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## Abstract

Given a graph  $G = (V, E)$  and a “cost function”  $f : 2^V \rightarrow \mathbb{R}$  (provided by an oracle), the problem [PCliqW] consists in finding a partition into cliques of  $V(G)$  of minimum cost. Here, the cost of a partition is the sum of the costs of the cliques in the partition. We provide a polynomial time dynamic program for the case where  $G$  is an interval graph and  $f$  belongs to a subclass of submodular set functions, which we call “value-polymatroidal”. This provides a common solution for various generalizations of the coloring problem in co-interval graphs such as max-coloring, “Greene-Kleitman’s dual”, probabilist coloring and chromatic entropy. In the last two cases, this is the first polytime algorithm for co-interval graphs. In contrast, NP-hardness of related problems is discussed. We also describe an ILP formulation for [PCliqW] which gives a common polyhedral framework to express min-max relations such as  $\bar{\chi} = \alpha$  for perfect graphs and the polymatroid intersection theorem. This approach allows to provide a min-max formula for [PCliqW] if  $G$  is the line-graph of a bipartite graph and  $f$  is submodular. However, this approach fails to provide a min-max relation for [PCliqW] if  $G$  is an interval graphs and  $f$  is value-polymatroidal.

**Keywords:** Partition into cliques; Interval graphs; Circular arc graphs; Max-coloring; Probabilist coloring; Chromatic entropy; Partial  $q$ -coloring; Batch-scheduling; Submodular functions; Bipartite matchings; Split graphs.

## 1 Introduction

Let  $G = (V, E)$  be a simple graph. In the following, a **clique** of  $G$  refers to a non-empty subset of vertices inducing a complete subgraph (not necessarily maximal with this property). Let  $\mathcal{C}(G)$  denote the set of cliques of  $G$ . A partition into cliques of  $G$  is a partition  $\mathcal{Q} = (K_1, \dots, K_k)$  of  $V(G)$ , where  $K_1, \dots, K_k \in \mathcal{C}(G)$ . In other words it is a coloring of  $\bar{G}$ , the complementary graph of  $G$ . Let  $\mathcal{P}(G)$  denote the set of all partitions into cliques of  $G$ . A classical problem consists in determining  $\bar{\chi}(G)$ , the minimum number of cliques necessary to partition  $G$ . In several applications however (see section 3), there is a **cost**  $f(C)$  associated to every clique  $C \in \mathcal{C}(G)$ , and we are interested in partitioning  $G$  into cliques, minimizing the sum of the costs of the cliques in the partition. Let  $\bar{\chi}(G, f)$  denote this minimum:

$$\bar{\chi}(G, f) := \min_{\mathcal{Q} \in \mathcal{P}(G)} \sum_{K \in \mathcal{Q}} f(K). \quad (1)$$

In order to describe some properties of  $f$ , one may assume that  $f$  is not only defined on cliques but is a **set function on  $\mathbf{V}$** , that is  $f : 2^V \rightarrow \mathbb{R}$ . This has no consequences for the definitions

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of  $\bar{\chi}(G, f)$  and [PCliqW] below. Notice that if  $f(C) = 1$  for all cliques  $C$ , we get the classical problem of coloring  $\bar{G}$  and we have  $\bar{\chi}(G, \mathbf{1}) = \bar{\chi}(G)$ . Determining  $\bar{\chi}(G, f)$  is therefore an NP-hard problem. Moreover, since  $|\mathcal{C}(G)|$  is usually exponential in  $|V|$  (the complete graph  $K_n$  on  $n$  vertices has  $|\mathcal{C}(K_n)| = 2^n$ ), encoding  $f$  itself raises complexity issues. In several applications however, both  $G$  and  $f$  have structural properties that allow to solve problem [PCliqW] in time polynomial in  $|V|$ .

**[PCliqW] Partition into cliques with weights**

**INPUT :** A graph  $G = (V, E)$  and a value oracle, providing  $f(K)$  in constant time for each  $K \in \mathcal{C}(G)$ .

**OUTPUT :** A partition into cliques of cost  $\bar{\chi}(G, f)$ .

[PCliqW] can also be described in terms of batch scheduling with compatibility graphs [12]. In this terminology (see [4] for batch scheduling problems not involving compatibility graphs and [16] for a classification of chromatic scheduling problems), each clique of a partition into cliques of  $G$  is called a *batch*. The operating time of a batch  $K$  is then  $f(K)$  and our objective is to minimize the makespan  $C_{\max}$  (whence the batches are ordered arbitrarily on the batch machine). Talking about cliques and batches allows to distinguish easily between cliques of  $G$  and cliques in a partition of  $V(G)$ . Two famous polytime cases of [PCliqW] are when

- $G$  is perfect and  $f \equiv 1$  [17],
- $G$  is complete and  $f$  is submodular set function [17]

Our solution for [PCliqW] for interval graphs and value-polymatroidal functions can be seen as a compromise between these two classical cases. Moreover, [PCliqW] enjoys a simple min-max formula in both cases [17] ( $\bar{\chi}(G) = \alpha(G)$  in the first case and ‘‘Dilworth’s truncation’’ in the second). One could therefore expect a common generalized min-max formula to hold in other cases for which [PCliqW] is polynomial. We deal with this issue in section 7.

