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λ-supermodular functions and dual packing theory

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Abstract

The main usage of supermodular functions is to cross out tight sets, that is to use only the following important property. The intersection and the union of two sets with a maximal function value also have maximal function value. But for this purpose we do not need the full strength of supermodularity, some weaker concept suffices. We introduce here a new property, the so called λ-supermodularity. A set function f is λ-supermodular if for any sets X and Y there is number $0 < \lambda \leq 2$ such that

$$f(X) + f(Y) \leq \lambda \cdot f(X \cap Y) + (2 - \lambda) \cdot f(X \cup Y).$$

Function f is strongly λ-supermodular if we always find a $\lambda < 2$ and weakly if sometimes $\lambda = 2$ is needed. Clearly, if f is strongly λ-supermodular and X and Y are sets with $f(X) = f(Y) =$maximum (called usually tight sets), then $X \cap Y$ and $X \cup Y$ are also tight. If f is weakly λ-supermodular then we may only conclude that the intersection is tight, however, for many purposes, this fact is enough.

We show several examples on how to use λ-supermodularity related to duals of packings (called barriers). The first and simplest example specializes to the case of star-packings, induced-star packings and long path packing. More and more complicated versions are developed for propeller packings, f-factors and "even factors".

Keywords: packings, supermodular functions, barriers

1 Notation

Throughout the paper (except Section 5) G denotes a simple undirected connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $d(v)$ denote the degree of vertex v, and for a subgraph F of G let $d_F(v)$ denote the degree of v in F. For a subset X of vertices $G[X]$ denotes the subgraph induced by X. For subsets X, Y of vertices $d(X, Y)$ denotes the number of edges going from $X - Y$ to $Y - X$.

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Let f be a set-function with domain $\mathrm{dom}(f) \subseteq 2^S$ where S is a finite set. We always assume that $\mathrm{dom}(f)$ is closed under intersection and union. Function f is called strongly λ-supermodular if for any sets $X, Y \in \mathrm{dom}(f)$ there is number $0 < \lambda < 2$ such that

$$f(X) + f(Y) \leq \lambda \cdot f(X \cap Y) + (2 - \lambda) \cdot f(X \cup Y).$$

Function f is called (weakly) λ-supermodular if for any sets $X, Y \in \mathrm{dom}(f)$ there is number $0 < \lambda \leq 2$ such that the same inequality holds.

λ-supermodularity can also be characterized by the following way. Function f is strongly λ-supermodular if for any sets $X, Y \in \mathrm{dom}(f)$ either $\max\{f(X \cap Y), f(X \cup Y)\} > (f(X) + f(Y))/2$, or $f(X \cap Y) = f(X \cup Y) = (f(X) + f(Y))/2$. Function f is weakly λ-supermodular if for any sets $X, Y \in \mathrm{dom}(f)$ either $f(X \cap Y) \geq (f(X) + f(Y))/2$, or $f(X \cup Y) > (f(X) + f(Y))/2$.

A set $X \in \mathrm{dom}(f)$ is called tight, if $f(X)$ is maximal. Clearly, if X and Y are tight sets, and f is strongly λ-supermodular then $X \cap Y$ and $X \cup Y$ are also tight sets; while if f is known to be only weakly λ-supermodular then $X \cap Y$ is also a tight set.

Sometimes we relax this property. We call some sets in $\mathrm{dom}(f)$ *essential*, taking care that all tight sets should be essential. We say that f is (strongly/weakly) λ-supermodular on essential sets, if the inequality is known to be true if X and Y are essential.

Unfortunately, for some of the applications we need a more difficult concept. We allow $\mathrm{dom}(f)$ to consist of pairs of sets, and we define a $\wedge$ (meet) and a $\vee$ (join) operation for these pairs, and replace intersection and union in the definition by meet and join.

Let G be a graph and $\mathcal{H}$ be a family of its subgraphs. We call a subgraph F of G an *$\mathcal{H}$-packing* if each connected component of F belongs to $\mathcal{H}$. The size of the packing is the number of covered vertices, that is vertices with $d_F(v) > 0$. The packing is maximum if it reaches the maximal possible size. The packing is a factor if its size is $|V(G)|$.

2 Basic generalization of matchings

As in this paper we are going to deal with a dual approach we first define critical subgraphs and later, after proving the central theorem, we show how these things are connected to packings.

Let $G = (V, E)$ be a fixed graph and m be a positive, integer valued function on V. We also consider m as a modular set-function: $m(X) = \sum_{x \in X} m(x)$. We think of $m(v)$ as the capacity of node v (essentially the maximum allowed degree in a packing).

