Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2006-17. Published by the Egrervàry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Mader matroids are gammoids

Gyula Pap

November 2006

Mader matroids are gammoids *

Gyula Pap **

Abstract

Schrijver [3, 4] asks: Is each Mader matroid a gammoid? We prove that the answer is positive.

1 Introduction

We consider the problem of packing fully node-disjoint $\mathcal{A}$-paths. Let $G = (V, E)$ be an undirected graph $G = (V, E)$. Consider a set $A \subseteq V$ of so-called **terminals**, and a partition of A into disjoint sets $\mathcal{A} = \{A_1, \cdots, A_k\}$. A path is called an **$\mathcal{A}$-path** if its ends are in two distinct A_i's, and its internal nodes are non-terminals. We will use the expression "**packing** of paths" for a family of pairwise fully node-disjoint paths. Let $\nu(G, \mathcal{A})$ denote the maximum cardinality of a packing of $\mathcal{A}$-paths. For a packing $\mathcal{P}$ of $\mathcal{A}$-paths, let $A(\mathcal{P})$ (resp. $V(\mathcal{P})$) denote the set of terminals (resp. nodes) covered by $\mathcal{P}$. A set $Q \subseteq A$ is called **coverable** if there is a packing $\mathcal{P}$ of $\mathcal{A}$-paths such that $Q \subseteq A(\mathcal{P})$. The family of coverable sets is the family of independent sets of a matroid, as was shown in [3] by Schrijver. This matroid is called the Mader matroid for $G, \mathcal{A}$.

Schrijver [3, 4] asks: Is each Mader matroid a gammoid? We prove that the answer is positive.

Theorem 1.1. *Mader matroids are gammoids.*

Gammoids are constructed as follows. Consider a digraph $D = (U, B)$ with nodeset U and arcset B. Let S, A be disjoint subsets of U. We say a set $Q \subseteq A$ is **linked to** S if there is a packing of $S \to Q$ directed paths such that S is the set of starting nodes, and Q is the set of ending nodes of paths in the family. The bases of the gammoid are the subsets of A linked to S.

The construction of the gammoid representation of a Mader matroid is in fact implicitly contained in Mader's original proof [2] of his min-max formula. Our proof of Theorem 1.1 below uses Mader's min-max formula, and the construction of an auxiliary digraph. Mader proved his formula by applying Menger's theorem to (almost) the same auxiliary digraph. This auxiliary digraph is constructed by taking a specially chosen dual optimum in Mader's formula.

*Research is supported by OTKA grants T 037547 and TS 049788, by European MCRTN Adonet, Contract Grant No. 504438 and by the Egerváry Research Group of the Hungarian Academy of Sciences.

**Dept. of Operations Research, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). e-mail: `gyuszko@cs.elte.hu`

2 Mader's formula

We need some more notation for the dual side in the formula. An **$\mathcal{A}$-partition** is a family $\mathcal{X} = \{X_0; X_1, \cdots, X_k\}$ of disjoint subsets of V such that $A_i \subseteq X_0 \cup X_i$ for $i = 1, \cdots, k$. Let $X := \bigcup_{i=1}^{k} X_i$. The **value** of an $\mathcal{A}$-partition is defined by

$$val(G, \mathcal{X}) := |X_0| + \sum_{K \in \mathcal{K}} \left\lfloor \frac{|V(K) \cap X|}{2} \right\rfloor \tag{1}$$

where $\mathcal{K}$ denotes the family of **components** of the subgraph $G - X_0 - \bigcup_{i=1}^{k} E[X_i]$. This notation is due to Sebő and Szegő [5], and provides an equivalent reformulation of Mader's formula.

Theorem 2.1 (Mader [2]). *Let $G = (V, E)$ be an undirected graph, and consider a set $A \subseteq V$ partitioned into $\mathcal{A} := \{A_1, \cdots, A_k\}$. Then*

$$\nu(G, \mathcal{A}) = \min val(G, \mathcal{X}) \tag{2}$$

where the minimum is taken over $\mathcal{A}$-partitions $\mathcal{X}$.

