the set of feasible pairings of G.

$$f_{\vec{M}}(X) \leq b_G(X) \quad \forall X \subseteq V, \tag{4}$$
$$d_M(X) \leq b_G(X) \quad \forall X \subseteq V. \tag{5}$$

We shall use that for subsets $X, Y, Z \subseteq V$ we have

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X,Y), \tag{6}$$
$$d_G(X) + d_G(Y) + d_G(Z) \geq d_G(X \cap Y \cap Z) + d_G(X - (Y \cup Z)) + d_G(Y - (X \cup Z)) + d_G(Z - (X \cup Y)). \tag{7}$$

3 Known results

The following theorems are due to Nash-Williams [19], [20].

Theorem 3.1. *A graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

Theorem 3.2. *Every graph has a best-balanced orientation.*

Theorem 3.3. *For every subgraph H of G, there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G.*

Theorem 3.4. *Every graph has a feasible pairing.*

The following results were shown in [14] by Király and Szigeti.

Claim 3.5. *The following statements are equivalent:*

$$\vec{G} \in \mathcal{O}_w(G), \tag{8}$$
$$\delta_{\vec{G}}(X) \geq \lfloor \frac{R(X)}{2} \rfloor \quad \forall\, X \subseteq V, \tag{9}$$
$$f_{\vec{G}}(X) \leq b_G(X) \quad \forall\, X \subseteq V. \tag{10}$$

Claim 3.6. *A pairing is feasible if and only if it is strong.*

Theorem 3.7. *Every pairing is well-orientable.*

Theorem 3.8. *For every partition* $\{E_1, E_2, ..., E_k\}$ *of* $E(G)$, *if* $G_i = (V, E_i)$ *then* G *has a best-balanced orientation* $\vec{G}$, *such that the inherited orientation of each* G_i *is also best-balanced.*

Theorem 3.9. *For every partition* $\{X_1, ..., X_l\}$ *of* $V = V(G)$, G *has an orientation* $\vec{G}$ *such that* $\vec{G}$, $((\vec{G}/X_1)...)/X_l$ *and* $\vec{G}/(V - X_i)$ $(1 \le i \le l)$ *are best-balanced orientations of the corresponding graphs.*

Theorem 3.9 implies the following slight refinement of the best-balanced orientation theorem.

Theorem 3.10. *Let* G *be an undirected graph and* M *be a pairing of* G. *Then* G *has a well-balanced orientation* $\vec{G}$ *with* $|f_{\vec{G}}(X)| \le 1$ *whenever* $d_M(X) \le 1$.

4 Well-balanced orientations with extra requirements

It is a natural question whether one can find a well-balanced orientation of minimum cost (with costs given for the two orientations of every edge) or whether one can find a well-balanced orientation satisfying some other constraints, for example lower and upper bounds on the out-degrees at each vertex. Here we present negative answers in this direction: we prove the NP-completeness of these problems. Let us introduce the problems we want to consider and give some motivation.
For well-balanced orientations we look at the following problems:

Problem 1. *:* MinCostWellBalanced
Instance*: A graph* G, *nonnegative integer costs for the two orientations of each edge, integer* K.
Question*: Is there a well-balanced orientation of* G *with total cost not more than* K*?*

Problem 2. *:* BoundedWellBalanced
Instance*: A graph* $G = (V, E)$, $l, u : V \mapsto \mathbb{Z}_+$ *bounds with* $l \le u$.
Question*: Is there a well-balanced orientation* $\vec{G}$ *of* G *with* $l(v) \le \delta_{\vec{G}}(v) \le u(v)$ *for every* $v \in V$*?*

Problem 3. *:* MinVertexCostWellBalanced
Instance*: A graph* G, *integer costs* $c : V \mapsto \mathbb{Z}$, *integer* B.
Question*: Is there a well balanced orientation* $\vec{G}$ *of* G *with* $\sum_{v\in V}(c(v)\delta_{\vec{G}}(v)) \le B$*?*

For best-balanced orientations we consider the following problems:

Problem 4. *:* MinCostBestBalanced
Instance*: A graph G, nonnegative integer costs for the two orientations of each edge, integer K.*
Question*: Is there a best-balanced orientation of G with total cost not more than K?*

Problem 5. *:* BoundedBestBalanced
Instance*: A graph $G = (V, E)$, $l, u : V \mapsto \mathbb{Z}_+$ bounds with $\lfloor d_G(v)/2 \rfloor \leq l(v) \leq u(v) \leq \lceil d_G(v)/2 \rceil$ for each $v \in V$.*
Question*: Is there a well-balanced orientation $\vec{G}$ of G with $l(v) \leq \delta_{\vec{G}}(v) \leq u(v)$ for every $v \in V$ (i.e. a best-balanced orientation with these bounds)?*

Problem 6. *:* MinVertexCostBestBalanced
Instance*: A graph G, integer costs $c : V \mapsto \mathbb{Z}$, integer B.*
Question*: Is there a best-balanced orientation $\vec{G}$ of G with $\sum_{v \in V}(c(v)\delta_{\vec{G}}(v)) \leq B$?*

Problems MinCostWellBalanced and MinCostBestBalanced are quite natural weighted versions of the original problem, the problem of finding a well-balanced or a best-balanced orientation. The constrained versions BoundedWell-Balanced and BoundedBestBalanced also arise naturally: they are mentioned in the survey paper of András Frank [8] and a related problem, when we have only bounds from one side (say, upper bounds) in a best-balanced orientation is still an open problem mentioned in [3] (though we have to mention that a related question was shown to be hard, namely it has been shown by [1] that it is NP-hard to decide whether a graph has an $\mathbf{r}$-arc-connected orientation with upper bounds on the out-degrees even for a $0-1$-valued symmetric function $\mathbf{r}$). The third approach is motivated by the following observation: in an orientation problem with arc-connectivity requirements, finding the out-degree function of a solution is polynomially equivalent with finding a solution. The authors of [13] introduce the following polyhedron for a graph $G = (V, E)$ (see Section 9 in [13]):

$$\begin{aligned} P := \quad & \{x \in \mathbb{R}^V : x(Z) \geq i_G(Z) + \lfloor R_G(Z)/2 \rfloor \ \forall Z \subseteq V, x(V) = |E|, \\ & \lfloor d_G(v)/2 \rfloor \leq x(v) \leq \lceil d_G(v)/2 \rceil \ \forall v \in V\}. \end{aligned}$$

This polyhedron corresponds to the fractional relaxations of good out-degree functions of a best-balanced orientation. It is proved in [13] that this polyhedron is not necessarily integral: here we prove that optimization over the integer hull of this polyhedron (that is, problem MinVertexCostBestBalanced) is NP-complete. Problem MinVertexCostWellBalanced is just the counterpart of this problem for well-balanced orientations.

Now we give some known results that will be needed later. The following is a simple observation: the proof is left to the reader.