Let $\mathcal{C}$ be a family of subgraphs of G. We call $\mathcal{C}$ a *critical family of subgraphs* (shortly CFS), if there is an integer $l \geq 1$ such that for any member $C \in \mathcal{C}$ the following property holds. For any $\emptyset \neq X \subseteq V(C)$

$$q(C - X) \leq m(X) - l, \tag{1}$$

where $q(C - X)$ denotes the number those components of $C - X$ that are elements of

$\mathcal{C}$. If there is a number $l \geq 2$ for which (1) is always satisfied then we call $\mathcal{C}$ a strong CFS, otherwise a weak CFS. If $\mathcal{C}$ is a CFS then its elements are called *critical* graphs.

We define a deficiency of a set $X \subseteq V$ as follows.

$$\mathrm{def}_G^{\mathcal{C}}(X) = q(G - X) - m(X).$$

When our set-function is of some deficiency-type, we call tight sets (that is sets with maximum deficiency) *barriers*. A set $X \subseteq V$ is *essential* if there is no $Y \supsetneq X$ with $\mathrm{def}_G^{\mathcal{C}}(Y) > \mathrm{def}_G^{\mathcal{C}}(X)$.

Theorem 2.1 (Main Theorem). *If $\mathcal{C}$ is a CFS then $\mathrm{def}_G^{\mathcal{C}}(\cdot)$ is λ-supermodular on essential sets. Moreover if $\mathcal{C}$ is a strong CFS then $\mathrm{def}_G^{\mathcal{C}}(\cdot)$ is strongly λ-supermodular on essential sets.*

Proof. Let A and B be two essential sets. First let $D_1, \ldots, D_\gamma$ denote those critical components of $G - A$ that are also components of $G - B$ (i.e., are not connected to any vertex in $A - B$). Next let $A_1, \ldots, A_\alpha$ denote those critical components of $G - A$ that lie entirely in $V - A - B$ (i.e., do not intersect $B - A$) and are *not* components of $G - B$ (i.e., are connected to some vertices in $A - B$), and similarly let $B_1, \ldots, B_\beta$ denote those critical components of $G - B$ that lie entirely in $V - A - B$ and are *not* components of $G - A$.

Next consider a component $G[Z]$ of $G - A$ intersecting $B - A$. If $G[Z]$ is not critical then we claim that $\beta_Z \leq m(Z \cap B)$, where β_Z denotes the number B_i's that are connected to $Z \cap B$ (i.e., are contained in Z). To see this we use that A is essential, consequently $\mathrm{def}_G^{\mathcal{C}}(A \cup (Z \cap B)) \leq \mathrm{def}_G^{\mathcal{C}}(A)$, and the fact that $q(G - (A \cup (Z \cap B))) = q(G - A) + \beta_Z$. In the case when $G[Z]$ is critical we have (using that $\mathcal{C}$ is a CFS) $\beta_Z \leq m(Z \cap B) - l$. If r_A denotes the number of critical components of $G - A$ intersecting $B - A$ then, by summing up the inequalities above for all possible Z, we get: $\beta \leq m(B - A) - r_A \cdot l$, such that $r_A \leq \frac{m(B-A)}{l} - \frac{\beta}{l}$. Thus we have

$$\mathrm{def}_G^{\mathcal{C}}(A) = \alpha + \gamma + r_A - m(A) \leq \alpha + \gamma + \frac{m(B - A)}{l} - \frac{\beta}{l} - m(A).$$

Similarly we also get

$$\mathrm{def}_G^{\mathcal{C}}(B) = \beta + \gamma + r_B - m(B) \leq \beta + \gamma + \frac{m(A - B)}{l} - \frac{\alpha}{l} - m(B).$$

The inequalities $\mathrm{def}_G^{\mathcal{C}}(A \cap B) \geq \gamma - m(A \cap B)$ and $\mathrm{def}_G^{\mathcal{C}}(A \cup B) \geq \gamma + \alpha + \beta - m(A \cup B)$ are easy to show. With $\lambda = \frac{l+1}{l}$ we get

$$\mathrm{def}_G^{\mathcal{C}}(A) + \mathrm{def}_G^{\mathcal{C}}(B) - \lambda \cdot \mathrm{def}_G^{\mathcal{C}}(A \cap B) - (2 - \lambda) \cdot \mathrm{def}_G^{\mathcal{C}}(A \cup B) \leq 0.$$

and this finishes the proof. □

Corollary 2.2. *If $\mathcal{C}$ is a weak CFS then the intersection of barriers is a barrier. If $\mathcal{C}$ is a strong CFS then also the union of barriers is a barrier.*

The proof also gives the following.

Corollary 2.3. *If $\mathcal{C}$ is a CFS and A and B are barriers then the critical components of $G - (A \cap B)$ are exactly the common critical components of $G - A$ and $G - B$. If, moreover, $\mathcal{C}$ is a strong CFS then the critical components of $G - (A \cup B)$ are exactly the critical components of either $G - A$ or $G - B$ lying entirely in $V - A - B$.*

2.1 Matchings

If $m(v) = 1$ for all $v \in V$ and $\mathcal{C}$ consists of the factor-critical (hypomatchable) subgraphs of G then we arrive at packing of K_2 subgraphs, that is matchings.