Later we will also use the following notation. For some $K \in \mathcal{K}$, let $A^K := V(K) \cap X$ denote the **border of** K. Thus X is in fact equal to the union of all the borders. Let us call $K \in \mathcal{K}$ **even/odd** depending on the cardinality of its border. We define a partition $\mathcal{A}^K = \{A_1^K, \cdots, A_k^K\}$ of the border by $A_i^K := X_i \cap V(K)$. See Figures 1, 2.

The easy part of Mader's theorem follows from the following observation. Any $\mathcal{A}$-path either uses at least one node of X_0, or uses at least two nodes of the border of some component. This observation also implies the following. If $\mathcal{P}$ is a maximum packing of $\mathcal{A}$-paths and $\mathcal{X}$ is a minimum value $\mathcal{A}$-partition, then

$$\text{the nodes in } A - V(\mathcal{P}) \text{ are in the border of odd components, and} \tag{3}$$

$$\text{if } A^K = \emptyset \text{ for some } K \in \mathcal{K}, \text{ then } V(\mathcal{P}) \cap V(K) = \emptyset. \tag{4}$$

Such optimality criteria are called slackness conditions.

3 A special optimal $\mathcal{A}$-partition

Mader's theorem determines the optimum by means of an $\mathcal{A}$-partition. Sebő and Szegő [5], and independently Chudnovsky et al. [1] gave a structural description, generalizing the Gallai-Edmonds decomposition. Their results imply that there is an optimum $\mathcal{A}$-partition having some specific properties. For our purposes we need an optimum $\mathcal{A}$-partition such that $K, \mathcal{A}^K$ is critical (defined below) for each component $K \in \mathcal{K}$. This can also be derived from results in [5, 1], instead we include direct proofs only using Mader's formula. We will prove this property for an optimum $\mathcal{A}$-partition with X_0 inclusionwise maximal and then X inclusionwise maximal.

We recall some notation introduced in [5] – analoguous notation was introduced in [1], too. A path P is called an A_i–v path if one of its ends is v, its other end is $a \in A_i$, and $\{a\} = V(P) \cap A$. (Here the nodes a and v are not necessarily distinct, i.e. P may be a zero-length path starting and ending in $a = v \in A_i$.) We call a node $v \in V$ i**-rooted** if there is an A_i–v path P such that $\nu(G, \mathcal{A}) = \nu(G - V(P), \mathcal{A})$. A node is called **rooted** it is i-rooted for some i, and is called **multi-rooted** if it is i-rooted and j-rooted for some $i \neq j$. Notice, a terminal in A_i may not be j-rooted for $j \neq i$, thus no terminal is multi-rooted. The instance $G, \mathcal{A}$ is called **critical** if $|A| = 2\nu(G, \mathcal{A}) + 1$, all the nodes are rooted and all the nodes in $V - A$ are multi-rooted. See Figure 3.

Lemma 3.1. *If $|A| = 2\nu(G, \mathcal{A}) + 1$ and $G, \mathcal{A}$ is not critical, then there is an optimal $\mathcal{A}$-partition $\mathcal{X} = \{X_0; X_1, \cdots, X_k\}$ such that $X_0 \neq \emptyset$, or $X \not\subseteq A$.*

Proof. Suppose some node $a \in A$ is not rooted. Let $G' := (V + a', E + aa')$, $A' := A + a'$, $A'_i := A_i$ for $i = 1, \cdots, k$, $A_{k+1} := \{a'\}$, and $\mathcal{A}' := \{A'_1, A'_2, \cdots, A'_{k+1}\}$. Then $\nu(G, \mathcal{A}) = \nu(G', \mathcal{A}')$. Consider an optimal $\mathcal{A}'$-partition $\mathcal{X}' = \{X'_0; X'_1, \cdots, X'_{k+1}\}$. Define $\mathcal{X} := \{X'_0 + a; X'_1, \cdots, X'_k\}$. Slackness condition (3) implies that a' is is in the border of an odd component. Thus $val(G, \mathcal{X}) \leq val(G', \mathcal{X}')$.

