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# The generic rank of body-bar-and-hinge frameworks

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# The generic rank of body-bar-and-hinge frameworks

Bill Jackson* and Tibor Jordán**

**Abstract**

Tay [6] characterized the multigraphs which can be realized as infinitesimally rigid $d$-dimensional body-and-bar frameworks. Subsequently, Tay [7] and Whiteley [11] independently characterized the multigraphs which can be realized as infinitesimally rigid $d$-dimensional body-and-hinge frameworks. We adapt Whiteley's proof technique to characterize the multigraphs which can be realized as infinitesimally rigid $d$-dimensional body-bar-and-hinge frameworks. More importantly, we obtain a sufficient condition for a multigraph to be realized as an infinitesimally rigid $d$-dimensional body-and-hinge framework in which all hinges lie in the same hyperplane. This result is related to a long-standing conjecture of Tay and Whiteley [8] which would characterize when a multigraph can be realized as an infinitesimally rigid $d$-dimensional body-and-hinge framework in which all the hinges incident to each body lie in a common hyperplane. As a corollary we deduce that if a graph $G$ has two spanning trees which use each edge of $G$ at most twice, then its square can be realized as an infinitesimally rigid 3-dimensional bar-and-joint framework.

## 1 Introduction

Informally, a $d$-dimensional body-bar-and-hinge framework consists of a set of $d$-dimensional rigid bodies in $d$-dimensional Euclidean space $\mathbb{R}^d$ connected by bars and hinges. The bodies are free to move continuously in $\mathbb{R}^d$ subject to the constraints that the distance between any two points joined by a bar is fixed and that the relative motion of any two bodies joined by a hinge is a rotation about the hinge. The framework is rigid if every such motion preserves the distances between all pairs of points belonging to different rigid bodies, i.e. the motion extends to an isometry of $\mathbb{R}^d$. We consider the framework as a pair $(G, q)$ where $G = (V, E_B, E_H)$ is a 2-edge-coloured

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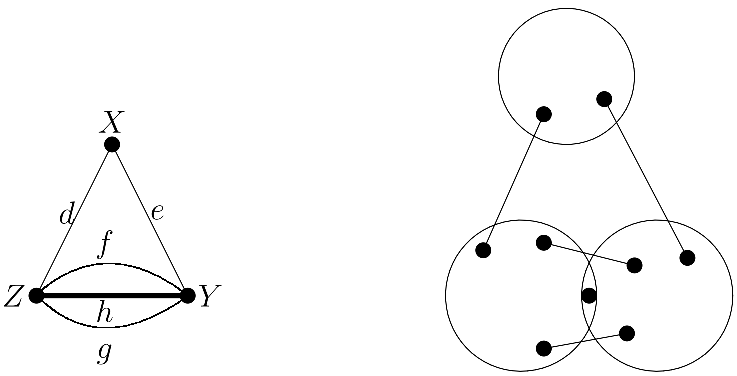


Figure 1: A 2-edge-coloured multigraph $G$ and an image of a body-bar-and-hinge realization of $G$ in $\mathbb{R}^2$. The realization consists of three rigid bodies connected by four bars, corresponding to the thin edges $d, e, f, g$, and one two-dimensional hinge (i.e. a pin) corresponding to the thick edge $h$. The points of attachment of the bars on the bodies act as ball joints, as does the two-dimensional hinge.

multigraph, and $q$ is a map which associates a rigid body with each vertex of $G$, a bar with each edge $e \in E_B$ and a hinge with each edge $e \in E_H$, both attached to the bodies corresponding to the endvertices of $e$. We say that the framework $(G, q)$ is a *body-bar-and-hinge realization* of the 2-edge-coloured multigraph $G$ in $\mathbb{R}^d$.