The results of this subsection are from [7], as well as the proof method of the Main Theorem in the previous subsection.

There are several possible ways to define factor-critical graphs, here we give a recursive definition.

Definition 2.4. A graph having one vertex is factor-critical. A connected graph F is factor-critical iff $|V(F)|$ is odd and for any $\emptyset \neq X \subseteq V(F)$ we have $q(F-X) < |X|$, where $q(F-X)$ denotes the number of factor-critical components of the graph $F-X$.

Such a way the definition ensures that $\mathcal{C}$ is a weak CFS. The following theorem shows the connection to matchings. Let $\mathrm{def}_G^{\mathcal{C}} = \max_{X\subseteq V} \mathrm{def}_G^{\mathcal{C}}(X)$.

Theorem 2.5 (Tutte-Berge formula). *The size of a maximum matching is* $|V(G)| - \mathrm{def}_G^{\mathcal{C}}$.

Proof. First we prove the easy direction, that is we cannot cover more vertices than $|V(G)| - \mathrm{def}_G^{\mathcal{C}}$. Take a barrier X and observe that each critical subgraph has an odd number of vertices. Consequently each of the $q(X)$ critical components of $G - X$ either has an uncovered vertex or a matching edge going to X, but we may have at most $|X|$ components of the second type.

We prove the important part by induction on the number of vertices. If $|V| = 1$ then the statement is trivial.

Let $u \in V$ be any vertex. If u is not contained in any barrier then evidently $\mathrm{def}_{G-u}^{\mathcal{C}} < \mathrm{def}_G^{\mathcal{C}}$, so the matching given by the inductive hypothesis is satisfactory. Otherwise let A denote the intersection of all the barriers containing u. By the Main Theorem A is a barrier. Clearly u has a neighbor v that lies in a critical component D of $G - A$. We claim that $\mathrm{def}_{G-u-v}^{\mathcal{C}} \leq \mathrm{def}_G^{\mathcal{C}}$, using the inductive hypothesis this would conclude the statement. Otherwise we have an essential set B containing both u and v having $\mathrm{def}_G^{\mathcal{C}}(B) \geq \mathrm{def}_G^{\mathcal{C}} - 1$. By the weak λ-supermodularity we have $\mathrm{def}_G^{\mathcal{C}}(A \cap B) \geq (1/2)(\mathrm{def}_G^{\mathcal{C}}(A) + \mathrm{def}_G^{\mathcal{C}}(B)) = \mathrm{def}_G^{\mathcal{C}} - 1/2$, that is $A \cap B$ is a barrier, so $B \supseteq A$. As D is critical, $\mathrm{def}_D^{\mathcal{C}}(B \cap V[D]) \leq -1$ and for any other component Z of $G - A$ we have $\mathrm{def}_Z^{\mathcal{C}}(B \cap V[Z]) \leq 0$ which gives $\mathrm{def}_G^{\mathcal{C}}(B) \leq \mathrm{def}_G^{\mathcal{C}}(A) - 2$, a contradiction. □

Remark 1. *Observe that using parity considerations, such that* $\mathrm{def}_G^{\mathcal{C}}(X)$ *has the same parity as* $|V|$*, would shorten the proof. We avoided this because this argument cannot be used in the more general cases that follows.*

Define $A(G)$ as the intersection of all the barriers, $D(G)$ as the union of critical components of $G - A(G)$ and $C(G) = V - A(G) - D(G)$. In this setup the following is obvious (for the last statement use that $A - A'$ cannot be a barrier).

Theorem 2.6 (Gallai-Edmonds). *The set* $A(G)$ *is a barrier and components in* $D(G)$ *are factor-critical.* $G[C(G)]$ *has a perfect matching. Every barrier is disjoint* *from $D(G)$, consequently $D(G)$ is the set vertices uncovered by some maximum matching, If $\emptyset \neq A' \subseteq A(G)$ then the number of components in $D(G)$ that are connected to some vertex in A' is strictly greater than $|A'|$.*

2.2 Stars

Let $\mathcal{C} = \{\{v\} |\ v \in V\}$. Clearly $\mathcal{C}$ is a CFS for any m (m is positive). Moreover $\mathcal{C}$ is a strong CFS if $m(v) \geq 2$ for all $v \in V$. This critical system corresponds to the m-star factors. A subgraph F of G is an m-star factor if every component of F is a star and for each vertex $d_F(v) \leq m(v)$. The main theorem immediately gives

Corollary 2.7. *The intersection of barriers is a barrier. If for each vertex $m(v) \geq 2$ then the union of barriers is a barrier.*

Using this corollary it is easy to get the corresponding Berge-Tutte type formula and Gallai-Edmonds type theorem as in the previous section. If $m(v) \geq 2$ for all $v \in V$ then we also have

Theorem 2.8. *Suppose $m(v) \geq 2$ for all $v \in V$ and let $\hat{A}$ denotes the union of all the barriers. Then $\hat{A}$ is a barrier, and it contains a vertex v iff for every maximum m-star packing F we have $d_F(v) = m(v)$.*

The case $m \equiv 1$ gives back the matchings, while the case $m(v) \equiv k > 1$ was examined in [9].