Suppose some node $v \in V - A$ is not i-rooted for any $i \neq 1$. Let $G' := (V + v', E + vv')$, $A' := A + v'$, $A'_1 := A_1 + v'$, $A'_i := A_i$ for $i \neq 1$, and $\mathcal{A}' := \{A'_1, A'_2, \cdots, A'_k\}$. Then $\nu(G, \mathcal{A}) = \nu(G', \mathcal{A}')$. Consider an optimal $\mathcal{A}'$-partition $\mathcal{X}' = \{X'_0; X'_1, \cdots, X'_k\}$. $|A'| \geq 2\nu(G', \mathcal{A}') + 2$, hence $X'_0 \neq \emptyset$, or $\bigcup_{i=1}^k X'_i \not\subseteq A'$. Define $\mathcal{X} := \{X'_0; X'_1 - v', \cdots, X'_k\}$. Slackness condition (3) implies that v' must be in the border of an odd component. Thus $val(G, \mathcal{X}) \leq val(G', \mathcal{X}')$. □

Lemma 3.2. *Let $\mathcal{X}$ be an optimal $\mathcal{A}$-partition such that X_0 is inclusionwise maximal and then X is inclusionwise maximal. Then for any component $K \in \mathcal{K}$ either $A^K = \emptyset$, or $K, \mathcal{A}^K$ is critical.*

Proof. If some even component K has a non-empty border, then pick an arbitrary node $x \in A_K$ of its border, and define $X'_0 := X_0 + x$ and $X'_i := X_i - x$. It is easy to check that $val(G, \mathcal{X}') = val(G, \mathcal{X})$, in contradiction with our choice of $\mathcal{X}$.

Now suppose K is an odd component, and thus $|A_K| \geq 2\nu(K, \mathcal{A}^K) + 1$. Suppose for contradiction, $K, \mathcal{A}^K$ is not critical. Let $\mathcal{X}^K$ be an optimal $\mathcal{A}^K$-partition with respect to $K, \mathcal{A}^K$, and choose $\mathcal{X}^K$ such as in Lemma 3.1. Define $X'_0 := X_0 \cup X_0^K$, $X'_i := (X_i \cup X_i^K) - X_0^K$ (for $i = 1, \cdots, k$), $\mathcal{X}' = \{X'_0; X'_1, \cdots, X'_k\}$, and $X' := \bigcup_{i=1}^k X'_i$. A straightforeward calculation gives $val(G, \mathcal{X}') \leq val(G, \mathcal{X})$. Equality must hold throughout, hence $|A_K| = 2\nu(K, \mathcal{A}^K) + 1$. Then by Lemma 3.1 we get $X'_0 \supsetneq X_0$ or $X' \supsetneq X$, a contradiction. □

Let us call a component $K \in \mathcal{K}$ trivial if $A^K = \emptyset$. Lemma 3.2 provides an optimum $\mathcal{A}$-partition such that all non-trivial components are critical. We reduce Mader matroids to the case when there is an optimum $\mathcal{A}$-partition such that all components $K \in \mathcal{K}$ are critical – i.e. we get rid of trivial components. Consider a representation $G = (V, E), \mathcal{A}$ of a Mader matroid such that $|V|$ is minimal. Consider an arbitrary optimal $\mathcal{A}$-partition $\mathcal{X}$. Suppose there is a trivial component $K \in \mathcal{K}$. Slackness condition (4) implies that $G - V(K), \mathcal{A}$ represents the same Mader matroid, a contradiction.

Next we reduce Mader matroids to the case when there is an optimum $\mathcal{A}$-partition $\mathcal{X}$ such that all components are critical, and $X_0 \subseteq V - A$. We get a triple $G, \mathcal{A}, \mathcal{X}$ from above such that all the components are critical. Construct $G' := (V', E')$ with $V' := V \cup \{a' : a \in A\}$, $E' := E \cup \{aa' : a \in A\}$. Let $A' := \{a' : a \in A\}$, $A'_i := \{a' : a \in A_i\}$, and $\mathcal{A}' = \{A'_1, \cdots, A'_k\}$. Clearly, $G', \mathcal{A}'$ represents the same Mader matroid, just use the bijection between A and A'. Define $\mathcal{X}' = \{X'_0; X'_1, \cdots, X'_k\}$ by $X'_0 := X_0$ and $X'_i := X_i \cup A'_i$. Notice that $\mathcal{K}' = \mathcal{K} \cup \{\{a'\} : a \in A\}$. Thus $val(G', \mathcal{X}') = val(G, \mathcal{X})$.

