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A note on kernels in h-perfect graphs

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Abstract

Boros and Gurvich [3] showed that every clique-acyclic superorientation of a perfect graph has a kernel. We prove the following extension of their result: if G is an h-perfect graph, then every clique-acyclic and odd-hole-acyclic superorientation of G has a kernel. We propose a conjecture related to Scarf's Lemma that would imply the reverse direction of the Boros-Gurvich theorem without relying on the Strong Perfect Graph Theorem.

1 Introduction

Let $D = (V, A)$ be a directed graph. The *out-neighbourhood* $O_D(v)$ of a node $v \in V$ is the set of nodes consisting of v and the nodes $w \in V$ for which $vw \in A$. A subset X of nodes is said to *dominate* a node $v \in V$ if $X \cap O_D(v) \neq \emptyset$. X is called *dominating* if it dominates every node. A *kernel* of D is a dominating independent set of nodes. Kernels have several applications in combinatorics and game theory, and there has been extensive work on the characterization of digraphs that have kernels. See [4] for a survey on the topic.

One approach is to identify undirected graphs for which every "nice" orientation has a kernel. Let $G = (V, E)$ be an undirected graph. A *superorientation* of G is a directed graph obtained by replacing each edge uv of G by an arc uv or an arc vu or both. A *proper directed cycle* in a superorientation is a directed cycle consisting of arcs that are not present reversed in the digraph. A superorienation is *clique-acyclic* if no oriented clique contains a proper directed cycle. Boros and Gurvich [3] proved the following.

Theorem 1.1 ([3])**.** *If G is a perfect graph then every clique-acyclic superorientation of G has a kernel.*

Sbihi and Uhri [8] introduced the class of *h-perfect graphs* as the graphs for which the stable set polyhedron is described by the following set of inequalities:

$$x_v \geq 0 \qquad \text{for every } v \in V, \tag{1}$$

$$x(C) \leq 1 \qquad \text{for every maximal clique } C, \tag{2}$$

$$x(Z) \leq \frac{|Z| - 1}{2} \qquad \text{for every odd hole } Z. \tag{3}$$

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In addition to perfect graphs, it is known that the class of h-perfect graphs includes

- all graphs containing no odd-K_4-subdivision (see [6]),
- all near-bipartite graphs containing no odd wheel and no prime antiweb except for cliques and odd holes (this is implicitly in [10]),
- line graphs of graphs that contain no odd subdivision of $C_5 + e$ (see [5]).

It follows from the Strong Perfect Graph Theorem that the property in Theorem 1.1 does not hold for non-perfect graphs. To extend the theorem to h-perfect graphs, let us call a superorientation of a graph *odd-hole-acyclic* if no oriented odd hole is a proper directed cycle. Obviously a superorientation of a perfect graph is always odd-hole-acyclic. Our result is as follows.

Theorem 1.2. *If G is an h-perfect graph then every clique-acyclic and odd-hole-acyclic superorientation of G has a kernel.*

Our proof, described in the next section, is a slight modification of the proof of Aharoni and Holzman for Theorem 1.1 [1]. It applies the following result of Scarf [9].

Theorem 1.3 (Scarf's Lemma)**.** *Let A be a non-negative $m \times n$ matrix and $b \in \mathbb{R}^m_+$ with the property that the polyhedron $P = \{x \in \mathbb{R}^n_+ :\ Ax \leq b\}$ is non-empty and bounded. Let $<_i$ be a total order on $\{1, 2, \dots, n\}$ for $i = 1, \dots, m$. Then the polytope P has a vertex x^* with the property that for each $1 \leq j \leq n$ there is an index $i \in \{1, \dots, m\}$ such that $a_i x^* = b_i$ and $x^*_k = 0$ for every $k <_i j$ (where a_i is the i-th row of A).*

Section 3 contains two generalizations of Theorem 1.2 and a conjecture concerning the "reverse direction" of Scarf's Lemma. In Section 4 we show counterexamples for some related questions.