The rigidity of a $d$-dimensional body-bar-and-hinge framework can be investigated using an associated rigidity matrix. The framework is infinitesimally rigid if this matrix has rank $\binom{d+1}{2}(|V| - 1)$, and this is known to be a sufficient condition for the rigidity of the framework. Tay [6] characterized the multigraphs which can be realized as infinitesimally rigid $d$-dimensional body-and-bar frameworks. Subsequently, Tay [7] and Whiteley [11] independently characterized the multigraphs which can be realized as infinitesimally rigid $d$-dimensional body-and-hinge frameworks. We present an elementary constructive proof of their results which easily extends to characterize the 2-edge-coloured multigraphs which can be realized as infinitesimally rigid $d$-dimensional body-bar-and-hinge frameworks. Our construction is closely related to one given given by Whiteley in [11]. The main difference in our proof techniques is that Whiteley uses a 'coordinate-free approach' while we use an explicit coordinate system to define a rigidity matrix for a body-bar-and-hinge framework. This enables us to discuss 'generic frameworks' without recourse to the algebraic geometry used in [11].

Tay and Whiteley made the following conjecture in [8].

**Conjecture 1.1.** *If a graph $G$ can be realized as a $d$-dimensional infinitesimally rigid body-and-hinge framework, then $G$ can be realized as a $d$-dimensional infinitesimally rigid body-and-hinge framework in which the hinges incident to each body lie in a common hyperplane.*

We recently verified this conjecture for the special case when $d = 2$ in [2]. We will adapt the construction we use for infinitisimally rigid body-bar-and-hinge realizations of graphs in the present paper to obtain our main result: a sufficient condition for a multigraph to have an infinitesimally rigid realization with all of its hinges in the same hyperplane. For the special case when $d = 3$, projective duality allows us to deduce that the same condition implies the graph can be realized as a 3-dimensional body-and-hinge framework in which the lines containing the hinges incident to each body are concurrent at a point. Such frameworks are of special interest since they are used to model the flexibility of molecules by representing atoms as rigid bodies and bonds as hinges in such a way that the lines containing the hinges incident to each body are concurrent at the centre of the body.

Throughout this paper we shall assume that $d \geq 2$ is a fixed integer and use $D$ to denote $\binom{d+1}{2}$. For $S \subseteq \mathbb{R}^d$, we use $\langle S \rangle$ to denote the subspace of $\mathbb{R}^d$ spanned by $S$.

# 2 Infinitesimal motions of a rigid body

We consider a *rigid body* $Z$ in $\mathbb{R}^d$ to be a set of points whose affine span is $\mathbb{R}^d$. An *infinitesimal motion* of $Z$ is a map $\nu : Z \to \mathbb{R}^d$ such that $(p_1 - p_2) \cdot (\nu(p_1) - \nu(p_2)) = 0$ for all $p_1, p_2 \in Z$. For each $p \in Z$, we refer to the vector $\nu(p)$ as the *instantaneous velocity* of the point $p$. It is known that the infinitesimal motions of $Z$ form a vector space of dimension $D$ over $\mathbb{R}$, and that this space is spanned by the *instantaneous rotations and translations*, i.e. the particular infinitesimal motions corresponding to rotations and translations of $\mathbb{R}^d$. This space can be coordinatized using screw centres (real vectors of length $D$ which represent $(d-1)$-tensors in projective $d$-space). This coordinatization was used by White and Whiteley [10] to model the infinitesimal motions of body-and-bar and body-and-hinge frameworks and will be used throughout this paper. We will describe it in detail in the remainder of this section. Our approach differs from White and Whiteley in that they develop a 'coordinate free' approach while our proof technique requires an explicit definition of the coordinization.

Lemmas 2.1 to 2.5 below follow from [10]. We give proofs using elementary linear algebra in an appendix to this paper for the sake of completeness.