Lemma 4.1. *If $\vec{G}$ and $\vec{G}'$ are two orientations of a graph $G = (V, E)$ with $\delta_{\vec{G}}(x) = \delta_{\vec{G}'}(x)$ for all $x \in V$ then $\vec{G}'$ can be obtained from $\vec{G}$ by reversing directed cycles.* □

Corollary 4.2. *If $\vec{G}$ and $\vec{G}'$ are two orientations of a graph $G = (V, E)$ with $\delta_{\vec{G}}(x) = \delta_{\vec{G}'}(x)$ for all $x \in V$ then*

$$\vec{G} \text{ is well-balanced} \iff \vec{G}' \text{ is well-balanced.}$$

Proof. Directly from lemma 4.1. Alternatively, we can show that $\lambda_{\vec{G}}(x, y) = \lambda_{\vec{G}'}(x, y)$ for all $x, y \in V$ using the fact $\delta_{\vec{G}}(X) = \sum_{x \in X} \delta_{\vec{G}}(x) - i_G(X) = \delta_{\vec{G}'}(X)$ for any $X \subseteq V$. □

For well-balanced orientations we have the following results.

Theorem 4.3. *Problems* MinCostWellBalanced, BoundedWellBalanced *and* MinVertexCostWellBalanced *are NP-complete.*

Proof. The problems are clearly in NP. In order to show their completeness we will give a reduction from Vertex Cover (see [11], Problem GT1). For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of the Vertex Cover problem consider the following undirected graph $G = (V, E)$. The vertex set V will contain one designated vertex s, $d_{G'}(v) + 1$ vertices $x_0^v, x_1^v, x_2^v, \ldots, x_{d_{G'}(v)}^v$ for every $v \in V'$, and one vertex x_e for every $e \in E'$. Let us fix an ordering of V', say $V' = \{v_1, v_2, \ldots, v_n\}$. The edge set E contains a circuit on $s, x_0^{v_1}, x_0^{v_2} \ldots, x_0^{v_n}$ in this order, one edge from s to x_1^v for every $v \in V'$, edges between x_i^v and x_{i+1}^v for every $v \in V'$ and every i between 0 and $d_{G'}(v) - 1$, two parallel edges between s and x_e for every $e \in E'$ and finally for each $v \in V'$ take an arbitrary order of the $d_{G'}(v) = d$ edges of G' incident to v, say $e^1, e^2, \ldots, e^d$ and include the edge $(x_i^v, x_{e^{i-1}})$ for any $2 \le i \le d - 1$ and the edges $(x_d^v, x_{e^{d-1}})$ and (x_d^v, x_{e^d}) (i.e. distribute the edges of G' incident to v arbitrarily among vertices $x_2^v, \ldots, x_d^v$ resulting $d_G(x_i^v) = 3$ for each $2 \le i \le d$).

The construction is illustrated in Figure 1. The edges drawn bold indicate a multiplicity of 2.

Notice that for every $v \in V'$ and $0 \le i \le d_{G'}(v)$ we have $d_G(x_i^v) = 3$ and for every $e \in E'$ we have $d_G(x_e) = 4$. What is more, it is easy to check, that $\lambda_G(x, y) = \min(d_G(x), d_G(y))$ for every $x, y \in V$ (for example one can check that this is true if $y = s$ from which it follows for arbitrary x, y).

Define a partial orientation of G: orient the circuit $s, x_0^{v_1}, x_0^{v_2} \ldots, x_0^{v_n}$ to become a directed circuit in this order, orient the edges from x_i^v to x_{i+1}^v for every $v \in V'$ and every i between 0 and $d_{G'}(v) - 1$, orient the two parallel edges from x_e towards s for every $e \in E'$ and finally for each $v \in V'$, $2 \le i \le d_{G'}(v)$ and $e \in E'$ if there is an edge between x_i^v and x_e then orient this edge from x_i^v to x_e (so we have given the orientation of every edge except those of form (s, x_1^v) for $v \in V'$). Figure 2 is an illustration.

Let us call the subgraph $G - \{(s, x_1^v) : v \in V'\}$ by G_1 and the above given orientation of this graph by $\vec{G}_1$. Observe that $\vec{G}_1$ is a strongly connected graph and $\lambda_{\vec{G}_1}(x_e, s) = 2$ for each $e \in E'$.

Claim 4.4. *Problem* MinCostWellBalanced *is NP-complete.*

Proof: For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinCostWellBalanced: let the graph G be as described

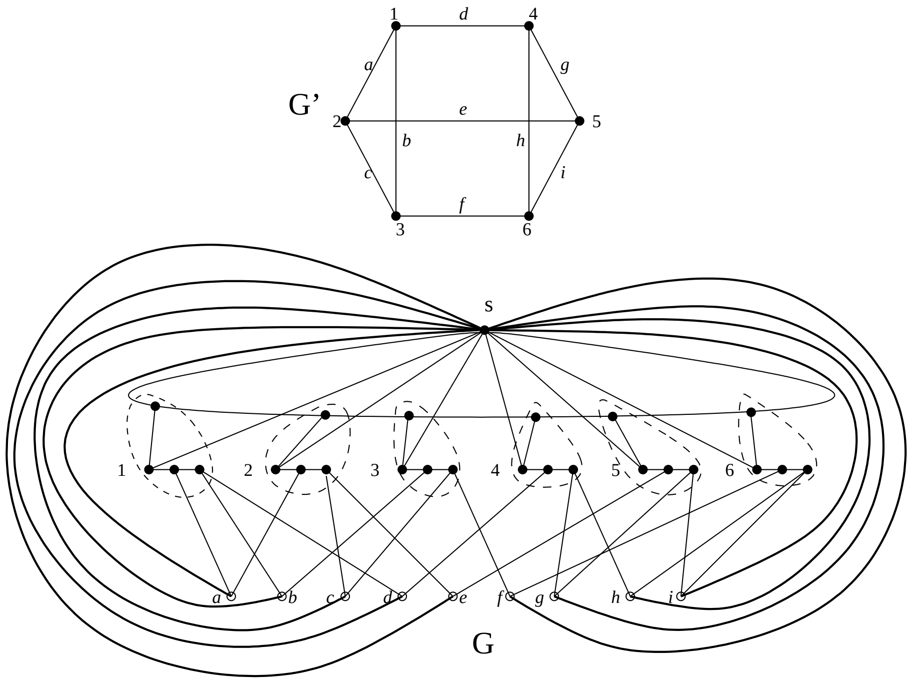

Figure 1: Construction of graph G

above, let $K = k$ be the bound on the total cost and define the orientation-costs as follows. For the edges of G_1 orienting these edges costs nothing in the way as given in $\vec{G}_1$, but reversing any one will cost exactly $k+1$. It remains to define the costs of orientations of edges between s and x_1^v for each $v \in V'$: such an edge costs 1, if oriented from s to x_1^v and 0 in the other direction. So we only have freedom choosing the orientation of these edges, if we don't want to exceed the cost limit k.

First we claim that if there is a vertex cover $S \subseteq V'$ of size not more than k then there is a well-balanced orientation $\vec{G}$ of G of cost not more than k: for each $v \in S$ orient the edge (s, x_1^v) from s to x_1^v and orient the other edges in the direction which costs nothing. This has clearly cost at most k and it is easy to check that $\lambda_{\vec{G}}(s, x_e) = 2$ for each $e \in E'$ which together with the former observations gives that $\vec{G}$ is well-balanced.