In section 2, we define polymatroid rank functions and motivate the definition of value-polymatroidal set functions in the context of [PCliqW]. In section 3, we provide examples of value-polymatroidal set functions. In section 4, we discuss value-polymatroidal functions whose values  $f(U)$  depend only on the size  $|U|$ . In section 5, we provide a dynamic program which solves [PCliqW] for interval graphs in polytime if  $f$  is value-polymatroidal. The algorithm extends to the minimum cost partition problem for circular arc graphs, when we only consider cliques in which the arcs share a common point. As a counterpart, we mention NP-hardness of [PCliqW] for interval graphs if  $f$  is only assumed to be polymatroidal [2]. In section 6, we discuss NP-hardness of [PCliqW] on split graphs for subclasses of value-polymatroidal set functions. In section 7, we deal with some polyhedral issues and provide a min-max formula for [PCliqW] in line-graphs of bipartite graphs.

## 2 Value-polymatroidal set functions

A set function  $f : \mathcal{P}(V) \rightarrow \mathbb{R}$  is *submodular* if it satisfies one of the following equivalent properties [17]:

$$f(S \cup T) + f(S \cap T) \leq f(S) + f(T) \quad \text{for all } S, T \subseteq V, \quad (2)$$

$$f(S + u) + f(T) \leq f(S) + f(T + u) \quad \text{for all } T \subseteq S \subseteq V \text{ and } u \in V \setminus S, \quad (3)$$

$$f(S + u + v) + f(S) \leq f(S + u) + f(S + v) \quad \text{for all } S \subseteq V \text{ and } u, v \in V \setminus S. \quad (4)$$

A set function  $f$  is *non-negative* if all its values are, *non-decreasing* if  $S \subseteq T \implies f(S) \leq f(T)$ , *subcardinal* if  $f(U) \leq |U|$  for all  $U \subseteq V$ . A *polymatroid* rank function is a submodular, non-negative, non-decreasing set function such that  $f(\emptyset) = 0$ . A *matroid* rank function is a subcardinal, integral polymatroid rank function.

In some graph classes, submodularity of  $f$  is enough to ensure polynomiality of [PCliqW] (see section 7 and [16]). Although submodularity is not sufficient for interval graphs (see Theorem 5.5),

a stronger exchange property will do. We say that  $f$  is a **value-polymatroidal** set function if  $f(\emptyset) = 0$ ,  $f$  is non-decreasing and for every  $S$  and  $T$  subsets of  $V$  such that  $f(S) \geq f(T)$  and every  $u \in V \setminus (T \cup S)$ , we have

$$f(S + u) + f(T) \leq f(S) + f(T + u). \quad (5)$$

**Proposition 2.1.** *Every value-polymatroidal set function is a polymatroid rank function.*

**Proof** Let  $f$  be value-polymatroidal. Since  $f$  is non-decreasing, we have  $f(S) \geq f(T)$  for every  $T \subseteq S \subseteq V$  and therefore  $f(S + u) + f(T) \leq f(S) + f(T + u)$  for every  $u \in V \setminus S$ .  $\square$

By a **maximal clique**, we mean a clique maximal for inclusion (not necessarily for cardinality). The main motivation behind the definition of value-polymatroidal set functions is given by the following proposition.

**Proposition 2.2.** *For any graph  $G$  and any value-polymatroidal set function  $f$  on  $V(G)$ , there is a partition  $\mathcal{Q}$  of cost  $\bar{\chi}(G, f)$  in which one of the cliques in  $\mathcal{Q}$  is a maximal clique of  $G$ .*

**Proof** Let  $\mathcal{Q}$  be a minimum cost partition of  $G$  and choose any clique  $K \in \mathcal{Q}$ , such that  $f(K) \geq f(T)$  for all  $T \in \mathcal{Q}$ . If  $K$  is not a maximal clique of  $G$ , there exists some  $t \in V \setminus K$  such that  $K + t$  is a clique in  $G$ . Now,  $t$  belongs to some  $T \in \mathcal{Q} - K$ . Since  $f$  is non-decreasing,  $f(K) \geq f(T) \geq f(T - t)$ . Since  $f$  is value-polymatroidal,  $f(K + t) + f(T - t) \leq f(K) + f(T)$ . Repeat the process until  $K$  becomes a maximal clique of  $G$ .  $\square$

In general, rank functions of (poly)matroids are not value-polymatroidal, and the conclusion of Proposition 2.2 doesn't hold as shown in Figure 1.