2 Proof of Theorem 1.2

Let G be an h-perfect graph and D a clique-acyclic and odd-hole-acyclic superorientation of G. Let c and o denote the number of maximal cliques and odd holes in D, respectively. Let $C_1, \dots C_c$ denote the maximal cliques in D and $C_{c+1}, \dots C_{c+o}$ the odd holes in D. Let A be the matrix of size $(c + o) \times n$ whose i-th row, a_i is the characteristic vector of C_i ($1 \leq i \leq c + o$). Finally let $b \in \mathbb{R}^{(c+o)}_+$ be the vector whose i-th component is 1 if $i \leq c$ and $\frac{|C_i|-1}{2}$ if $i > c$. Since G is h-perfect, the polyhedron $P = \{x \in \mathbb{R}^V :\ x \geq 0,\ Ax \leq b\}$ is the convex hull of the stable sets.

Because D is clique-acyclic and odd-hole-acyclic, if C_i is a maximal clique or an odd hole, its nodes have an order with the property that there is no edge in C_i which is oriented only backwards. Let $<_i$ be an ordering of V such that the first $|C_i|$ nodes are those of C_i in the above given order.

Applying Scarf's Lemma for this instance we get that there is a vertex x^* of P with the property that for each node $v \in V$ there is a maximal clique or odd hole $C_{i(v)}$

such that $a_{i(v)}x^* = b_{i(v)}$ and $x^*_{v'} = 0$ for every $v' <_{i(v)} v$. These imply that v is in $C_{i(v)}$, because otherwise for every node v' of $C_{i(v)}$, $v' <_{i(v)} v$ would hold, and so $a_{i(v)}x^*$ would be zero.

The vector x^* is the characteristic vector of a stable set because it is a vertex of P. We want to show that it is the characteristic vector of a kernel.

Let v be a node. Scarf's lemma implies that if $x^*_w = 1$ for a node w, then $w \geq_{i(v)} v$ holds. If $C_{i(v)}$ is a clique, then because of $a_{i(v)}x^* = b_{i(v)} = 1$, there is a node w in $C_{i(v)}$ with $x^*_w = 1$. Hence $w \geq_{i(v)} v$ holds, so $w \in O_D(v)$.

If $C_{i(v)}$ is an odd hole, then $a_{i(v)}x^* = b_{i(v)} = \frac{|C_{i(v)}|-1}{2}$ implies that every second value of x^* on the circuit $C_{i(v)}$ is 1, the others are 0 (so there are two consecutive 0-s). This means that v has at least one neighbour w on the circuit whose value is 1. Like above, $w \geq_{i(v)} v$, so w must be in $O_D(v)$. This concludes the proof of the theorem. □

3 Further results and conjectures

A stronger version of the theorem can also be proved with the same method. Let $\mathrm{STAB}(G)$ denote the polytope of the stable sets. We say that a digraph is *acyclic* in a subset of nodes if there is no proper directed cycle in the subset.

Theorem 3.1. *If $\{x \in \mathbb{R}^V_+ : Ax \leq b\} = \mathrm{STAB}(G)$ for an undirected graph $G = (V, E)$ and D is a superorientation of G which is acyclic in $\mathrm{supp}(a)$ for every row a of A, then there is a kernel in D.*

Proof. We can assume that every inequality is facet-defining in the system $\{x \in \mathbb{R}^V_+ : Ax \leq b\}$. Then A is nonnegative and b is positive.

In $<_a$ let the elements of $\mathrm{supp}(a)$ be the smallest ones, in a topological order of the one-way edges. The other nodes can be in arbitrary order. Scarf's lemma implies that there exists a vertex x^* of $\mathrm{STAB}(G)$ such that for every $v \in V$ there is a row a of A for which

(i) $ax^* = b_a$, and

(ii) if $w \in \mathrm{supp}(x^*)$ then $w \geq_a v$.

Since the system describes $\mathrm{STAB}(G)$, x^* is the characteristic vector of some stable set X. We want to show that X dominates every node v. For a given $v \in V$, let a be a row like above. Then v is in $\mathrm{supp}(a)$ because otherwise ax^* would be 0 contradicting (i). Moreover, (i) implies that there is a node $w \in X \cap \mathrm{supp}(a) \cap N_G(v)$ (where $N_G(v)$ denotes the neighbourhood of v in G with v) because else $(X \cap \mathrm{supp}(a)) \cup \{v\}$ would be a stable set which violates the inequality of a. From (ii) w is an outneighbour of v or v itself. □

We have mentioned that the reverse direction of Theorem 1.1 is also true. The same does not hold for Theorem 1.2, and a counterexample is given in the last section. Nevertheless, one may hope for a stronger theorem where the reverse direction also holds. We give here a less elegant but stronger theorem for which we conjecture that this is the case.