We have proved above that every Mader matroid can be represented by some $G, \mathcal{A}$ such that there exists an optimal $\mathcal{A}$-partition with the following properties.

$$X_0 \subseteq V - A. \tag{5}$$

$$K, \mathcal{A}^K \text{ is critical for each component } K \in \mathcal{K}. \tag{6}$$

4 The gammoid representation of the Mader matroid

Consider an optimal $\mathcal{A}$-partition $\mathcal{X}$ having properties (5) and (6). Let $\beta_K := |X \cap V(K)| - 1$. By (6), β_K is even. Let $[\beta_K] := \{1, 2, \cdots, \beta_K\}$. We define an auxiliary directed graph $D = (U, B)$. See Figure 6 for an example, and (7), (8) for the formal definition. For the nodeset U, we keep the old nodes of X, and delete all other nodes. For any node $x \in X_0$ we introduce $k + 2$ new nodes $x', x'', (x, 1), \cdots, (x, k)$. For any component K we introduce β_K new nodes $(K, 1), \cdots, (K, \beta_K)$.

$$U := \big\{x',\ x'' : x \in X_0\big\} \ \cup\ \big\{(K, m) : K \in \mathcal{K}, m \in [\beta_K]\big\} \cup \\ \big\{(x, i) : x \in X_0, i \in [k]\big\} \ \cup\ X \tag{7}$$

and arcset is defined by

$$\begin{aligned} B := &\big\{x'(x, i),\ x''(x, i)\ :\ x \in X_0, i \in [k]\big\} \ \cup\ \big\{(x, i)v\ :\ xv \in E, v \in X_i\big\} \cup \\ \cup\ &\big\{(x, i)v\ :\ \exists K \in \mathcal{K}, v \in V(K) \cap X, \exists xu \in E, u \in V(K) - X\big\} \cup \\ \cup\ &\big\{(K, m)v\ :\ v \in V(K) \cap X, m \in [\beta_K]\big\} \ \cup\ \textstyle\bigcup_{i=1}^{k} \overleftrightarrow{E[X_i]}, \end{aligned} \tag{8}$$

where $\overleftrightarrow{E[\cdot]}$ is obtained by replacing all edges $uv \in E[\cdot]$ by the two arcs uv, vu. Define

$$S := \big\{x',\ x'' : x \in X_0\big\} \ \cup\ \big\{(K, m) : K \in \mathcal{K}, m \in [\beta_K]\big\}$$

Notice that $|S| = 2\, val(G, \mathcal{X})$ holds. In Figure 6, nodes of S are shown in clouds. The following claim completes the proof of Theorem 1.1.

Claim 4.1. *A set $Q \subseteq T$ is a maximum coverable set if and only if it is linked to S in D.*

Proof. To prove this claim we need to formulate slackness conditions in more detail. As already observed, any $\mathcal{A}$-path uses at least one node of X_0, or at least two nodes of the border of some component. This observation, and our choice of $\mathcal{X}$ imply that if $\mathcal{P}$ is a maximum packing, and $P \in \mathcal{P}$, then P can be decomposed into segments defined by (9) or (10). See Figures 4, 5.

$P = a_i$–P_i–r_i–R–r_j–P_j–a_j, where $a_i \in A_i$, $r_i \in X_i \cap V(K)$, $r_j \in X_j \cap V(K)$, $a_j \in A_j$, and these nodes split P into segments P_i, R, P_j such that $V(P_i) \subseteq X_i$, $V(R) \subseteq V(K)$, $V(P_j) \subseteq X_j$. See Figure 4. (9)

The notation in (9) implies that $i \neq j$, $r_i \neq r_j$, thus R has positive length. However P_i, P_j may have zero length, so a_i, r_i and a_j, r_j may coincide.