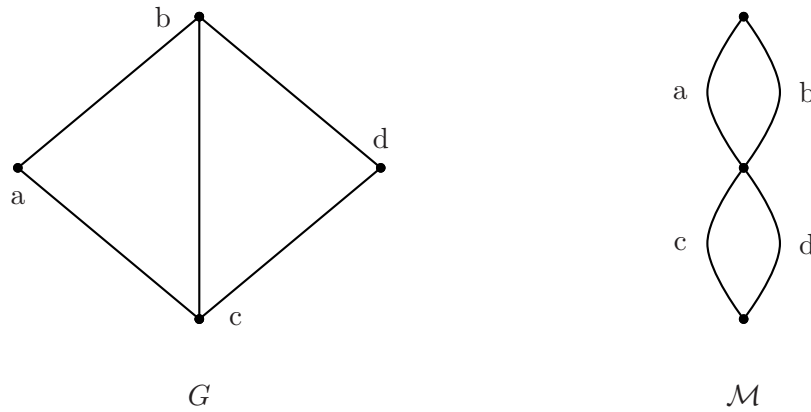


Figure 1: A graph  $G$  and a graphic matroid  $\mathcal{M}$  (whose rank function is not value-polymatroidal) such that  $\bar{\chi}(G, r(\mathcal{M})) = 2 = r(\{a, b\}) + r(\{c, d\})$ . No optimal partition contains a maximal clique of  $G$ .

### 3 Examples of value-polymatroidal set functions

In this section we mention some (coloring) problems that have been studied in the literature, and that amount to solving [PCliqW] for special subclasses of value-polymatroidal set functions. These problems are often formulated in terms of finding a minimum cost partition into stable sets, which is equivalent to [PCliqW] by taking the complementary graph.

**Maximum** Let  $p : V \rightarrow \mathbb{R}_+$  and define

$$f(U) := \max_{u \in U} p(u) \quad (6)$$

for any  $U \subseteq V$ . Then  $f$  is value-polymatroidal. Indeed, let  $S, T \subseteq V$  with  $f(S) \geq f(T)$ , and let  $u \in V \setminus (S \cup T)$ . Then, since  $p(s) = f(S) \geq f(T) = p(t)$  for some  $s \in S$  and  $t \in T$ , we have

$$f(S + u) + f(T) = \max\{p(s), p(u)\} + p(t) \leq p(s) + \max\{p(t), p(u)\} = f(S) + f(T + u).$$

A set function arising as in (6) is called a **max-batch cost function**. When restricted to max-batch cost functions, the corresponding problem of finding a minimum cost partition into stable sets is called [max-coloring] and is strongly-NP-hard for split graphs [8, 3], for bipartite graphs [8] and for interval graphs [11]. However, [max-coloring] is polynomial for  $P_4$ -free graphs [8] as well as for co-interval graphs [12, 2, 9].

**Independent probabilities** Let  $q : V \rightarrow [0, 1]$  and for  $U \subseteq V$ , let

$$f(U) := 1 - \prod_{u \in U} q(u) \quad (7)$$

Let  $S, T \subseteq V$  with  $f(S) \geq f(T)$ , and  $u \in V \setminus (S \cup T)$ . Write  $f(S) = 1 - \sigma$  and  $f(T) = 1 - \tau$  (so  $\sigma \leq \tau$ ). Then

$$\begin{aligned} f(S) + f(T + u) &= (1 - \sigma) + (1 - q(u)\tau) \\ &\geq (1 - q(u)\sigma) + (1 - \tau) = f(S + u) + f(T). \end{aligned}$$

Hence  $f$  is value-polymatroidal. A set function arising as in (7) is a **probabilistic cost function**. Transitive references for applications of probabilist optimization can be found in [7].

When restricted to probabilistic cost functions, [PCliqW] is strongly NP-hard in split graphs [7]. The corresponding problem of partitioning into stable sets is called [probabilist coloring].

**Chromatic Entropy** Let  $p : V \rightarrow [0, 1]$  and for  $U \subseteq V$ , let

$$c_U := \sum_{u \in U} p(u) \quad (8)$$

$$f'(U) := -c_U \log(c_U). \quad (9)$$

If  $c_V = 1$ ,  $f'$  is a **chromatic entropy** cost function. Although  $f'$  is not value-polymatroidal (it is not non-decreasing), the function  $f := f' + c$  is value-polymatroidal as can be derived from the concavity of the function  $x \mapsto x - x \log(x)$  [1]. Since for any partition  $V = K_1 \cup \dots \cup K_k$  of  $V$  into cliques, we have  $\sum_i f(K_i) = c(V) + \sum_i f'(K_i)$ , the two functions  $f'$  and  $f$  yield the same optimal partitions.

The corresponding problem of partitioning into stable sets is called [chromatic entropy] [1, 6] and is strongly NP-hard for interval graphs [6].