On the other hand suppose that we have found a well-balanced orientation $\vec{G}$ of G of cost at most k: this is possible only if there are at most k vertices in V' such that the edges (s, x_1^v) are oriented from s to x_1^v exactly for these edges and all the other edges are oriented in the direction which costs 0. We claim that these vertices form a vertex cover of G': if edge $e = (v_j, v_k) \in E'$ was not covered (where $j < k$ are the indices of the vertices in the fixed ordering), then $\varrho_{\vec{G}}(X) = 1$ would contradict the well-balancedness of $\vec{G}$, where

$$\begin{aligned} X &= \{x_e\} \bigcup \{x_0^{v_i} : j \le i \le k\} \\ &\quad \bigcup \{x_i^{v_j} : 1 \le i \le d_{G'}(v_j)\} \bigcup \{x_i^{v_k} : 1 \le i \le d_{G'}(v_k)\} \end{aligned}$$

(Figure 2 illustrates the cut, too). $\square$

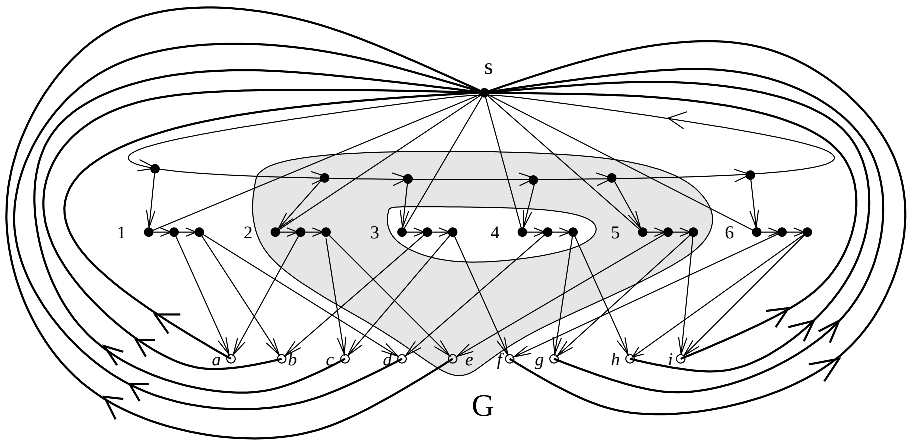

Figure 2: The partial orientation and the cut

Claim 4.5. *Problem* BoundedWellBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of BoundedWellBalanced: let the graph G be as described above and upper bound on the out-degree of s given by $u(s) = k + 1$, and lower bounds $l(x_i^v) = 2$ for each $v \in V'$ and $i \in \{0, 2, \dots, d_{G'}(v)\}$ (observe that these are in fact exact prescriptions for these out-degrees, notice, that we excluded $i = 1$): the other bounds can be trivial, that is $l(x) = 0$ and $u(x) = d_G(x)$ if it was not specified otherwise. We refer the reader to [2] for the details. □

Claim 4.6. *Problem* MinVertexCostWellBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinVertexCostWellBalanced: let the graph G be as described above and vertex-costs the following: let $c(s) = 1$ and $c(x_i^v) = -k$ for each $v \in V'$ and $i \in \{0, 2, \dots, d_{G'}(v)\}$ (and zero for the rest of the vertices). Finally, let $B = -4k|E'| + k + 1$. For more details see [2]. □

For best-balanced orientations we have the following results.

Theorem 4.7. *Problems* MinCostBestBalanced*,* BoundedBestBalanced *and* MinVertexCostBestBalanced *are NP-complete.*

Proof. The problems are clearly in NP. To show completeness we reduce Vertex Cover as before, but we need to change the construction a bit. For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of the Vertex Cover problem, modify the construction of the graph $G = (V, E)$ as follows: add $2|E'| + |V'| - 2k = N$ new vertices $z_1, z_2, \dots, z_N$ and connect each of these vertices with s. So these new vertices will have degree 1 and s will have degree $4|E'| + 2|V'| + 2 - 2k$ in G. Denote this modified graph with $G = (V, E)$.

Define again a partial orientation of G: this is the same as the one defined above in the first construction, with the addition that for each i between 1 and N orient the edge (s, z_i) from s to z_i.

Again call the subgraph $G - \{(s, x_1^v) : v \in V'\}$ by G_1 and the above given orientation of this graph by $\vec{G}_1$. Again we have $\lambda_G(x, y) = \min(d_G(x), d_G(y))$ for every $x, y \in V$, $\lambda_{\vec{G}_1}(x, y) \geq 1$ for every $x, y \in V - \{z_1, z_2, \ldots, z_N\}$ and $\lambda_{\vec{G}_1}(x_e, s) = 2$ for each $e \in E'$.

Claim 4.8. *Problem* MinCostBestBalanced *is NP-complete, even for* $1 - 0$ *orientation costs.*

Proof: For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinCostBestBalanced: let the graph G be as described above, let $K = 0$ be the bound on the total cost and define the orientation-costs as follows. For the edges of G_1 orienting these edges costs nothing in the way as given in $\vec{G}_1$, but reversing any one will cost exactly 1. It remains to define the costs of orientations of edges between s and x_1^v for each $v \in V'$: these edges can be oriented in any direction with 0 cost. Details again can be found in [2]. □

Claim 4.9. *Problem* BoundedBestBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of BoundedBestBalanced: let the graph G be as described above and bounds on the out-degrees of odd degree vertices of G given as follows (of course, for even-degree vertices $x \in V$ one has $l(x) = d_G(x)/2 = u(x)$):

- $l(x_i^v) = 2 = u(x_i^v)$ for each $v \in V'$ and $i \in \{0, 2, \ldots, d_{G'}(v)\}$ (exact prescriptions),
- $l(z_i) = 0 = u(z_i)$ for each $i = 1, 2, \ldots, N$ (exact prescriptions),
- $l(x_1^v) = 1$ and $u(x_1^v) = 2$ for each $v \in V'$ (so we only have freedom here).

For the details see [2]. □

Claim 4.10. *Problem* MinVertexCostBestBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinVertexCostWellBalanced: let the graph G be as described above and vertex-costs the following: let $c(z_i) = 1$ for each $i = 1, 2, \ldots, N$ and $c(x_i^v) = -1$ for each $v \in V'$ and $i \in \{0, 2, \ldots, d_{G'}(v)\}$ (and zero for the rest of the vertices). Finally, let $B = -2\left(\sum(d_{G'}(v) : v \in V')\right) = -4|E'|$. Details again can be found in [2]. □

5 Mixed graphs

A *mixed graph* is determined by the triple (V, E, A) where V is the set of vertices, E is the set of undirected edges and A is the set of directed edges. The *underlying undirected graph* is obtained by deleting the orientation of the arcs in A. An *orientation of a mixed graph* means that we orient the undirected edges (and leave the directed ones).