$P = a_i$–P_i–y_i–R_i–z_i–x–z_j–R_j–y_j–P_j–a_j, where $a_i \in A_i$, $y_i \in V(K) \cap X_i$, $z_i \in V(K)$, $x \in X_0$, $z_i \in V(K')$, $y_j \in V(K') \cap X_j$, $a_j \in A_j$ for some $K \neq K'$, $K, K' \in \mathcal{K}$. Moreover, these nodes split P into segments P_i, R_i, z_ix, xz_j, R_j, P_j such that $V(P_i) \subseteq X_i$, $V(R_i) \subseteq V(K)$, $z_ix, xz_j \in E$, $V(R_j) \subseteq V(K')$, $V(P_j) \subseteq X_j$. See Figure 5. (10)

The notation in (10) implies $i \neq j$, $x \neq a_i, y_i, z_i, z_j, y_j, a_j$, and $a_i, y_i, z_i \neq z_j, y_j, a_j$. However, the nodes a_i, y_i, z_i (resp. z_j, y_j, a_j) are not necessarily distinct, and P_i, R_i, R_j, P_j may have zero length. The following conditions hold for a maximum packing $\mathcal{P}$.

For any component $K \in \mathcal{K}$ there are exactly $\beta_K/2$ paths in $\mathcal{P}$ of category (9). (11)

Every node $x \in X_0$ is traversed by a path in $\mathcal{P}$ of category (10). (12)

Suppose a maximum packing $\mathcal{P}$ such that $Q = A(\mathcal{P})$. We replace paths $P \in \mathcal{P}$ by two directed S–Q paths in digraph D as follows. If P is a category (9) path, then we replace P by the two directed paths (K,m)–r_i–$\overrightarrow{P_i}$–a_i and (K,m')–r_j–$\overrightarrow{P_j}$–a_j. (11) implies that we can choose m's such that all (K,m)'s are used by exactly one such directed path. Now suppose P is of category (10). Then the two paths in exchange for P will be x'–(x,i)–z_i–$\overrightarrow{P_i}$–a_i and x''–(x,j)–z_j–$\overrightarrow{P_j}$–a_j. By (12), all then nodes $x', x'' \in S$ are use by exactly one such directed path.

Consider a family $\mathcal{R}$ of directed paths linking Q to S. Consider a directed path $R \in \mathcal{R}$. It follows from the construction of D that there is a unique component K such that R traverses an arc $(K,m)v$ or $(x,i)v$ with $v \in V(K) \cap X$. We say R **enters** K **via** $(K,m)v$, or via $(x,i)v$, respectively. The paths in $\mathcal{R}$ entering K are the following: exactly β_K paths starting in the nodes $\{(K,m) : m \in [\beta_K]\}$, and at most one additional path starting in some x' or x''. So all but one of the nodes in the border of K are traversed by the paths entering K via some $(K,i)v$. The remaining border node z may be traversed by a path entering K via some $(x,i)v$, or a path not entering K.

Let $R^{(1)}, \cdots, R^{(\beta_K)}$ denote the paths entering K via some $(K,i)v$. Let $P^{(i)}$ (for $i \in [\beta_K]$) denote the undirected path in G which we get from $R^{(i)}$ by deleting its starting node (K, m). If there is a path entering K via some $(x, i)v$, then let $P^{(0)}$ denote the path we get from it by deleting its starting segment x'–(x, i)–v (or x''–(x, i)–v). Thus we have either β_K or $\beta_K + 1$ paths joining nodes in the border A^K to terminals in A such that nodes in A_i^K are joined to nodes in A_i.

Consider a component for which there are β_K paths joining A^K to A (i.e. $P^{(i)}$ for $i \in [\beta_K]$). Choose a packing $\mathcal{P}_K$ in $K, \mathcal{A}^K$ (i.e. of $\beta_K/2$ paths) such that $A^K(\mathcal{P}_K)$ is equal to the set of starting nodes of the paths $P^{(i)}$. We compose $\beta_K/2$ $\mathcal{A}$-paths from the union of the paths $\{R^{(1)}, \cdots, R^{(\beta_K)}\} + \mathcal{P}_K$. Clearly, this composition consist of $\mathcal{A}$-paths.