## 2.1 Infinitesimal rotations

We will define the screw centre corresponding to an infinitesimal rotation of a rigid body $Z$ about a $(d-2)$-dimensional affine subspace $A$ of $\mathbb{R}^d$ and show how it can be used to construct the instantaneous velocity induced by this rotation at each point $p \in Z$. Let $p_1, p_2, \ldots, p_{d-1}$ be points which span $A$. Let $M_A$ be the $(d-1) \times (d+1)$-matrix whose $i$'th row is the vector $(p_i, 1)$. Let $S(A)$ be the real vector of length $D$ whose coordinates are obtained from the $(d-1) \times (d-1)$-minors of $M_A$ as follows. We have $S(A) = (s_{i,j})$ where $s_{i,j} = (-1)^{i+j-1} \det M_{i,j}$, $M_{i,j}$ is obtained by deleting the $i$'th and $j$'th columns of $M_A$, and the coordinates $s_{i,j}$ are ordered lexicographically. The 1-dimensional subspace $\langle S(A) \rangle$ of $\mathbb{R}^D$ generated by $S(A)$ is uniquely determined by the subspace $A$, it is independent of the choice of $p_1, p_2, \ldots, p_{d-1}$. The vectors in $\langle S(A) \rangle$ are the *screw centres* corresponding to the rotations of $\mathbb{R}^d$ about $A$. They can be used to determine the instantaneous velocity induced by such a rotation at each point $p \in Z$ as follows.

For $p \in \mathbb{R}^d$, let $M_{A,p}$ be the $d \times (d+1)$-matrix obtained from $M_A$ by adding $(p, 1)$ as a new row. Let $v_{A,p} = (v_i)$ be the vector of length $d+1$ where $v_i = (-1)^i \det M_i$, $M_i$ is obtained by deleting the $i$'th column of $M_{A,p}$ and the coordinates $v_i$ are ordered lexicographically. Note that $v_{A,p}$ can be obtained directly from $S(A)$ and $p$, since its $i$'th component $v_i$ can be obtained by expanding $\det M_i$ along the row corresponding to $p$. We write $v_{A,p} = S(A) \vee p$.

Let $v_{A,p} = (v^*_{A,p}, v_{d+1})$ where $v^*_{A,p} \in \mathbb{R}^d$. We shall show that $v^*_{A,p}$ is proportional to the instantaneous velocity at $p$ induced by a rotation about $A$ and that $v_{d+1}$ is uniquely determined by $v^*_{A,p}$ and $p$.

**Lemma 2.1.** *$S(A) \vee p = (v^*_{A,p}, -v^*_{A,p} \cdot p)$, and there exists a constant $\lambda \in \mathbb{R}$ such that, for all $p \in \mathbb{R}^d$, the instantaneous velocity of $p$ under a fixed rotation of $\mathbb{R}^d$ about $A$ is equal to $\lambda v^*_{A,p}$.*

The screw centre $S(A)$ will be used to derive the constraint corresponding to a hinge in a body-bar-and-hinge framework. In order to derive bar-constraints we need to define another vector of length $D$.

Suppose that $p, p'$ are distinct points in $\mathbb{R}^d$. Let $M_{p,p'}$ be the $2 \times (d+1)$-matrix with $(p, 1)$ in its first row and $(p', 1)$ in its second row. Let $T(p, p') = (t_{i,j})$ be the vector of length $D$ where $t_{i,j} = \det M_{i,j}$, $M_{i,j}$ is the $2 \times 2$ matrix consisting of the $i$'th and $j$'th columns of $M_{p,p'}$, and in which the coordinates $t_{i,j}$ are ordered lexicographically.

**Lemma 2.2.** *Suppose $A$ is a $(d-2)$-dimensional affine subspace of $\mathbb{R}^d$ and $p, p' \in \mathbb{R}^d$. Then*

$$v^*_{A,p} \cdot (p' - p) = S(A) \cdot T(p, p').$$

**Lemma 2.3.** *Let $p, p'$ be distinct points in $\mathbb{R}^d$ and $A$ be a $(d-2)$-dimensional affine subspace of $\mathbb{R}^d$. Suppose that the line spanned by $p, p'$ has a non-empty intersection with $A$. Then $S \cdot T(p, p') = 0$ for all $S \in \langle S(A) \rangle$.*

## 2.2 Infinitesimal translations

We will define the screw centre corresponding to an infinitesimal translation of a rigid body $Z$ in the direction of a vector $x \in \mathbb{R}^d$ and show that it gives rise to the instantaneous velocity induced by this translation at each point $p \in Z$, in the same way as the screw centre for an instantaneous rotation.