A possible way to prove the well-balanced orientation theorem could be to characterize mixed graphs whose undirected edges can be oriented to have a well-balanced orientation of the underlying undirected graph. The following problem was mentioned in Section 4.2 of [13]:

Problem 7. *Given a mixed graph, decide whether it has an orientation that is a well-balanced orientation of the underlying undirected graph.*

The following question is an open problem:

Question 1. *Is Problem 7 NP-complete?*

While we don't know the answer to Question 1, the proof of Claim 4.8 immediately gives the NP-completeness of the following, related problem.

Problem 8. *Given a mixed graph, decide whether it has an orientation that is a best-balanced orientation of the underlying undirected graph.*

We have to mention that the global case of these questions can be solved: one can decide whether a mixed graph has a k-arc-connected orientation, even with the presence of lower and upper bounds on the out-degrees of the required orientation.

6 Splitting off

We have seen in the introduction that splitting off theorems are very useful in the proof of the global or local well-balanced orientation theorem. We also mention that Mader's proof [17] for the well-balanced orientation theorem as well as Frank's proof [8] for Theorem 3.4 uses Mader's splitting off theorem.

The odd vertex pairing theorem would be an easy task if the following was true.

Question 2. *For every 2-edge-connected graph G there exists a pair of adjacent edges rs, st such that for $G_{rt} := G - \{rs, st\} + rt$ we have:*

$$b_G(X) \geq b_{G_{rt}}(X) \quad \forall X \subseteq V. \tag{11}$$

Counter-example 2 Let $G = (U, V; E)$ be the complete bipartite graph $K_{3,4}$. Let us denote the vertices as follows: $U := \{a, b, c, d\}$ and $V := \{x, y, z\}$. By symmetry, $\{rs, st\}$ is either $\{xd, dy\}$ or $\{az, zb\}$. In the first case $b_G(z) = 0 < 2 = b_{G_{xy}}(z)$ and in the second case $b_G(\{a, x, y\}) = 3 < 5 = b_{G_{ab}}(\{a, x, y\})$. In both cases (11) is violated. □

Question 3. *If $\vec{G}$ is a best-balanced orientation of $G := (V + s, E)$ and $\varrho_{\vec{G}}(s) = \delta_{\vec{G}}(s)$ then there exist $rs, st \in A(\vec{G})$ so that for $\vec{G}_{rt} := \vec{G} - \{rs, st\} + rt$ we have*

$$\lambda_{\vec{G}_{rt}}(x, y) \geq \lambda_{\vec{G}}(x, y) \; \forall (x, y) \in V \times V. \tag{12}$$

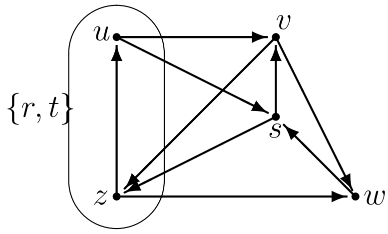

Figure 3: There is no good directed splitting-off at s

Counter-example 3 Let $G := (V+s,E)$ and $\vec{G} := (V+s,A)$ be defined as follows (see Figure 3): $V := \{u,v,w,z\}$, $E := \{uv,us,uz,vz,vs,vw,ws,wz,zs\}$, $A := \{uv,us,zu,vz,sv,vw,ws,zw,sz\}$. It is easy to check that $\vec{G} \in \mathcal{O}_b(G)$. In particular $\lambda_{\vec{G}}(v,z) = \lambda_{\vec{G}}(z,v) = 2$. Suppose that for some $(r,t) \in \{(u,z),(u,v),(w,z),(w,v)\}$, (12) is satisfied. Then, by (12), $3 = \varrho_{\vec{G}_{rt}}(\{r,t\}) + \delta_{\vec{G}_{rt}}(\{r,t\}) \geq \lambda_{\vec{G}_{rt}}(v,z) + \lambda_{\vec{G}_{rt}}(z,v) \geq \lambda_{\vec{G}}(v,z) + \lambda_{\vec{G}}(z,v) = 4$, a contradiction. □

We note that this is a smaller counter-example for a conjecture of Jackson than Enni's one: for details see [4].

Question 4. *If $\vec{G}$ is a best-balanced orientation of $G := (V+s,E)$ and $\varrho_{\vec{G}}(s) = \delta_{\vec{G}}(s)$ then there exist $rs, st \in A(\vec{G})$ so that $\vec{G}_{rt}$ is a best-balanced orientation of G_{rt}.*

Question 4 is an open problem. However the following is true.

Theorem 6.1. *For every pair rs, st of edges of a graph $G := (V+s,E)$ there exists a best-balanced orientation $\vec{G}$ of G so that $rs, st \in A(\vec{G})$ and $\vec{G}_{rt}$ is a best-balanced orientation of G_{rt}.*

Proof. By Theorem 3.4, there exists a feasible pairing M of G. Then M is a pairing of G_{rt} and hence, by Theorem 3.7, $G_{rt} + M$ has an Eulerian orientation $\vec{G}_{rt} + \vec{M}$ so that $\vec{G}_{rt} \in \mathcal{O}_b(G_{rt})$. Wlog. assume that rt is directed as $\vec{rt}$ in $\vec{G}_{rt}$. Then, for $\vec{G} := \vec{G}_{rt} - rt + rs + st$, $\vec{G} + \vec{M}$ is Eulerian, that is, since $M \in P_f(G)$, $\vec{G} \in \mathcal{O}_b(G)$. □

Question 5. *For every graph $G = (V+s,E)$ with $d(s) \geq 4$ there exist $rs, st \in E$ such that for every best-balanced orientation $\vec{G}_{rt}$ of G_{rt}, $\vec{G} := \vec{G}_{rt} - rt + rs + st$ is a best-balanced orientation of G.*

Question 5 is an open problem. If Question 5 had an affirmative answer, it would give us a possible way to prove the best-balanced orientation theorem.

7 Simultaneously well-balanced orientations

In this section we consider some possible generalizations of Theorem 3.3 and Theorem 3.8. Here we consider the statements of these theorems as assuring simultaneous (compatible) best-balanced orientations of some graphs.

The first two questions correspond to the global and local cases related to Theorem 3.3, i.e. the subgraph-chain-property.

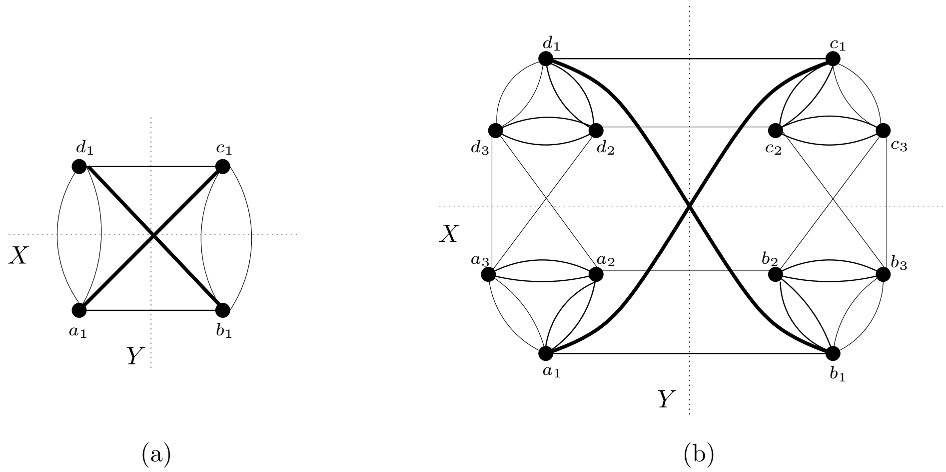

Figure 4: Counterexamples to the subgraph-chain-property

Question 6. *Let G_3 be a subgraph of G_2 and G_2 a subgraph of G_1. Then, for $i = 1, 2, 3$, there exists an orientation $\vec{G}_i$ of G_i such that $\vec{G}_j$ is a restriction of $\vec{G}_i$ if $1 \le i < j \le 3$ and*

(a) Local case: $\vec{G}_i \in \mathcal{O}_w(G_i)$.