We call an $\mathcal{H}$-packing F induced, if every component of F is an induced subgraph of G. A graph is an odd-clique-tree (shortly oct) if each of its two-connected component is a clique on an odd number of vertices. If $\mathcal{C}$ consists of the oct subgraphs of G we arrive at the induced m-star packing problem. We also have the corollary above. The case $m(v) \equiv k > 1$ was examined in [6], [2] and [8].

2.3 Long path packing

A path is called long if it contains at least two edges. This problem was considered in [4], see also in [5]. Here $\mathcal{C}$ consists of the sun subgraphs, defined as follows. K_1 is a sun, moreover if we take any factor-critical graph and put one new pendant edge on each vertex we arrive at a sun. We have $m(v) \equiv 2$. Is it easy to conclude that $\mathcal{C}$ is a weak CFS. It is almost strong, we need to define $l = 1$ only for K_2 subgraphs. Consequently

Theorem 2.9. *The intersection of barriers is a barrier. If A and B are barriers so that neither $G-A$ has a K_2 component intersecting B nor $G-B$ has a K_2 component intersecting A then $A \cup B$ is also a barrier.*

¿From this, Gallai-Edmonds type structure theorem can be deduced. We omit it because the theorem above was also discovered in [3] where they proved all the consequences even the more general problem of k-piece packing.

2.4 A more complicated form

In this section we give a seemingly unnecessarily complicated form of the Main Theorem. This type of formalism will be necessary for the next sections.

We call a pair $(X, \mathcal{D})$ valid, if $X \subseteq V$ and each element $D \in \mathcal{D}$ is a critical component of $G - X$. Define $\mathrm{def}_G^{\mathcal{C}}(X, \mathcal{D}) = |\mathcal{D}| - m(X)$. We call a valid pair $(X, \mathcal{D})$ essential whenever for any valid pair $(Y, \mathcal{E})$ if $Y \supseteq X$ and $\mathcal{E} \supseteq \mathcal{D}$ then $\mathrm{def}_G^{\mathcal{C}}(Y, \mathcal{E}) \leq \mathrm{def}_G^{\mathcal{C}}(X, \mathcal{D})$.

Remark 2. *To be correct, we should redefine what a CFS is. Essentially the same definition will do the job. The requirement is that for any $\emptyset \neq X \subseteq V(C)$ and any valid pair $(X, \mathcal{D})$ in C inequality $|\mathcal{D}| \leq m(X) - l$ must be satisfied.*

We define the meet and join operation on valid pairs. $(X, \mathcal{D}) \wedge (Y, \mathcal{E}) = (X \cap Y, \mathcal{D} \cap \mathcal{E})$ and $(X, \mathcal{D}) \vee (Y, \mathcal{E}) = (X \cup Y, (\mathcal{D} - \mathcal{D}_Y) \cup (\mathcal{E} - \mathcal{E}_X))$ where $\mathcal{D}_Y \subseteq \mathcal{D}$ denotes the set of critical components in $\mathcal{D}$ intersecting Y; and $\mathcal{E}_X$ is defined accordingly.

The same proof gives the following variation of the Main Theorem (and this includes Corollary 2.3 as well).

Theorem 2.10. *If $\mathcal{C}$ is a CFS then $\mathrm{def}_G^{\mathcal{C}}$ is λ-supermodular on essential valid pairs wrt. meet and join. Moreover if $\mathcal{C}$ is a strong CFS then $\mathrm{def}_G^{\mathcal{C}}$ is strongly λ-supermodular on essential valid pairs wrt. meet and join.*

3 Propellers

Propeller families were first defined in [11]. The non-induced case was examined in [11, 10, 12] while the induced packing was examined in [8].

We use the definitions of the previous subsection with the main difference that m is a much more general function. Here we have $m : V \times 2^{\mathcal{C}} \to \mathbb{Z}^+$ is a positive function of two variables, satisfying the following property: for any $v \in V$, $\mathcal{D}, \mathcal{E} \subseteq \mathcal{C}$ if $\mathcal{E} \supseteq \mathcal{D}$ then $m(v, \mathcal{E}) \geq m(v, \mathcal{D})$ with equality if no subgraph in $\mathcal{E} - \mathcal{D}$ is connected to v. For a set $X \subseteq V$ we also define $m(X, \mathcal{D}) = \sum_{x \in X} m(x, \mathcal{D})$. Here $\mathcal{C}$ is a CFS if for $C \in \mathcal{C}$, $\emptyset \neq X \subseteq V(C)$ and for a valid pair $(X, \mathcal{D})$ if $\bigcup \mathcal{D} \subseteq V(C)$ then we have $|\mathcal{D}| \leq m(X, \mathcal{D}) - l$. For a valid pair we define $\mathrm{def}_G^{\mathcal{C}}(X, \mathcal{D}) = |\mathcal{D}| - m(X, \mathcal{D})$.