Let $x_1, \ldots, x_{d-1}$ be a basis for orthogonal complement of $\langle x \rangle$ in $\mathbb{R}^d$. Let $M_x$ be the $(d-1) \times (d+1)$-matrix whose $i$'th row is the vector $(x_i, 0)$. Let $S(x)$ be the real vector of length $\binom{d+1}{d-1} = D$ whose coordinates are obtained from the $(d-1) \times (d-1)$-minors of $M_x$ as follows. We have $S(x) = (s_{i,j})$ where $s_{i,j} = (-1)^{i+j-1} \det M_{i,j}$, $M_{i,j}$ is obtained by deleting the $i$'th and $j$'th columns of $M_x$ and the coordinates $p_{i,j}$ are ordered lexicographically. The 1-dimensional subspace $\langle S(x) \rangle$ of $\mathbb{R}^D$ generated by $S(x)$ is uniquely determined by the subspace $\langle x \rangle$, it is independent of the choice of $x_1, x_2, \ldots, x_{d-1}$. The vectors in $\langle S(x) \rangle$ are the *screw centres* corresponding to the translations of $\mathbb{R}^d$ in the direction of $x$. (They can be viewed as screw centres corresponding to infinitesimal rotations about the $(d-2)$-dimensional subspace of projective $d$-space which corresponds to the intersection of the hyperplane $< x_1, x_2, \ldots, x_{d-1} >$ and the hyperplane at infinity.)

For $p \in \mathbb{R}^d$, let $M_{x,p}$ be the $d \times (d+1)$-matrix obtained from $M_x$ by adding $(p, 1)$ as a new row. Let $v_{x,p} = (v_i)$ be the vector of length $d+1$ where $v_i = (-1)^i \det M_i$, $M_i$ is obtained by deleting the $i$'th column of $M_{x,p}$ and the coordinates of $v_i$ are ordered lexicographically. Note that $v_{A,p}$ can be obtained directly from $S(x)$ and $p$, since its $i$'th component $v_i$ can be obtained by expanding $\det M_i$ along the row corresponding to $p$. We write $v_{x,p} = S(x) \vee p$. Let $v^*_{x,p}$ be the vector containing the first $d$ coordinates of $v_{x,p}$.

**Lemma 2.4.** *There exists a constant $\lambda$ such that, for each $p \in \mathbb{R}^d$, we have $S(x) \vee p = \lambda(x, -x \cdot p)$.*

We also have

**Lemma 2.5.** *Suppose $x, p, p' \in \mathbb{R}^d$. Then*

$$v^*_{x,p} \cdot (p' - p) = S(x) \cdot T(p, p').$$

### 2.3 Arbitrary infinitesimal motions

An arbitrary infinitesimal motion of a rigid body $Z$ in $\mathbb{R}^d$ can be expressed as a linear combination of infinitesimal rotations and translations. Let $S_1, S_2, \ldots, S_m$ be the screw centres corresponding to these rotations and translations. We define the *screw centre* $S$ for the arbitrary infinitesimal motion of $Z$ by putting $S = \sum_{i=1}^m S_i$. The instantaneous velocity of a point $p \in Z$ can then be calculated by adding together the instantaneous velocities given by each of the $S_i$ on $p$. Let $S \vee p = \sum_{i=1}^m S_i \vee p$. Thus $S \vee p$ is a $(d+1)$-dimensional vector, say $S \vee p = (v^*, v_{d+1})$ where $v^* \in \mathbb{R}^d$. Then $v^*$ is proportional to the instantaneous velocity at $p$ induced by the infinitesimal motion of $Z$, and $v_{d+1} = -v^* \cdot p$. Furthermore, if $p' \in \mathbb{R}^d$ then Lemmas 2.2 and 2.5 imply

$$v^* \cdot p' - v^* \cdot p = S \cdot T(p, p'). \quad (1)$$

## 3 Infinitesimal motions of frameworks

Following White and Whiteley [10] we use the coordinatization of the infinitesimal motions of a rigid body in $\mathbb{R}^d$ by screw centres to model the infinitesimal motions of body-bar-and-hinge frameworks. We first consider the constraints due to hinges and bars separately.