(b) Global case: $\vec{G}_i$ *is a k_i-arc-connected orientation of G_i provided that G_i is $2k_i$-edge-connected.*

Counter-examples 6 Let $G_i := (V_i, E_i)$ $(i = 1, 2, 3)$ be defined as in Figure 4, that is

(a) $V_1 = V_2 = V_3 := \{a_1, b_1, c_1, d_1\}, E_3 := \{a_1d_1, a_1d_1, b_1c_1, b_1c_1\}, E_2 := E_3 \cup \{a_1b_1, c_1d_1\}, E_1 := E_2 \cup \{a_1c_1, b_1d_1\}$. Let $X := \{a_1, b_1\}, Y := \{a_1, d_1\}$.

(b) $V_1 = V_2 = V_3 := \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3\}$, $E_3 := \{a_2b_2, b_2c_3, c_3b_3, b_3c_2, c_2d_2, d_2a_3, a_3d_3, d_3a_2\} \cup \{x_3x_1, x_3x_1 : x \in \{a, b, c, d\}\}$, $E_2 := E_3 \cup \{a_1b_1, c_1d_1\} \cup \{x_1x_2, x_1x_2, x_2x_3, x_2x_3 : x \in \{a, b, c, d\}\}$, $E_1 := E_2 \cup \{a_1c_1, b_1d_1\}$ and $k_3 = 1$, $k_2 = 2$, $k_1 = 3$. Let $X := \{a_1, a_2, a_3, b_1, b_2, b_3\}$, $Y := \{a_1, a_2, a_3, d_1, d_2, d_3\}$.

We prove at the same time for (a) and (b) that the required orientations do not exist. Suppose that they do exist. It is easy to check that $\vec{G}_1$ and $\vec{G}_3$ are Eulerian orientations of G_1 and G_3, whence, by (1), $f_{\vec{G}_1}(X) = 0 = f_{\vec{G}_1}(Y)$ and $f_{\vec{G}_3}(X) = 0$. G_2 is $2k$-edge-connected and $d_{G_2}(Y) = 2k$, so $f_{\vec{G}_2}(Y) = 0$. Then $f_{\vec{G}_1 - \vec{G}_2}(X) = f_{\vec{G}_1 - \vec{G}_3}(X) = f_{\vec{G}_1}(X) - f_{\vec{G}_3}(X) = 0$ and $f_{\vec{G}_1 - \vec{G}_2}(Y) = f_{\vec{G}_1}(Y) - f_{\vec{G}_2}(Y) = 0$. Note that $E(G_1 - G_2) = E_1 - E_2 = \{a_1c_1, b_1d_1\}$ and $a_1 \in X \cap Y, c_1 \in V - (X \cup Y), b_1 \in X - Y, d_1 \in Y - X$, contradiction. □

Regarding general simultaneous orientations, we may ask the following question:

Question 7. *Given two graphs (neither edge-disjoint nor containing each other), is there a good characterization for having simultaneous best-balanced orientations?*

The next theorem and corollary shows that this problem is NP-complete even for Eulerian graphs.

Theorem 7.1. *Deciding whether two Eulerian graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have Eulerian orientations that agree on the common edges $E_1 \cap E_2$, is NP-complete.*

Proof. The problem is clearly in NP. For the completeness we show a reduction from ONE-IN-THREE 3SAT (see [11], Problem LO4). For a given 3SAT formula, n denotes the number of variables, the clauses are denoted by $C_1, \dots, C_m$, and J_i denotes the set of indices of the clauses that contain the variable x_i.

Construct first the graph $G_1 = (V_1, E_1)$ as follows. Each connected component $G_1^i = (V_1^i, E_1^i)$ of G_1 corresponds to a clause C_i namely V_1^i contains the vertices $\{C_i, C_i'\}$ and the vertices $\{x_j^i, \overline{x}_j^i : i \in J_j\}$ and E_1^i contains the edge C_iC_i', the edges $\{x_j^i\overline{x}_j^i : i \in J_j\}$, the edges $\{C_ix_j^i, C_i'\overline{x}_j^i\}$ if x_j occurs in C_i and the edges $\{C_i\overline{x}_j^i, C_i'x_j^i\}$ if $\overline{x}_j$ occurs in C_i. Note that vertices corresponding to literals are of degree two and vertices corresponding to clauses are of degree four.

The graph $G_2 = (V_2, E_2)$ is constructed in such a way that each connected component of G_2 is a cycle. One cycle has color classes $\{C_i : 1 \le i \le m\}$ and $\{C_i' : 1 \le i \le m\}$ and contains the edges $\{C_iC_i' : 1 \le i \le m\}$. We also have cycles for $1 \le i \le n$ with color classes $\{x_i^j : j \in J_i\}$ and $\{\overline{x}_i^j : j \in J_i\}$ containing the edges $\{x_i^j\overline{x}_i^j : j \in J_i\}$. The details of the proof can be found in [13]. $\square$

We remark that another construction can be made by adding some extra vertices, in which both graphs are connected.

Corollary 7.2. *Deciding whether two graphs have simultaneous best-balanced orientations is NP-complete.*

8 Feasible pairing defining a best-balanced orientation

Nash-Williams' original idea was that every feasible pairing provides a best-balanced orientation. In fact Theorem 3.6 shows that every feasible pairing provides lots of best-balanced orientations. A natural question is whether every best-balanced orientation can be defined by a feasible pairing.

Question 8. *For every best-balanced orientation $\vec{G}$ of G there exists a feasible pairing M and an orientation $\vec{M}$ of M such that $\vec{G} + \vec{M}$ is Eulerian.*

Counter-example 8 Let $G := (V, E)$ and $\vec{G} := (V, A)$ be defined as follows (see Figure 5):

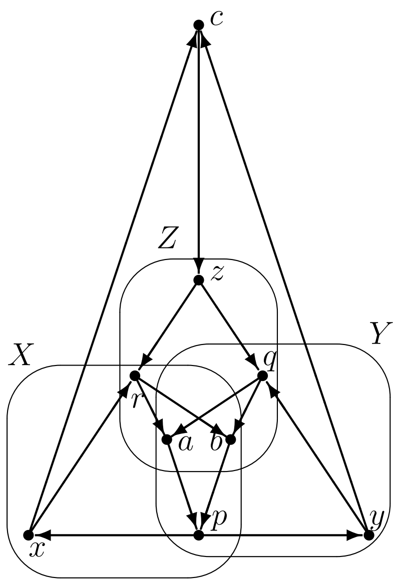

Figure 5: A best-balanced orientation which can not be defined with a feasible pairing

$V := \{a, b, c, p, q, r, x, y, z\}$, $E := \{ap, aq, ar, bp, bq, br, cx, cy, cz, xp, py, yq, qz, zr, rx\}$, $A := \{ap, qa, ra,\ bp, qb, rb, xc, yc, cz, px, py, yq, zq, zr, xr\}$. It is easy to check that $\vec{G} \in \mathcal{O}_b(G)$.