We call a non-singleton connected graph a *propeller* if it has a vertex c, the *center*, such that after deleting c, each connected component is factor-critical. These components are called *blades*. Let k be any positive integer and $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be arbitrary nonempty families of factor-critical graphs such that $K_1 \in \mathcal{F}_1$. Construct a propeller in the following way. For an integer $1 \leq t \leq k$ choose t distinct families $\mathcal{F}_i$ and choose $B_i \in \mathcal{F}_i$. Add a new vertex c and connect it to an arbitrary nonempty subset of vertices of each B_i. The set $\mathcal{P}$ of propellers arising in this way is called a *propeller family of order k*. A family $\mathcal{H}$ of graphs is called *admissible* (of order k) if $\mathcal{H} = \mathcal{P} \cup \mathcal{F}$, where $\mathcal{P}$ is a propeller family of order k and $\mathcal{F}$ is an arbitrary set of factor-critical graphs. In this section we are dealing with the more general induced $\mathcal{H}$-packings problem. Now $\mathcal{C}$, the set of critical subgraphs contains the $\mathcal{H}$-critical subgraphs of G, such that the subgraphs C having no induced $\mathcal{H}$-packing for which $C - v$ has an induced $\mathcal{H}$-packing for any $v \in V(C)$. By a lemma of [8] we have the fact that each $\mathcal{H}$-critical subgraph is factor-critical, and this fact ensures that $\mathcal{C}$ is a weak CFS. Here $m(v, \mathcal{D})$ is defined to be $|\{i :\ v$ is connected to a nice subgraph F of $D \in \mathcal{D}$ where $F \in \mathcal{F}_i\}|$.

The analogue of the Main Theorem can be proved by almost the same reasoning, we leave the details to the full version.

Theorem 3.1. $\mathrm{def}_G^{\mathcal{C}}$ *is weakly λ-supermodular on essential valid pairs wrt. meet and join.*

And we also have a Gallai-Edmonds type theorem

Theorem 3.2. *[8] If $(A(G), \mathcal{D}(G))$ denotes the meet of all the barriers then this valid pair is a barrier, each element of $\mathcal{D}$ is an $\mathcal{H}$-critical component of $G - A(G)$. Moreover $\bigcup \mathcal{D}$ consists of the vertices missed by some maximum packing.*

4 f-factors

Let f be a positive integer-valued function on V. A subgraph F of G is an f-factor, if for every vertex $d_F(v) = f(v)$. This problem was solved in [16]. Though the setup of this section can be extended for the more general cases as well (for example for the (f, g)-factor problem that was solved in [13]), in this extended abstract we remain at the case of f-factors.

We define $\hat{f}(v) = d(v) - f(v)$ and for a subset Y of vertices $f(Y) = \sum_{y \in Y} f(y)$ and $\hat{f}(Y) = \sum_{y \in Y} \hat{f}(y)$. A pair (X, Y) where $X, Y \subseteq V$ is valid if $X \cap Y = \emptyset$. For a valid pair we define $\mathrm{def}_G^f(X, Y) = q_Y(G - X - Y) + d(X, Y) - f(X) - \hat{f}(Y)$, where $q_Y(G - X - Y)$ denotes the Y-critical components of $G - X - Y$, to be defined later. For a subset Y of the vertices and a vertex $v \in V - Y$ we define $f_Y(v) = f(v) - d(v, Y)$. A component K of $G - X - Y$ is Y-critical if $f_Y(V(K))$ is an odd number and for any valid pair (X', Y') in K that is not the $(\emptyset, \emptyset)$ we have $\mathrm{def}_K^{f_Y}(X', Y') \leq -1$. This is again a recursive definition. On the light of the following theorem of Tutte it will be obvious that K is Y-critical iff it has no f_Y-factor, but changing f_Y at any vertex by one (in either direction) it will have the corresponding f'_Y-factor. A valid pair (X, Y) is essential if for any valid pair (X', Y') if $X' \supseteq X$ and $Y' \supseteq Y$ then $\mathrm{def}_G^f(X', Y') \leq \mathrm{def}_G^f(X, Y)$. As here we are going to prove only weak λ-supermodularity, we define just the meet operation: $(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2)$. Finally a valid pair (X, Y) is a barrier if $\mathrm{def}_G^f(X, Y)$ reaches the maximum.