Consider components K for which there are $\beta_K + 1$ paths joining A^K to A (i.e. $P^{(i)}$ for $i \in \{0\} \cup [\beta_K]$). These components come up in pairs K, K' corresponding to x' and x'', for some $x \in X_0$. Suppose K is entered by the path x'–(x, i)–v–R and K' is entered by the path x''–(x, j)–v'–R'. Let us recall the definition (8) of the arcset B. The arc $(x, i)v \in B$ implies that either

$$xv \in E, v \in X_i, \text{ or} \tag{13}$$

$$\exists xu \in E, v \in V(K) \cap X, u \in V(K) - X \text{ holds.} \tag{14}$$

If arc $(x, i)v$ is of case (13), then consider a maximum packing $\mathcal{P}_K$ in $K, \mathcal{A}^K$ (i.e. of $\beta_K/2$ paths) such that it is disjoint from v. Such a $\mathcal{P}_K$ exists because of the criticality of $K, \mathcal{A}^K$. Then we can compose a family of disjoint paths from $\{R^{(0)}, \cdots, R^{(\beta_K)}\} + \mathcal{P}_K + vx$, and this composition consists of $\beta_K/2$ $\mathcal{A}$-paths and one additional A_i–x path.

If both of the arcs $(x, i)v$ and $(x, j)v'$ are of case (13), then merge the additional A_i–x path and the additional A_j–x path. This gives a family of $\beta_K + \beta_{K'} + 1$ disjoint $\mathcal{A}$-paths. The terminals covered by these paths are those nodes which were linked to S by some paths entering K and K'.

If arc $(x, i)v'$ is of case (13), but $(x, j)v$ is of case (14), then we construct the family of $\beta_K + \beta_{K'} + 1$ disjoint $\mathcal{A}$-paths as follows. We do the above construction for $(x, i)v'$, thus we get a family of fully node-disjoint paths which consists of $\beta_K/2$ $\mathcal{A}$-paths and one additional A_i–x path. We do a different construction for $(x, j)v$, which is the following. Since $v \in V(K) - X = V(K) - A^K$, and $K, \mathcal{A}^K$ is critical, v is multi-reachable in $K, \mathcal{A}^K$. Thus there is a packing $\mathcal{P}_{K'}$ of $\mathcal{A}^{K'}$-paths such that there is an $A_m^{K'}$–u path Q disjoint from all the paths in $\mathcal{P}_{K'}$, and $m \neq i$. Then we can compose a packing of $\beta_K + \beta_{K'} + 1$ $\mathcal{A}$-paths from the union of $\{R^{(0)}, \cdots, R^{(\beta_K)}\} + \mathcal{P}_K + Q + ux$. This composition covers the nodes which were linked to S by some paths entering K and K'.

Suppose both of the arcs $(x, i)v$ and $(x, j)v'$ are of case (14). We do the second of the above constructions for both arcs. There it is possible to choose $m \neq m'$ such that x is reached by an additional A_m–x path (through K) and an additional $A_{m'}$–x path (through K'). From the union of these two constructions we compose a family of $\mathcal{A}$-paths as desired. □

5 Figures

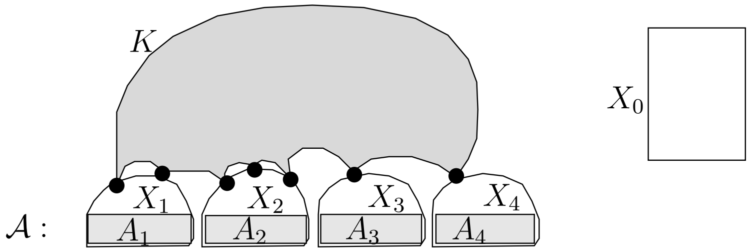

Figure 1: The component $K \in \mathcal{K}$ has a 7-node border.

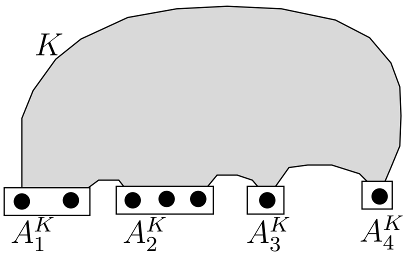

Figure 2: The instance $K, \mathcal{A}^K$ constructed for some component $K \in \mathcal{K}$ with respect to G, $\mathcal{X}$.

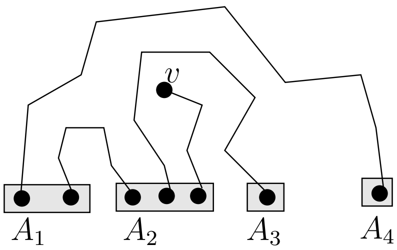

Figure 3: The node v is 2-rooted.