We show that if $M \in \mathcal{P}_f(G)$, then $ab \in M$. Note that $T_G = \{a, b, c, x, y, z\}$. Let $X := \{a, b, p, r, x\}$, $Y := \{a, b, p, q, y\}$, $Z := \{a, b, q, r, z\}$. Note that $d_G(W) = 5$ and $\hat{R}(W) = 4$ hence $b_G(W) = 1$ for $W \in \{X, Y, Z\}$. Then, by (5) and (7), $3{=}b_G(X) + b_G(Y) + b_G(Z) \geq d_M(X) + d_M(Y) + d_M(Z) \geq d_M(X \cap Y \cap Z) + d_M(X - (Y \cup Z)) + d_M(Y - (X \cup Z)) + d_M(Z - (X \cup Y)) = d_M(\{a, b\}) + d_M(x) + d_M(y) + d_M(z) \geq 0 + 1 + 1 + 1 = 3$, so $d_M(\{a, b\}) = 0$ that is $ab \in M$.

Then for every orientation $\vec{M}$ of any feasible pairing M of G either $\delta_{\vec{M}}(a) = 0$ or $\delta_{\vec{M}}(b) = 0$. Then, since $f_{\vec{G}}(a) = f_{\vec{G}}(b) = 1$, $\vec{G} + \vec{M}$ cannot be Eulerian. □

The following theorem shows that the answer for Question 8 is affirmative for global edge-connectivity.

Theorem 8.1. *Let $G := (V, E)$ be a $2k$-edge-connected graph and let $\vec{G} := (V, A)$ be a smooth k-arc-connected orientation of G. Then there is a pairing M of G and an orientation $\vec{M}$ of M so that*

$$d_M(X) \leq d_G(X) - 2k \quad \emptyset \neq \forall X \subset V \text{ and} \tag{13}$$

$$\vec{G} + \vec{M} \text{ is Eulerian.} \tag{14}$$

Proof. By induction on $|A|$. We shall apply the following stronger version of Mader's splitting off theorem [18] due to Frank [9].

Theorem 8.2. *Let $\vec{H} := (U+s, F)$ be k-arc-connected in U. If $\delta_{\vec{H}}(s) - \varrho_{\vec{H}}(s) < \varrho_{\vec{H}}(s) < 2\delta_{\vec{H}}(s)$ then there exist $rs, st \in F$ so that $\vec{H}_{rt} := \vec{H} - \{rs, st\} + rt$ is k-arc-connected in U.*

Case 1 If there is $s \in V$ with $d(s) \geq 2k+2$. Then, by (3) and Theorem 8.2, there exist $rs, st \in A$ so that $\vec{G}_{rt}$ is k-arc-connected in $V - s$. It follows, by the assumption of Case 1 and (3), that $\vec{G}_{rt}$ is k-arc-connected. Note that $T_{G_{rt}} = T_G$. $|A(\vec{G}_{rt})| < |A|$ so by induction there is a pairing M of G_{rt} and an orientation $\vec{M}$ of M so that (13) and (14) are satisfied for (G_{rt}, M) and for $(\vec{G}_{rt}, \vec{M})$ and hence for (G, M) and for $(\vec{G}, \vec{M})$ and we are done.

Case 2 If there is $s \in V$ with $d(s) = 2k$. This case can be handled in the same way as Case 1 but here we have to make a complete splitting off at s.

Case 3 Otherwise, $d(s) = 2k+1$ $\forall s \in V$. Then $T_G = V$. By a result of Mader [16], since there is no vertex v with $\varrho_{\vec{G}}(v) = \delta_{\vec{G}}(v)$, there exists $uv \in A$ such that $\vec{G}' := \vec{G} - uv$ is k-arc-connected. Note that, by the assumption of Case 3 and (3), $\vec{G}'$ satisfies (3). $|A(\vec{G}')| < |A|$ so by induction there is a pairing M' on $T_{G'} = T_G - \{u, v\}$ and an orientation $\vec{M}'$ of M' so that (13) and (14) are satisfied for (G', M') and for $(\vec{G}', \vec{M}')$. Let $M := M' + uv$ and $\vec{M} := \vec{M}' + vu$. Then $\vec{G} + \vec{M} = (\vec{G}' + \vec{M}') + uv + vu$ is Eulerian. Moreover, $\emptyset \neq \forall X \subset V$ either $d_M(X) = d_{M'}(X)$ and $d_G(X) = d_{G'}(X)$ or $d_M(X) = d_{M'}(X) + 1$ and $d_G(X) = d_{G'}(X) + 1$ so (13) is satisfied for G and M. $\square$

9 The structure of feasible pairings

We call a pairing M a **feasible matching** (in G), if M is a subgraph of G (i.e., a matching in G) and M is a feasible pairing.

Theorem 9.1. *Deciding whether a graph has a feasible matching is NP-complete, even for planar three-regular graphs.*

Proof. We claim that a 2-connected 3-regular graph $G = (V, E)$ has a feasible matching if and only if G is Hamiltonian. Indeed, for a perfect matching M of G, the 2-regular graph $G - M$ is Hamiltonian if and only if $G - M$ is 2-edge-connected that is if and only if $d_{G-M}(X) \geq 2$ for all $\emptyset \neq X \subset V$ or equivalently if M is feasible.

It is known that deciding whether a graph has a Hamiltonian cycle is NP-complete even for planar 2-connected 3-regular graphs [12]. $\square$

Corollary 9.2. *We are given a graph G and a weight on each pair of distinct odd-degree vertices. Finding the minimum weight strong pairing is NP-hard, even for planar 3-regular graphs and for $0-1$-valued weighting.*

We mention that the proof given here shows that the feasible matching problem and the minimum weight feasible pairing problem is NP-complete *even for the global case with $k = 1$.*

Another question on the structure of feasible pairings has been investigated in [13]. The following question was answered negatively there (see Question 13 in [13]):

Question 9. *Let $a, b, c, d \in T_G$. If there exist $M_1, M_2 \in \mathcal{P}_f(G)$ such that $ab, cd \in M_1$ and $ad, bc \in M_2$ then there exists $M_3 \in \mathcal{P}_f(G)$ such that $ac, bd \in M_3$.*

10 Feasible pairing for connectivity functions

A set function $b : V \to \mathbb{R}$ is called **skew-submodular** if for every $X, Y \subseteq V$, at least one of the following two inequalities is satisfied:

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y), \quad (15)$$
$$b(X) + b(Y) \geq b(X - Y) + b(Y - X). \quad (16)$$

A set function p is called **skew-supermodular** if $-p$ is skew-submodular. We mention that, by [19], $\hat{R}_G$ is skew-supermodular and hence b_G is skew-submodular. A set function b on V is called **crossing submodular** if (15) is satisfied for every $X, Y \subseteq V$ with $X \cap Y,\ X - Y,\ Y - X,\ V - (X \cup Y) \neq \emptyset$.