Theorem 4.1. def_G^f *is weakly λ-supermodular on essential valid pairs wrt. meet.*

We have not got a nice direct proof of this theorem yet. Using the reduction of Tutte [17] to the original matching problem, we observe that barriers of the reduced graph correspond to the barriers of the f-factor problem as follows. Let A be a barrier of the reduced graph G', and $\mathcal{D}$ is the set of factor-critical components of $G' - A$. The vertices of G having all copies in A make up X. The one-vertex elements of $\mathcal{D}$ that came from a vertex of G make up Y, while the bigger elements correspond to the Y-critical components of $G - X - Y$. Using Corollary 2.3 the proof is straightforward, we leave the details to the full version.

Corollary 4.2. *The meet of barriers is a barrier.*

It is possible to give a new, inductive proof for the following theorem of Tutte using this corollary, we also leave it to the full version.

Theorem 4.3. *[16] G has an f-factor iff no valid pair (X,Y) has positive deficiency.*

We may also define $(A(G), B(G))$ as the meet of all the barriers and prove the Gallai-Edmonds type structure theorem of Lovász [14].

5 Even factors

In this section we have a directed graph G with vertex set $V = V(G)$, which is weakly symmetric, that is for every arc uv either u is not reachable from v or vu is also an arc. Note that strongly connected subgraphs are symmetric (each arc has its opposite), we sometimes consider symmetric subgraphs as undirected graphs as well. If N is a subset of vertices then we call a pair $(N,\mathcal{D})$ valid, if each element of $\mathcal{D}$ is a strongly connected component of $G-N$, and it is factor-critical, and no path connects distinct elements of $\mathcal{D}$ in graph $G-N$. For a valid pair we define $\text{def}_G(N,\mathcal{D}) = |\mathcal{D}| - |N|$, $D = \bigcup \mathcal{D}$, $O_N^{\mathcal{D}}$ as the set of vertices in $G-N-D$ reachable from some element of $\mathcal{D}$ in graph $G-N$ and $R_N^{\mathcal{D}} = V - D - O_N^{\mathcal{D}}$. Observe that

Lemma 5.1. *No arc goes*

- *between distinct elements of $\mathcal{D}$*
- *from D to $R_N^{\mathcal{D}}$*
- *from $O_N^{\mathcal{D}}$ to $R_N^{\mathcal{D}}$, and*
- *from $O_N^{\mathcal{D}}$ to D.*

The meet operation is defined as $(N,\mathcal{D}) \wedge (M,\mathcal{E}) = ((N \cap M) \cup (N \cap O_M^{\mathcal{E}}) \cup (M \cap O_N^{\mathcal{D}}), \mathcal{D}^M \cup \mathcal{E}^N)$, where $\mathcal{D}^M \subseteq \mathcal{D}$ denotes the set of critical components in $\mathcal{D}$ lying entirely in $E \cup O_M^{\mathcal{E}}$ and $\mathcal{E}^N$ is defined accordingly. The join operation is $(N,\mathcal{D}) \vee (M,\mathcal{E}) = ((N \cap M) \cup (N \cap R_M^{\mathcal{E}}) \cup (M \cap R_N^{\mathcal{D}}), (\mathcal{D} - \mathcal{D}_M) \cup (\mathcal{E} - \mathcal{E}_N))$ where $\mathcal{D}_M \subseteq \mathcal{D}$ denotes the set of critical components in $\mathcal{D}$ intersecting $M \cup O_M^{\mathcal{E}}$ and $\mathcal{E}_N$ is defined accordingly. Observe that we are in harmony with the notation of Subsection 2.4 because in symmetric graphs $O_N^{\mathcal{D}}$ is always empty; except, of course, the operation join. But here this join operation will be much more convenient and ensures strong λ-supermodularity.

Theorem 5.2. def_G *is strongly λ-supermodular (actually is supermodular) on valid pairs wrt. meet and join.*

Proof. As here we are dealing with directed graphs now, we must repeat the proof though it is very similar to the proof of the Main Theorem. But with the tricky join operation we do not need essentiality.

Let $(N,\mathcal{D})$ and $(M,\mathcal{E})$ be valid essential pairs. Let $D^1_1,\dots,D^1_{n_1}$ denote the elements of $\mathcal{D}\cap\mathcal{E}$, $D^2_1,\dots,D^2_{n_2}$ denote the elements of $\mathcal{D}$ lying entirely in E but not being an element of $\mathcal{E}$, $D^3_1,\dots,D^3_{n_3}$ denote the elements of $\mathcal{D}$ intersecting M, $D^4_1,\dots,D^4_{n_4}$ denote the elements of $\mathcal{D}$ lying entirely in $R^{\mathcal{E}}_M$, and $D^5_1,\dots,D^5_{n_5}$ denote the elements of $\mathcal{D}$ lying entirely in $O^{\mathcal{E}}_M$. It is easy to see – using the symmetry of each element of $\mathcal{D}$ and Lemma 5.1 – that this is a partition of $\mathcal{D}$. Similarly we define the corresponding partition of $\mathcal{E}$.