**Uniform matroid and Partial  $q$ -coloring** Let  $q \in \mathbb{N}$  and let

$$f(U) := \min\{q, |U|\} \quad (10)$$

Then  $f$  is value-polymatroidal, and the proof is left as an exercise since a more general statement is given with the next example. Functions arising this way are exactly the rank functions of uniform matroids. [PCliqW] with such a cost function arises in Greene-Kleitman's min-max relations stating that for any (co)-comparability graph  $G$  and any integer  $q$ , the maximum cardinality  $\alpha_q(G)$  of the union of  $q$  stable sets of  $G$  satisfies  $\alpha_q(G) = \bar{\chi}(G, f)$  (see [5] and [17], sections 14.6 and 14.7 on unions of chains and antichains in posets and section 66.5e on " $k$ -perfect" graphs for more details and references).

**Size-defined concave** Assume that  $f(\emptyset) = 0$  and that

$$f(U) := \psi(|U|) \quad (11)$$

for some  $\psi : \mathbb{N} \rightarrow \mathbb{R}_+$ . Then  $f$  is value-polymatroidal if and only if  $f$  is the rank of a polymatroid and also if and only if  $\psi$  has a non-decreasing concave extension on the real segment  $[0, |V|]$  (see section 4). The rank function of a uniform matroid is a special case.

## 4 Size-defined submodular set functions

In this section, we notice that if  $f(U)$  only depends on  $|U|$ , then polymatroid ranks coincide with value-polymatroidal functions. Let  $[a..b]$  denote the set of integers in the interval  $[a, b]$ . A set function  $f$  on  $V$  is **size-defined** if there exists a function  $\psi : [0..|V|] \rightarrow \mathbb{R}$  such that  $f(U) = \psi(|U|)$ . The function  $\psi$  is then the **compact representation** of  $f$ . Recall that a function  $f : [a, b] \rightarrow \mathbb{R}$  is **concave** if for all  $c, d \in [a, b]$  we have  $f(c) + f(d) \leq 2f((c+d)/2)$

**Theorem 4.1.** *Let  $f$  be a size-defined, non-decreasing set function such that  $f(\emptyset) = 0$  and  $\psi$  be the compact representation of  $f$ . The following are equivalent:*

- i)  $f$  is value-polymatroidal
- ii)  $f$  is a polymatroid rank function
- iii)  $2\psi(i) \geq \psi(i-1) + \psi(i+1)$  for all  $i \in [1..|V|-1]$
- iv)  $\psi(i+1) - \psi(i) \geq \psi(j+1) - \psi(j)$  for all  $i, j \in [0..|V|-1]$ , with  $i < j$
- v)  $\exists \hat{\psi} : [0, |V|] \rightarrow \mathbb{R}$  concave such that  $\psi(i) = \hat{\psi}(i)$  for  $i \in [0..|V|]$

**Proof** i)  $\implies$  ii): Proposition 2.1

ii)  $\implies$  iii): Use definition (4) of polymatroids with  $|S| = i - 1$ .

iii)  $\implies$  iv): By induction on  $j - i$ . The case  $j - i = 1$  being exactly iii). Adding  $\psi(i+1) - \psi(i) \geq \psi(j+1) - \psi(j)$  and  $2\psi(j+1) \geq \psi(j) + \psi(j+2)$  gives  $\psi(i+1) - \psi(i) \geq \psi(j+2) - \psi(j+1)$ .

iv)  $\implies$  i): For  $S, T \subseteq V$ , since  $f$  is size-defined and non-decreasing,

$$f(S) \geq f(T) \iff \psi(|S|) \geq \psi(|T|) \iff |S| \geq |T|$$

Applying iv) to  $j = |S|$  and  $i = |T|$  gives i).

v)  $\implies$  iii): Apply the concavity condition to  $c = i - 1$  and  $d = i + 1$ .

iii)  $\implies$  v): Take  $\hat{\psi}$  as the piecewise linear interpolation of  $f$  (for any  $x \in [0..|V|]$ ,  $\hat{\psi}(x) := \lambda f(\lfloor x \rfloor) + (1 - \lambda)f(\lceil x \rceil)$  for  $\lambda := x - \lfloor x \rfloor$ ). One can check that the subgradient of  $-\hat{\psi}$  is nondecreasing.  $\square$

## 5 Partition into cliques in interval and circular arc graphs

A graph  $G = (V, E)$  is an **interval graph** [13, 17] if there exists a set  $\{\phi(v) \mid v \in V\}$  of closed intervals on the real line, such that two vertices  $u$  and  $v$  are adjacent in  $G$  if and only if the two corresponding intervals  $\phi(u)$  and  $\phi(v)$  have nonempty intersection. Observe that any maximal clique  $K$  in  $G$  is of the form  $\{v \in V \mid t \in \phi(v)\}$  for some endpoint  $t$  of one of the intervals.