Question 10. *Let $b : V \to \mathbb{Z}_0^+$ be a symmetric, skew-submodular function with $b(\emptyset) = 0$ and $b(X) \equiv |X \cap T_b|$ mod 2, where $T_b = \{v : b(v)$ is odd$\}$. Then there exists a pairing M on T_b that satisfies*

$$d_M(X) \leq b(X) \qquad \forall\ X \subseteq V. \quad (17)$$

Counter-example 10 Let $b(X)$ be defined on V with $|V| = 6$ as follows: $b(X) := 0$ if $X = \emptyset, V$; 1 if $|X|$ is odd and 2 otherwise. It is easy to see that b satisfies all the conditions. Note that $T_b = V$. For any pairing M on T_b, we may choose $X \subset V$ with $d_M(X) = 3$ but then X violates (17). □

Note that, by Theorem 3.4, the answer for Question 10 is affirmative for $b(X) = b_G(X)$.

The problem corresponding to the global case is the following open problem.

Question 11. *Let $b : V \to \mathbb{Z}_0^+$ be a symmetric crossing submodular function with $b(\emptyset) = 0$ and $b(X) \equiv |X \cap T_b|$ mod 2. Then there exists a pairing M on T_b that satisfies (17).*

If the answer to Question 11 was affirmative then it would imply the following theorem that can be proved directly.

Theorem 10.1. *Let $G = (V, E)$ be an undirected graph. Let $b : V \to \mathbb{Z}_0^+$ be a crossing submodular function with $b(X) + d(X)$ even for every $X \subseteq V$. Then there exists an orientation $\vec{G}$ of G*

$$f_{\vec{G}}(X) \leq b(X) \ \ \forall\ X \subseteq V. \quad (18)$$

Proof. Let $\vec{G} = (V, A)$ be an arbitrary orientation of G. Let $P := \{z \in \mathbb{R}^{|A|} : 0 \leq z_a \leq 1\ \forall a \in A,\ \delta^z_{\vec{G}}(X) - \varrho^z_{\vec{G}}(X) \leq (b(X) - f_{\vec{G}}(X))/2\ \ \forall X \subseteq V\}$. By the modularity of $f_{\vec{G}}$ and by the assumptions, $(b(X) - f_{\vec{G}}(X))/2$ is integral and crossing submodular. Then, by the Edmonds-Giles theorem, P is an integral polyhedron. The vector $\frac{1}{2}\mathbb{1}$

belongs to P because b is non-negative. Then P contains an integral vector $\overline{z}$. Let $\vec{G}'$ be the orientation obtained from $\vec{G}$ by reversing the arcs $a \in A(\vec{G})$ for which $\overline{z}(a) = 1$. Then, since $\overline{z}$ is a $0-1$ vector in P, $f_{\vec{G}'}(X) = \varrho_{\vec{G}'}(X) - \delta_{\vec{G}'}(X) = (\varrho_{\vec{G}}(X) - \varrho^{\overline{z}}_{\vec{G}}(X) + \delta^{\overline{z}}_{\vec{G}}(X)) - (\delta_{\vec{G}}(X) - \delta^{\overline{z}}_{\vec{G}}(X) + \varrho^{\overline{z}}_{\vec{G}}(X)) = f_{\vec{G}}(X) + 2(\delta^{\overline{z}}_{\vec{G}}(X) - \varrho^{\overline{z}}_{\vec{G}}(X)) \leq b(X) \ \ \forall X \subseteq V$, and hence $\vec{G}'$ is the desired orientation. □

Note that if G is $2k$-edge-connected and $b(X) = d_G(X) - 2k \ \forall \emptyset \neq X \subset V$ and $b(\emptyset) = b(V) = 0$, then Theorem 10.1 is equivalent to Theorem 3.1. We remark that Theorem 10.1 also follows from a theorem of Frank [5] on orientations satisfying a G-supermodular function.

Question 12. *Let $d : V \to \mathbb{Z}^+_0$ be a symmetric function that satisfies $d(\emptyset) = 0$ and $\forall X, Y \subseteq V$*

$$d(X) + d(Y) + d(X \triangle Y) = d(X \cap Y) + d(X \cup Y) + d(X - Y) + d(Y - X), \quad (19)$$

$$d(X) + d(Y) - d(X \cup Y) \text{ is even if } X \cap Y = \emptyset. \quad (20)$$

Let $\hat{R} : V \to \mathbb{Z}^+_0$ be an even valued, symmetric, skew-supermodular function. Suppose that $\hat{R}(X) \leq d(X) \ \ \forall X \subseteq V$. Then there exists a pairing M on T_d that satisfies

$$d_M(X) \leq d(X) - \hat{R}(X) \qquad \forall \ X \subseteq V. \quad (21)$$

Counter-example 12 Let $V := \{u, v, w, z\}$, $G := (V, \{uw, uz, vw, vz, wz\})$, $H := (V, \{uv\})$, $d(X) := d_G(X) - d_H(X)$, $\hat{R}(X) := 2$ if $|X \cap \{w, z\}| = 1$ and 0 otherwise. Since for a proper subset X, $d_G(X) \geq 1$ and $d_H(X) \leq 1$, $d(X) \geq 0 \ \forall X \subseteq V$. Clearly, d is integer valued and symmetric. d_G and d_H satisfy (19) and (20), consequently d also satisfies them. It is easy to see that $\hat{R}$ satisfies all the conditions. Note that $T_d = V$. Let M be an arbitrary pairing on T_d. Let e be the edge of M incident to w. Let $X := \{u, w\}$ and let $Y := \{v, w\}$. Then e leaves either X or Y but $d(X) - \hat{R}(X) = 0 = d(Y) - \hat{R}(Y)$ so either X or Y violates (21). □

Note that, by Theorem 3.4, the answer for Question 12 is affirmative for $d(X) = d_G(X)$ and $\hat{R}(X) = \hat{R}_G(X)$.

Question 13. *Let $G = (V, E)$ be a graph and $\hat{R} : V \to \mathbb{Z}^+_0$ an even valued, symmetric, skew-supermodular function. Suppose that $\hat{R}(X) \leq d_G(X) \ \ \forall X \subseteq V$. Then there exists a pairing M on T_G that satisfies*

$$d_M(X) \leq d_G(X) - \hat{R}(X) \qquad \forall \ X \subseteq V. \quad (22)$$

Question 13 is an open problem. If $\hat{R}$ satisfies $\hat{R}(X \cup Y) \leq \max\{\hat{R}(X), \hat{R}(Y)\}$ $\forall X, Y \subset V$ then $\hat{R}(X) = \max\{\mathbf{r}(x, y) : x \in X, y \in V - X\}$ for some symmetric, even valued $\mathbf{r} : V \times V \to \mathbb{Z}^+_0$ and hence, by Theorem 3.4, such a pairing exists.