Let G be an h-perfect graph, and let D be a clique-acyclic superorientation of G. Some odd holes of G may become proper directed cycles; let us denote these by $Z_1, \ldots, Z_k$. Let us select nodes $v_1, \ldots, v_k$ such that $v_i \in Z_i$ for $i = 1, \ldots, k$ (the selected nodes need not be distinct). We call this a *superorientation with special nodes*. An *almost-kernel* for a superorientation with special nodes is an independent set S with the following property:

> If a node v is not dominated by S, then $v = v_i$ for some i and $|Z_i \cap S| = (|Z_i| - 1)/2$.

Theorem 3.2. *If G is an h-perfect graph then every clique-acyclic superorientation with special nodes has an almost-kernel.*

Proof. We use Scarf's Lemma in a similar way as in the proof of Theorem 1.2. The orderings $<_i$ associated to the lines of the matrix can be defined the same way as there, except for the odd holes which are proper directed cycles. For these, we can define the ordering so that the special node is the first node of the ordering, and the only edge oriented backwards is the one entering the special node.

Using Scarf's lemma as in the proof of Theorem 1.2, we get that the only possible case when a node v is not dominated by x^* is when $C_{i(v)}$ is an odd hole which is a proper directed cycle and v is its special node. This implies that x^* is the characteristic vector of an almost-kernel. $\square$

Note that this theorem is stronger than Theorem 1.2 since every almost-kernel in a clique-acyclic and odd-hole-acyclic orientation is a kernels. We conjecture that here the converse also holds:

Conjecture 3.3. *A graph G is h-perfect if and only if every clique-acyclic superorientation with special nodes has an almost-kernel.*

In fact, we can formulate a more general conjecture, which is a kind of converse of Scarf's Lemma. A vertex of a polyhedron is called *maximal* if every other point of the polyhedron has at least one coordinate which is smaller.

Conjecture 3.4. *Let A be a non-negative $m \times n$ matrix and let $b \in \mathbb{R}^m$ be a positive vector so that the polyhedron $P = \{x : \ Ax \leq b, \ x \geq 0\}$ is bounded. Let x^* be a maximal vertex of P. Then for each inequality in $Ax \leq b$ we can give a total order of the variables that have positive coefficients, so that the following hold:*

1. *For every variable x_j, there is an inequality which is tight at x^*, and in whose total ordering the variables in supp(x^*) are greater or equal to x_j.*
2. *For every vertex $x' \neq x^*$ of P, there exists $j \in \{1, \ldots, n\}$ so that for any tight inequality at x' there is a variable x_l with $x'_l > 0$ that precedes x_j in the total order of the inequality.*

In fact, the second condition implies the first by Scarf's Lemma. To see that Conjecture 3.3 follows from Conjecture 3.4, consider a non-h-perfect graph G. The polyhedron P defined by inequalities (1) − (3) has a non-integral vertex, hence it has a non-integral maximal vertex x^*. Let $<_i$ $(i = 1, \ldots, m)$ denote the total orders given by Conjecture 3.4. These total orders define a clique-acyclic superorientation with special vertices:

- For each maximal clique, we orient the edges of the clique according to the total ordering of the clique. (An edge may appear in two cliques and its endpoints may be in different order in the two total orders; in this case, we orient the edge in both directions.) This defines the superorientation.
- If an odd hole is a proper directed cycle in this superorientation, we define its special node to be the first node in its total order.

Let S be an arbitrary stable set of G. The characteristic vector of S is an integral vertex of the polyhedron P. By the properties of the partial orders, there is a node $v \in V$ with the following properties:

- If there is a maximal clique C_i with $|C_i \cap S| = 1$ and $v \in C_i$, then there is a node $u \in C_i \cap S$ with $u <_i v$.
- If there is an odd hole Z_i with $|Z_i \cap S| = (|Z_i| - 1)/2$ and $v \in Z_i$, then there is a node $u \in Z_i \cap S$ with $u <_i v$.