In [12, 9, 2], [PCliqW] is solved in polytime for interval graphs and max-batch cost functions. These algorithms use the fact that there exists an optimal solution in which a vertex of maximum cost is contained in a batch inducing a maximal clique. Based on this fact, a dynamic program is proposed. This fact is no longer true for value-polymatroidal costs as shown by the example in Figure 2. Nonetheless, based on Lemma 5.2, we describe a generalization of the algorithm proposed in [12], which provides an optimal solution for any value-polymatroidal cost function.

**Theorem 5.1.** *For any interval graph  $G = (V, E)$  and any value-polymatroidal set function  $f$  on  $V$  given by a value oracle, we can compute a partition into cliques of  $G$  of cost  $\bar{\chi}(G, f)$  in time  $O(n^3)$ .*

**Proof** Let  $\{I_i = [a_i, b_i]\}_{i=1, \dots, n}$  be a set of intervals on the real line representing graph  $G$ . We consider the set  $X$  of **endpoints** of the intervals:

$$X = \{a_i\}_{i=1, \dots, n} \cup \{b_i\}_{i=1, \dots, n} = \{1, \dots, q\}.$$

Let the **subproblem**  $\mathcal{I}(i, j)$  denote the set of all intervals completely contained in the closed interval  $[i, j]$ . For every pair of values  $i \leq j \in X$ , let  $F(i, j) := \bar{\chi}(G[\mathcal{I}(i, j)], f)$ , be the optimum cost of a partition of the subgraph induced by  $\mathcal{I}(i, j)$  (by definition of  $\bar{\chi}(G, f)$ ,  $F(i, j) = 0$  if  $\mathcal{I}(i, j) = \emptyset$ ). Our Dynamic Programming approach is based on Lemma 5.2 below, which implies that we can separate the problem restricted to  $\mathcal{I}(i, j)$  into two subproblems.

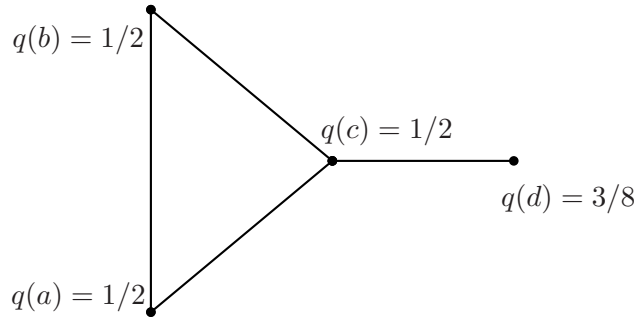


Figure 2: Let  $f$  be the probabilist cost defined by  $p$ . Vertex  $d$  has maximum cost  $f(\{d\}) = 1 - q(d) = 5/8$ . However, in an optimal partition, vertex  $d$  cannot be placed in a maximal clique since  $25/16 = f(\{a, b\}) + f(\{c, d\}) > \bar{\chi}(G, f) = f(\{a, b, c\}) + f(\{d\}) = 12/8$ .

**Lemma 5.2.** *For every  $i, j \in X$  there is an optimal partition into cliques of  $G[\mathcal{I}(i, j)]$  in which at least one batch induces a maximal clique of  $G[\mathcal{I}(i, j)]$ .*

**Proof** Directly from Proposition 2.2 □

Given  $i < z < j \in X$ , let  $K_{i,j}^z$  be the set of intervals of  $\mathcal{I}[i, j]$  containing point  $z$ . Notice that  $K_{i,j}^z$  is a clique for all  $i \leq z \leq j \in X$ .

**Lemma 5.3.** *For arbitrary fixed  $i < j$  in  $X$ , the following recursion holds:*

$$F(i, j) = \min_{z \in [i, j]} \{f(K_{i,j}^z) + (F(i, z-1) + F(z+1, j))\}. \quad (12)$$

**Proof** By Lemma 5.2, there is an optimal partition of  $G[\mathcal{I}(i, j)]$  in which a batch is a maximal clique  $B^*$ . All maximal cliques of  $G[\mathcal{I}(i, j)]$  are browsed while considering the minimum in (12). Hence  $B^* = K_{i,j}^{z^*}$  for some  $z^*$ . Given such point  $z^*$ , every interval in  $\mathcal{I}[i, z^* - 1]$  has its terminal endpoint before the initial endpoint of every interval in  $\mathcal{I}[z^* + 1, j]$ . Hence, the graph  $G(\mathcal{I}[i, j] \setminus B^*)$  decomposes into two disconnected subgraphs:  $G(\mathcal{I}[x_i, z^* - 1])$  and  $G(\mathcal{I}[z^* + 1, j])$ . One can therefore solve the problems on these two subgraphs independently. □