11 Matroid property

If $\vec{G}$ is an orientation of G then let $T^+_{\vec{G}} := \{v \in V(G) : \varrho_{\vec{G}}(v) > \delta_{\vec{G}}(v)\}$. Note that if $\vec{G}$ is smooth, then $|T^+_{\vec{G}}| = |T_G|/2$.

The following *strict reorientation property* was proved for k-arc-connected orientations by Frank in [6]: if $\vec{G}_1$ and $\vec{G}_2$ are k-arc-connected orientations of a graph $G=(V,E)$ and $\varrho_{\vec{G}_1}(u) < \varrho_{\vec{G}_2}(u)$ at a vertex $u \in V$ then there exists a directed path in $\vec{G}_1$ from u to some vertex $v \in V$ with $\varrho_{\vec{G}_1}(v) > \varrho_{\vec{G}_2}(v)$ such that reversing this path in $\vec{G}_1$ results in a k-arc-connected digraph. This result has interesting consequences, for example when restricted to smooth k-arc-connected orientations (which is not destroyed by such a reorientation) then it is equivalent with the following statement: for a $2k$-edge-connected graph G the family $\mathcal{T} := \{T^+_{\vec{G}} : \vec{G}$ is a smooth k-arc-connected orientation of $G\}$ is the base family of a matroid. Another consequence of the strict reorientation property is that k-arc-connected orientations of a graph satisfy the so called *linkage property*. In this section we investigate whether any of the above properties hold for well-balanced orientations.

First we investigate the matroid property:

Question 14. *$\mathcal{T} := \{T^+_{\vec{G}} : \vec{G} \in \mathcal{O}_b(G)\}$ is the base family of a matroid.*

Counter-example 14 Let G, $\vec{G} \in \mathcal{O}_b(G)$, X, Y and Z be as in Figure 5. Then $\overleftarrow{G} \in \mathcal{O}_b(G)$ hence $B_1 := \{a,b,c\}$ and $B_2 := \{x,y,z\}$ are in $\mathcal{T}$. Suppose that $\mathcal{T}$ is the base family of a matroid. Then for $c \in B_1 - B_2$ there must exist $u \in B_2 - B_1$ such that $B_1 - \{c\} + \{u\} \in \mathcal{T}$, by symmetry we may suppose that $\{a,b,x\} \in \mathcal{T}$. Then there exists $\vec{G}' \in \mathcal{O}_b(G)$ so that $T^+_{\vec{G}'} = \{a,b,x\}$. Whence, by (10) and (1), $1{=}b_G(X){\geq}f_{\vec{G}'}(X){=}\sum_{u\in X} f_{\vec{G}'}(u){=}3$, contradiction. □

We can see from the previous proof that the reorientation property in the strict sense as introduced above is not true for well-balanced orientations. A weakening of the reorientation property is formulated by the following question:

Question 15. *Let $\vec{G}^a, \vec{G}^b \in \mathcal{O}_w(G)$. Then there exist $\vec{G}^0 = \vec{G}^a, \vec{G}^1, ..., \vec{G}^l = \vec{G}^b \in \mathcal{O}_w(G)$ such that $\vec{G}^k$ is obtained from $\vec{G}^{k-1}$ by reversing a directed path or a directed cycle $(1 \leq k \leq l)$.*

Question 15 is an open problem. However, a less weak reorientation property was shown not to hold in [1], namely the following statement was disproved there: if $\vec{G}_1, \vec{G}_2 \in \mathcal{O}_w(G)$ such that $\exists x \in V(G)$ with $\varrho_{\vec{G}_1}(x) \neq \varrho_{\vec{G}_2}(x)$ then $\exists u, v \in V(G)$ with $\varrho_{\vec{G}_1}(u) < \varrho_{\vec{G}_2}(u)$ and $\varrho_{\vec{G}_1}(v) > \varrho_{\vec{G}_2}(v)$ such that reversing a directed path in $\vec{G}_1$ from u to v results in another well-balanced orientation. We recall that, by Frank [6], the answer for Question 15 is affirmative for global edge-connectivity.

Now we investigate whether the linkage property holds for well-balanced orientations.

Question 16. *Let $l, u : V \to \mathbb{Z}^+_0$ such that $l(v) \leq u(v)$ for all $v \in V$. Then there exists $\vec{G} \in \mathcal{O}_w(G)$ such that $l(v) \leq \varrho_{\vec{G}}(v) \leq u(v)$ $\forall v \in V$ if and only if there exist $\vec{G}^1, \vec{G}^2 \in \mathcal{O}_w(G)$ such that $l(v) \leq \varrho_{\vec{G}^1}(v)$ $\forall v \in V$ and $\varrho_{\vec{G}^2}(v) \leq u(v)$ $\forall v \in V$.*

Counter-example 16 Let G, $\vec{G}^1 := \vec{G}, \vec{G}^2 := \overleftarrow{G} \in \mathcal{O}_w(G)$, X, Y and Z as in Figure 5. Let the functions l and u be defined as follows: $l(a) = l(b) = 2$ and

$l(t) = \lfloor \frac{d_G(t)}{2} \rfloor$ $\forall t \in V - a - b$, $u(c) = 1$ and $u(t) = \lceil \frac{d_G(t)}{2} \rceil$ $\forall t \in V - c$. Then $l(v) \leq \varrho_{\vec{G}^1}(v)$ $\forall v \in V$ and $\varrho_{\vec{G}^2}(v) \leq u(v)$ $\forall v \in V$. Let $\vec{G}^3 \in \mathcal{O}_w(G)$ such that $l(v) \leq \varrho_{\vec{G}^3}(v)$ $\forall v \in V$. Recall that $b_G(X) = b_G(Y) = b_G(Z) = 1$. Then, by Claim 3.5, $1 = b_G(X) \geq f_{\vec{G}^3}(X) = f_{\vec{G}^3}(x)+f_{\vec{G}^3}(p)+f_{\vec{G}^3}(a)+f_{\vec{G}^3}(b)+f_{\vec{G}^3}(r) = f_{\vec{G}^3}(x)+0+1+1+0$, so $f_{\vec{G}^3}(x) \leq -1$ and hence $f_{\vec{G}^3}(x) = -1$. Similarly, $f_{\vec{G}^3}(y) = f_{\vec{G}^3}(z) = -1$. Then, since $f_{\vec{G}^3}(V) = 0$, $f_{\vec{G}^3}(c) = 1$, that is $\varrho_{\vec{G}^3}(c) = 2 > 1 = u(c)$. Thus there is no well-balanced orientation of G whose in-degree function satisfies both the lower and upper bounds. □

Question 16 is valid for the global case by Frank [5]. This follows from the facts that the in-degree vectors of k-arc-connected orientations form a base-polyhedron and for such polyhedra the linkage property holds, but as mentioned before it follows easily from the strict reorientation property, too.

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