The first property means that $v \notin S$ and the outneighbours of v in the superorientation are not in S, so v is not dominated by S. The second property implies that if v is the special node of an odd hole Z (i.e. it is the first node in the total order) then $|Z \cap S| < (|Z| - 1)/2$. Therefore the existence of v proves that S is not an almost-kernel.

4 Counterexamples to related questions

It is a well-known result in the theory of stable matchings that a clique-acyclic and odd-hole-acyclic *orientation* of a line graph always has a kernel (it follows for example from the stable roommates algorithm of Irwing [7]). However, this is not true for *superorientations*, as the superorientation of the line graph of $C_5 + e$ on Figure 1 shows.

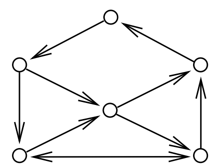

Figure 1: A kernel-less superorientation of the line graph of $C_5 + e$

Furthermore it is also false that a graph is h-perfect if and only if every clique-acyclic and odd-hole-acyclic superorientation has a kernel, for the graph on Figure

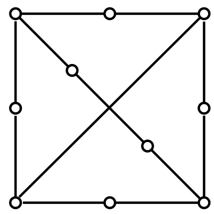

Figure 2: A non-h-perfect graph whose clique- and odd-hole-acyclic superorientations all have kernels

2 is not h-perfect (this follows from the results of Barahona and Mahjoub [2]), but every clique- and odd-hole-acyclic superorientation of it has a kernel.

Consider Conjecture 3.4 in the special case when P is 3-dimensional and A is a positive matrix. We can prove the claim of the conjecture the following way. The skeleton of P has 3 internally vertex-disjoint paths between x^* and 0. These paths divide the surface of the polyhedron into 3 parts. We can use the total ordering $(x_1 < x_2 < x_3)$ on the facets of the part containing the facet $\{x \in P : x_3 = 0\}$, $(x_2 < x_3 < x_1)$ on the facets of the part containing the facet $\{x \in P : x_1 = 0\}$, and $(x_3 < x_1 < x_2)$ on the facets of the part containing the facet $\{x \in P : x_2 = 0\}$. It is easy to check that these total orders satisfy the requirements of the conjecture.

This fact led us to ask the following question:

> *Let P be a d-dimensional polyhedron, and let x^1 and x^2 be two distinct vertices of P. Is it true that the facets of P can be coloured by d colours so that x^1 and x^2 are precisely the vertices that are incident to facets of all colours?*

This is true in 3 dimensions by the above argument; furthermore, if it was true in higher dimensions, it would imply Conjecture 3.4 in the case when A is a positive matrix. However, it turned out to be false in 4 dimensions, as the following polyhedron shows:

Facets:
$$\begin{aligned}
-x_1 - x_3 + x_4 &\le 1\\
x_1 + x_2 + x_4 &\le 1\\
x_2 - x_3 + x_4 &\le 1\\
-x_1 - x_2 - x_3 + x_4 &\le 1\\
x_1 - x_2 - x_3 - x_4 &\le 1\\
-x_1 - x_3 - x_4 &\le 1\\
-x_1 - x_4 &\le 1\\
-x_1 - x_2 + x_3 - x_4 &\le 1
\end{aligned}$$

Vertices:
(0,0,0,1), (0,0,0,-1),
(-2,2,2,1), (2,2,2,-3),
(1,2,0,-2), (2/3,4/3, -2/3,-1),
(0,2,0,-1), (-1,1,0,0),
(2,-3,2,2), (2/3,-1/3,-2/3,2/3),
(-1,0,2,2), (-1/3,-2/3,-1/3,-1/3),
(0,0,-1,0), (-1,0,0,0).

The first four facets of this polyhedron are incident to the vertex $x^1 = (0, 0, 0, 1)$, while the last four facets are incident to the vertex $x^2 = (0, 0, 0, -1)$. It can be shown by case analysis that no matter how we colour the first four facets by four different colours and the last four facets by the same four colours, there will be another vertex incident to facets of all four colours.

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