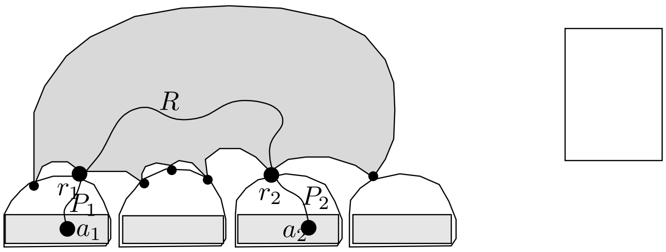

Figure 4: A path of category (9).

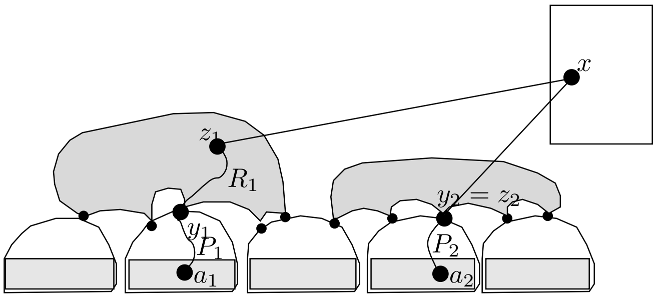

Figure 5: A path of category (10).

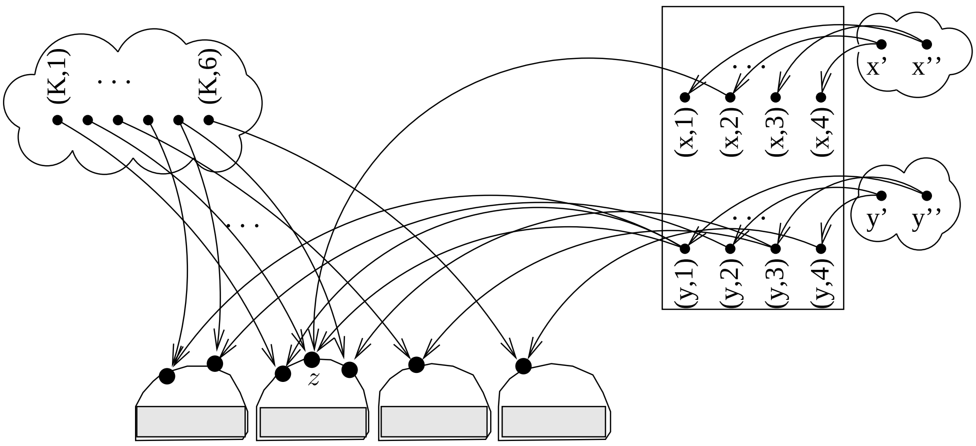

Figure 6: The auxiliary directed graph D with respect to G, $\mathcal{A}$, $\mathcal{X}$, assuming there is an edge $zx \in E$ such that $z \in V(K) \cap X_2$, and some edge $wy \in E$ such that $w \in V(K) - X$. We have only depicted a couple of arcs from bunches of arcs $(K,i)v$ and $x'(x,i), x''(x,i)$ in order not to make the figure confusing.

References

[1] M. Chudnovsky, J.F Geelen, B. Gerards, L. Goddyn, M. Lohman, P. Seymour, *Packing non-zero A-paths in group-labeled graphs*, to appear in the Journal of Combinatorial Theory Ser. B.

[2] W. Mader, *Uber die Maximalzahl kreuzungsfreier H-Wege*, Archiv der Mathematik (Basel) **30** (1978) 387-402.

[3] A. Schrijver, *Combinatorial Optimization*, Springer, 2003, pages 1292, 1459.

[4] A. Schrijver, *Is each Mader matroid a gammoid?*, www.cwi.nl/lex.

[5] A. Sebő, L. Szegő, *The path-packing structure of graphs*, in: Integer programming and combinatorial optimization, D. Bienstock, G. Nemhauser, eds., 256–270, Lecture Notes in Comput. Sci., 3064, Springer, Berlin, 2004.