The Dynamic Programming algorithm starts from the initial conditions

$$F(i, i) = f(\mathcal{I}[i, i]) \quad \text{for all } i = 1, \dots, q.$$

Applying the recursion (12) with increasing subproblem width  $x_j - x_i$ , it computes an optimal schedule

$$S(x_i, x_j) = \begin{cases} \emptyset & \text{if } \mathcal{I}[i, j] = \emptyset; \\ S(i, z^* - 1) \cup B^* \cup S(z^* + 1, j) & \text{otherwise.} \end{cases}$$

The optimum value is  $\bar{\chi}(G, f) = F(1, q)$ , and  $S(1, q)$  is an optimal solution. Since there are  $O(q^2) = O(n^2)$  subproblems and  $O(q) = O(n)$  candidate values for  $z$  in each subproblem, the resulting Dynamic Programming algorithm solves the problem in  $O(n^3)$  time. This completes the proof of Theorem 5.1. □

Theorem 5.1 and the associated algorithm can be extended in the following way. A graph  $G = (V, E)$  is a **circular arc graph** [13] if there exists a set  $\{\phi(v) \mid v \in V\}$  of closed arcs of the unit circle, such that two vertices  $u$  and  $v$  are adjacent in  $G$  if and only if the two corresponding arcs  $\phi(u)$  and  $\phi(v)$  have nonempty intersection. Call a clique  $K$  of  $G$  a **Helly clique** if  $\bigcap_{v \in K} \phi(v)$  is nonempty.

**Corollary 5.4.** *For any circular arc graph  $G$ , and any value-polymatroidal function  $f$  on  $V(G)$  given by a value oracle, we can compute an optimum partition into Helly cliques in time  $O(n^3)$ .*

**Proof** Let  $X$  be the set of endpoints of the arcs  $\phi(v)$ , (as in Theorem 5.1). For  $i, j \in X$ , let  $\mathcal{I}[i, j]$  be the set of arcs contained in the portion of the circle in clockwise order between  $i$  and  $j$ . Note that after removing any maximal Helly clique, the remaining arcs are contained in some set  $\mathcal{I}[i, j]$ . Compute all  $O(n^2)$  values as in Theorem 5.1. Compute the best maximal Helly clique afterwards.  $\square$

On the other hand, we have the following negative result:

**Theorem 5.5.** *[2] [PCliqW] is NP-hard even if  $G$  is an interval graphs and  $f$  is a polymatroid cost (even if  $f$  is given by a rooted-TSP on a tree).*

**Rooted-TSP on trees** Let  $T = (W, A)$  be a tree,  $l : A \rightarrow \mathbb{N}$  and  $r \in W$  be the root of  $T$ . For  $U \subseteq W$ , let  $A(U)$  be the set of arcs spanning  $U + r$  and  $f(U) := 2 \sum_{a \in A(U)} l(a)$ . The function  $f$  is called a rooted-TSP cost since it is the cost of visiting all nodes in  $U \subseteq V$ , moving along edges of  $A$ , starting and finishing the tour from node  $r$  (see Figure 3). Such a cost function can easily be shown to be polymatroidal<sup>1</sup>. Complementing Theorem 5.5, [2] gave a 2-approximation for [PCliqW] when  $G$  is an interval graphs and  $f$  is rooted-TSP on a tree. This has applications in vehicle routing problems with time windows (where the length  $l(a)$  represents a travel cost and we assume that the traveling times are negligible compared to the size of the time windows [9]).

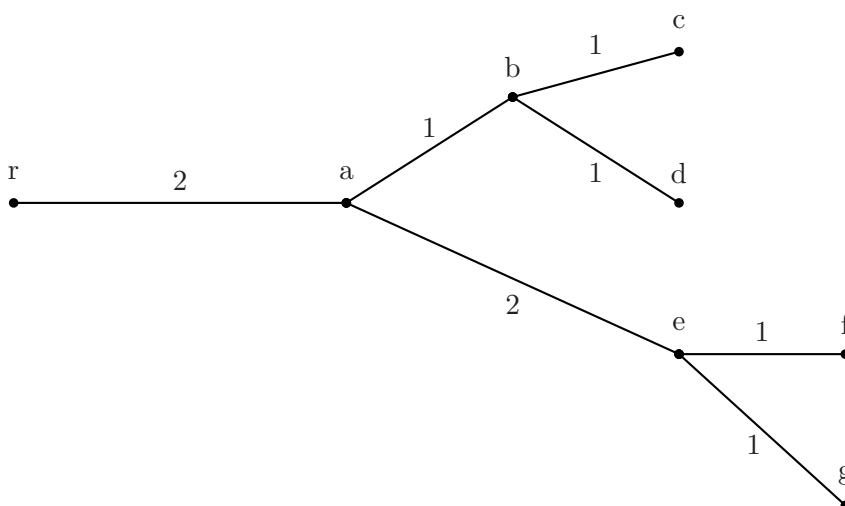


Figure 3: A rooted tree with a length function  $l : A \rightarrow \mathbb{R}$ . The cost associated with a subset  $U \subseteq V$  is twice the length of the arcs spanning  $U + r$ . For example  $f(\{a\}) = 4$ ,  $f(\{a, b, f\}) = 12$  and  $f(\{c, d, e, f\}) = 16$ .