$$\mathrm{def}_G(N,\mathcal{D}) = n_1+n_2+n_3+n_4+n_5-|N| \tag{2}$$

$$\mathrm{def}_G(M,\mathcal{E}) = m_1+m_2+m_3+m_4+m_5-|M| \tag{3}$$

$$\begin{aligned}\mathrm{def}_G((N,\mathcal{D})\wedge(M,\mathcal{E})) = \mathrm{def}_G((N\cap M)\cup(N\cap O^{\mathcal{E}}_M)\cup(M\cap O^{\mathcal{D}}_N),\mathcal{D}^M\cup\mathcal{E}^N)\geq\\ \geq n_1+n_5+m_5-|(N\cap M)\cup(N\cap O^{\mathcal{E}}_M)\cup(M\cap O^{\mathcal{D}}_N|\end{aligned} \tag{4}$$

$$\begin{aligned}&\mathrm{def}_G((N,\mathcal{D})\vee(M,\mathcal{E})) =\\ =\;&\mathrm{def}_G((N\cap M)\cup(N\cap R^{\mathcal{E}}_M)\cup(M\cap R^{\mathcal{D}}_N),(\mathcal{D}-\mathcal{D}_M)\cup(\mathcal{E}-\mathcal{E}_N))\geq\\ \geq\;&m_1+n_4+m_4-|(N\cap M)\cup(N\cap R^{\mathcal{E}}_M)\cup(M\cap R^{\mathcal{D}}_N)|\end{aligned} \tag{5}$$

To see these later two inequalities it is enough to observe that $((N,\mathcal{D})\wedge(M,\mathcal{E}))$ and $((N,\mathcal{D})\vee(M,\mathcal{E}))$ are valid pairs. Using Lemma 5.1 this is easy.

By summing up on the components $Z=D^3_1,\dots,D^3_{n_3}$ the fact, that – due to factor-criticality – the number of E^2_i's inside Z is strictly less than $|Z\cap M|$ we get

$$m_2\leq|M\cap D|-n_3 \tag{6}$$

and similarly

$$n_2\leq|N\cap E|-m_3 \tag{7}$$

By summing up equations (4) and (5) and subtracting (2), (3), (6) and (7) we get

$$\mathrm{def}_G((N,\mathcal{D})\wedge(M,\mathcal{E}))+\mathrm{def}_G((N,\mathcal{D})\vee(M,\mathcal{E}))-\mathrm{def}_G(N,\mathcal{D})-\mathrm{def}_G(M,\mathcal{E})\geq 0 \tag{8}$$

and this finishes the proof. □

We call a valid pair a barrier if its deficiency reaches the maximum (over valid pairs).

Corollary 5.3. *The meet of barriers is a barrier and also the join of barriers is a barrier.*

Even factors was defined in [1]. We call a subgraph P of G an even factor, if P consists of pairwise vertex-disjoint directed paths (of any length) and directed circles of even length; and $V(P) = V$ (we allow path of length zero). The size of an even factor P is the number of arcs it contains. Let $\nu(G)$ denote the maximum size of an even factor and def_G denote the maximal value of the deficiency over valid pairs. Tutte-Berge type min-max formula and Gallai-Edmonds type structure theorem was given in [15].

Theorem 5.4. *[15]* $\nu(G) = |V| - \mathrm{def}_G$.

For the proof we need some preparation. Let $\Delta^+(X)$ denote the set of outgoing edges from vertex set X and $\delta(X) = |\Delta^+(X)|$ and $I(X)$ denote the set of edges induced by X. Let $(A, \mathcal{D})$ be the meet of all the barriers, by Corollary 5.3 this is a barrier.

Lemma 5.5. *For a vertex* v $\mathrm{def}_{G-\Delta^+(v)} \le \mathrm{def}_G$ *if and only iff* $v \in D \cup O_A^{\mathcal{D}}$*.*