<sup>1</sup>In fact, several characterizations of the graphs for which rooted TSP costs are polymatroidal for all edge length can be found in [15]. Based on [15], Jost [16] characterized these graphs as the graphs without  $K_{2,3}$  minors.



## 6 Partition into cliques in split graphs

One may wonder if Proposition 2.2 could be applied in more general graphs than interval graphs. A property of interval graphs which is used to prove polynomiality in Theorem 5.1 is that they have a polynomial number of maximal cliques. In this section, we illustrate that this property is not sufficient to ensure polytime solvability of [PCliqW] restricted to value-polymatroidal costs.

A graph  $G = (V, E)$  is a *split graph* if  $V$  can be partitioned into two sets  $S$  and  $K$  such that  $S$  is a stable set and  $K$  is a clique. Notice that split graphs have a polynomial number of maximal cliques (at most  $|S| + 1$ ). However, [max-coloring] and [probabilist coloring] are (strongly) NP-hard in split graphs ([3, 8] and [7] respectively). Since the class of split graphs is self-complementary, [PCliqW] is also NP-hard if we restrict to maximum or probabilist cost functions. Moreover, Yannakakis and Gavril [18] proved that the maximum  $q$ -chromatic subgraph problem is NP-hard for split graph. Unsurprisingly then, Greene-Kleitman's relation doesn't hold for split graphs [5]. However, the "dual problem", that is [PCliqW] with  $f(U) := \min\{q, |U|\}$  is trivial. If  $q = 1$  this is equivalent to find a partition of  $G$  into a minimum number of cliques. If  $q \geq 2$ , we may assume  $\omega(G) = |K|$  (in general, the bipartition  $(S, K)$  of a split graph is not unique). Then the partition consisting of all elements of  $S$  alone and all vertices of  $K$  together in a unique class is optimal. This fact however, does not extend to size-defined cost functions.

**Theorem 6.1.** *[PCliqW] is strongly NP-hard even if we restrict  $G$  to be a split graph and  $f$  to be size-defined and value-polymatroidal.*

**Proof** We reduce the NP-complete problem [X3C] to [PCliqW].

[X3C] **Exact three-set cover**

**INPUT :** A finite set  $X$  of size  $3m$  and a set  $T$  of triples of  $X$ .

**OUTPUT :** Does there exists a partition of  $X$  into  $m$  elements of  $T$ ?

Given an instance of [X3C], build the split graph  $G = ((T, X), E)$  where  $G[T]$  is a stable set and  $G[X]$  a clique and  $(t, x) \in E$  iff  $x \in t$ . Let  $\psi(0) := 0$ ,  $\psi(1) := \alpha = m + 1$  and  $\psi(i) := \beta = m + 2$  for all  $i \geq 2$ . We claim that there is a partition of cost not exceeding  $m\beta + (|T| - m)\alpha$  if and only if  $X$  has a partition into triples of  $T$ . A partition into triples yields such a cost. Now, assume that  $X$  has no partition into triples. Since  $T$  induces a stable set, any partition of  $V(G)$  into cliques contains at least  $|T|$  classes. Those partitions which consist in exactly  $|T|$  cliques, are of cost at least  $(m + 1)\beta + (|T| - (m + 1))\alpha > m\beta + (|T| - m)\alpha$ . Those consisting in at least  $|T| + 1$  cliques are of cost at least  $(|T| + 1)\alpha > m\beta + (|T| - m)\alpha$ .  $\square$

## 7 ILP formulation and min-max formula for [PCliqW]

Seen as a partition problem, [PCliqW] can be formulated as an integer linear program, with variables  $y$  in  $\mathbb{R}^{\mathcal{C}(G)}$  (where  $\mathcal{C}(G)$  is the set of cliques of  $G$ ):

$$(13) \quad \begin{array}{ll} \text{(i)} & \min f^T y \\ \text{(ii)} & \sum_{C \ni v} y_C = 1 \text{ for all } v \in V \\ \text{(iii)} & y_C \in \{0, 1\} \text{ for all } C \in \mathcal{C}(G) \end{array}$$

Clearly, if  $f$  is non-negative, there is no advantage in taking  $y_C > 1$ . Therefore,  $y_C \in \{0, 1\}$  can be replaced by  $y_C \geq 0$  and  $y_C \in \mathbb{Z}$ . Also, if  $f$  is non-decreasing, (13) (ii) can be replaced by  $\sum_{C \ni v} y_C \geq 1$  (if  $y_A = y_B = 1$ ,  $A, B \in \mathcal{C}(G)$  and  $A \cap B \neq \emptyset$  then  $B \setminus A$  is still a clique of  $G$  and we can reset  $y_B := 0$  and  $y_{B \setminus A} := 1$ ).

If  $f$  is non-negative and non-decreasing, the dual of the linear relaxation of (13) can therefore be written as maximizing  $\mathbf{1}^T x$  subject to<sup>2</sup>:

$$(14) \quad (i) \quad \sum_{v \in C} x_v \leq f(C) \text{ for all } C \in \mathcal{C}(G)$$

$$(ii) \quad x_v \geq 0 \text{ for all } v \in V(G)$$

If  $G$  is perfect and  $f \equiv 1$ , (14) is TDI. Also if  $G$  is complete and  $f$  is submodular, (14) is box-TDI. So in both cases, (14) yields a min-max formula for [PCliqW]. But there are other famous cases where (14) yields a min-max formula. Greene-Kleitman's theorems can be restated in the following terms: if  $G$  is a comparability graph or the complement of such a graph and if  $f$  is the rank function of a uniform matroid, system (14) is TDI. Alternatively, Greene-Kleitman's theorems can be stated as the box-TDIness of (14) if  $G$  is (co-)comparability and  $f \equiv 1$  [5]. Note that cliques of the line-graph of a bipartite graph  $G$  correspond to subsets of  $\delta(v)$  (the set of edges incident with  $v$ ), for some  $v \in V(G)$ . Now, a common generalization of the polymatroid intersection theorem, of Dilworth's truncation and of min-max relations for bipartite  $b$ -matching can be stated as box-TDIness of (14) if  $G$  is the line-graph of a bipartite multigraph and  $f$  is submodular. More precisely we have (see section 48.3 of [17] for an idea of the proof and Chapter 60 for extensions),

**Theorem 7.1 (Submodular bipartite matchings polyhedron).** [16]

Let  $G = ((A, B), E)$  be a bipartite multi-graph and for all  $v \in A \cup B$  let  $f_v$  be a submodular function on  $\delta(v)$ , then the following system is box-TDI

$$\sum_{e \in F} x_e \leq f_v(F) \text{ for all } v \in A \cup B \text{ and } \emptyset \neq F \subseteq \delta(v) \quad (15)$$

In view of these results, it seems reasonable to expect system (14) to provide other min-max relations for [PCliqW]. However, the linear relaxation of (13) does not always have an integral optimal solution, even if  $G$  is an interval graph and  $f$  is a value-polymatroidal set function as shown in Figure 4 (other examples for which  $G$  is perfect,  $f$  is a submodular but the linear relaxation of (13) has no integral optimal solution are provided in [16]).

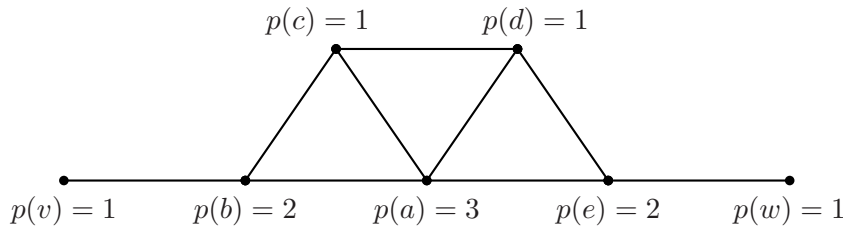


Figure 4: Let  $f$  be the max-batch cost defined by  $p$ . An optimal solution to the linear relaxation of (13) is given by  $y_C = 1/2$  if  $C \in \{\{v\}, \{b, v\}, \{a, b, c\}, \{a, d, e\}, \{c, d\}, \{e, w\}, \{w\}\}$  and  $y_C = 0$  otherwise. The cost of this fractional partition is  $13/2$ . Optimality can be checked using an  $x$  maximizing  $\mathbf{1}^T x$  subject to (14), for instance  $x(a) := 3/2$ ,  $x(c) = x(d) := 1/2$  and  $x(b) = x(e) = x(v) = x(w) := 1$ .

<sup>2</sup>An interpretation of system (14) within the framework of cooperative game theory with cooperation restricted to the cliques of a graph is described in [16].

## 8 Conclusion and extension

Although we were able to compute an optimum solution for [PCliqW] when  $G$  is an interval graph and  $f$  is value-polymatroidal, we were unable to complement this result by a min-max formula. This issue could be linked with the following extension: consider the problem of multi-partition into cliques, that is, generalize the ILP (13) by replacing constraints (ii) by  $\sum_{C \ni v} y_C = d_v$ , where  $d_v \in \mathbb{N}$  is the covering demand associated to vertex  $v$ . The complexity of this problem is left open and, to the best of our knowledge, is beyond the scope of our dynamic program. A polytime algorithm for this last problem might shed new light on the structure of interval graphs and therefore be useful to solve various problems on interval graphs.

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