Proof. Let $G' = G - \Delta^+(v)$. For the “only if” part suppose $v \notin D \cup O_A^{\mathcal{D}}$. If $v \in A$ then $(A-v, \mathcal{D})$ is a valid pair in G' with a bigger deficiency. If $v \in R_A^{\mathcal{D}}$ then $(A, \mathcal{D}+\{v\})$ is a valid pair in G' with a bigger deficiency. For the “if” part suppose first that $v \in D \cup O_A^{\mathcal{D}}$ but $\mathrm{def}_{G'} > \mathrm{def}_G$. Let $(M', \mathcal{E}')$ be a barrier in G'. Then, using that v is a sink in G', by the previous argument either $\{v\} \in \mathcal{E}'$ or $v \in O_{M'}^{\mathcal{E}'}$. In the first case define $(M, \mathcal{E}) = (M', \mathcal{E}' - \{v\})$ and in the second case define $(M, \mathcal{E}) = (M'+v, \mathcal{E}')$. In both cases $(M, \mathcal{E})$ is a valid pair in G having deficiency $\mathrm{def}_G(M, \mathcal{E}) = \mathrm{def}_{G'}(M', \mathcal{E}') - 1$, that is a barrier by the assumption above. By the definition of $(A, \mathcal{D})$ we have $(A, \mathcal{D}) \wedge (M, \mathcal{E}) = (A, \mathcal{D})$. For the both cases above this contradicts to the fact that $v \in D \cup O_A^{\mathcal{D}}$. □

Proof of Theorem 5.4. Denote $D \cup O_A^{\mathcal{D}}$ by X. For every even factor P we have (counting the heads of edges in P in the first two cases and the tails in the third case)

$$|P \cap I(X)| \le |X| - |\mathcal{D}|$$

$$|P \cap \Delta^+(X)| \le |A|$$

$$|P - (I(X) \cup \Delta^+(X))| \le |A| + |R_A^{\mathcal{D}}|$$

Summing up we get $|P| \le |V| + |A| - |\mathcal{D}| = |V| - \mathrm{def}_G$.

The hard part is proved by induction on the number of edges.

CASE 1. $\mathcal{D} = \emptyset$.

Now $\mathrm{def}_G = 0$. We claim that every strongly connected component of G has a perfect matching. Otherwise there is a strongly connected component Y such that Y has a barrier $M \subseteq V(Y)$ such that $Y - M$ has more than $|M|$ factor-critical components, $\mathcal{E} = \{D_1, D_2, \ldots, D_k\}$. Now $(M, \mathcal{E})$ is a valid pair having positive deficiency, a contradiction.

CASE 2. $\exists v \in X$ that is not a sink.

By Lemma 5.5 $\mathrm{def}_{G-\Delta^+(v)} \le \mathrm{def}_G$, so, by induction, $G - \Delta^+(v)$ has an even factor of the desired size.

CASE 3. X contains only sinks and is not empty.

In this case we have $O_A^{\mathcal{D}} = \emptyset$ and $\mathcal{D}$ is the set of all the sinks of G and also $A = \emptyset$. Suppose $v \in D$ is a sink. If $\mathrm{def}_{G-v} < \mathrm{def}_G$ then, by induction, $G - v$ has an even factor of the desired size. So now on we suppose $\mathrm{def}_{G-v} \geq \mathrm{def}_G$.

Let $\hat{G}$ denote the graph $G - v$ and $(\hat{A}, \hat{\mathcal{D}})$ denote the meet of all the barriers in $\hat{G}$, finally $\hat{X} = \hat{D} \cup \hat{O}_{\hat{A}}^{\hat{\mathcal{D}}}$. We claim that there exists an arc uv in G such that $u \in \hat{X}$. To see this, observe that otherwise $\mathrm{def}_G(\hat{A}, \hat{\mathcal{D}} + \{v\}) > \mathrm{def}_{\hat{G}}(\hat{A}, \hat{\mathcal{D}})$ contradicting to $\mathrm{def}_{G-v} \geq \mathrm{def}_G$.

Using Lemma 5.5 again for $\hat{G}$ we see that $(G - v) - \Delta^+(u)$ has also deficiency def_G. Thus, by induction, it has an even factor P' of size $|V| - 1 - \mathrm{def}_G$. Adding edge uv to P' we get an even factor P of G, and $|P| = |V| - \mathrm{def}_G$. □

As a consequence we get (using Lemma 5.5 once more)

Theorem 5.6. *[15] For a graph G there is a canonical barrier $(A, \mathcal{D})$ with property $D \cup O_A^{\mathcal{D}} = \{v \in V :$ there exists a maximum even factor P such that $\delta_P(v) = 0\}$.*

The following theorem is an easy consequence of the theorems is this section, we leave the detailed proof for the full version.

Theorem 5.7. *Let $(\bar{A}, \bar{\mathcal{D}})$ denote the join of all the barriers and $X = (D \cup O_A^{\mathcal{D}}) \cap (\bar{D} \cup \bar{R}_{\bar{A}}^{\bar{\mathcal{D}}})$. Then $X = \{v \in V : \ \mathrm{def}_{G-v} < \mathrm{def}_G\}$. Moreover X is the union of vertex sets of some elements of $\mathcal{D}$.*

Note that the statement says X consists of exactly those vertices that can be isolated in a maximum even factor.

6 Future work

We are sure that with some work this concept can be used for all other packing-type problems (most importantly, for Mader's A-path theorems).

But the really tempting research issue is to find other areas where λ-supermodularity can be used, that is some theory independent of packings.

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