### 3.1 Hinge constraints

Suppose two rigid bodies $Z_1, Z_2$ are joined to a hinge which constrains that their relative motion is a rotation about a given $(d-2)$-dimensional affine subspace $A$ of

$\mathbb{R}^d$. Suppose further that the infinitesimal motion of $Z_i$ is represented by the screw centre $S_i$. Then the constraint on the relative motion implies the vector constraint that

$$S_1 - S_2 \in \langle S(A) \rangle. \tag{2}$$

## 3.2 Bar constraints

Suppose two rigid bodies $Z_1, Z_2$ are connected by a rigid bar which is attached to $Z_i$ at a point $p_i$, $i = 1, 2$, and constrains that the motions of $Z_1, Z_2$ preserve $\|p_1 - p_2\|$. Suppose further that $Z_i$ undergoes an infinitesimal motion which is represented by the screw centre $S_i$. Let $v_i^*$ be the resultant instantaneous velocity at $p_i$. The constraint imposed by the bar joining $p_1$ and $p_2$ implies that

$$0 = (v_1^* - v_2^*) \cdot (p_1 - p_2) = v_1^* \cdot p_1 - v_1^* \cdot p_2 - v_2^* \cdot p_1 + v_2^* \cdot p_2.$$

Using (1) and the fact that $T(p_2, p_1) = -T(p_1, p_2)$, this is equivalent to the vector constraint:

$$(S_1 - S_2) \cdot T(p_1, p_2) = 0. \tag{3}$$

Note that replacing $p_1, p_2$ by any other pair of distinct points on the line through $p_1, p_2$ will result in multiplying $T(p_1, p_2)$ by a non-zero scalar, and hence will not change the constraint (3). Note also that Lemma 2.3 implies that two screw centres $S_1, S_2$ which satisfy the hinge constraint (2), must also satisfy the bar constraint (3) whenever the bar-line intersects the hinge-space. This fact will be used later in the paper to convert a 'hinge constraint' into several 'bar constraints'.

## 3.3 Body-bar-and-hinge frameworks

We can now give a formal definition for a body-bar-and-hinge framework and its infinitesimal motions.

A *d-dimensional body-bar-and-hinge framework* $(G, q)$ is a 2-edge-coloured multigraph $G = (V, E_B, E_H)$ together with a map $q$ which associates a line segment, $q_e$, of $\mathbb{R}^d$ with each edge $e \in E_B$ and a $(d-2)$-dimensional affine subspace, $q_e$, of $\mathbb{R}^d$ with each edge $e \in E_H$. An *infinitesimal motion* of $(G, q)$ is a map $S$ from $V$ to $\mathbb{R}^D$ such that, for every edge $e = uv \in E_B$, $(S(u) - S(v)) \cdot T_{q_e} = 0$ and, for every edge $e = uv \in E_H$, $S(u) - S(v) \in \langle S(q_e) \rangle$. (The vector $S(v)$ is the screw centre representing the infinitesimal motion of the rigid body corresponding to $v$.)

Note that the positions of the rigid bodies do not appear in the above definition. Note also that our definition of a bar and a hinge is more restrictive than the definition given in [10, 12], which allows bars joining two points at infinity in projective $d$-space, and 'prismatic hinges' which constrain the relative motion of the pair of bodies they are incident with to be a translation in a fixed direction.

An infinitesimal motion $S$ is *trivial* if $S(u) = S(v)$ for all $u, v \in V$ and $(G, q)$ is said to be *infinitesimally rigid* if all its infinitesimal motions are trivial.

## 3.4 The rigidity matrix

We shall see that the set of infinitesimal motions of a body-bar-and-hinge framework $(G, q)$ is the null space of a matrix. For each $e \in E_H$, let $R_1(q_e), R_2(q_e), \ldots, R_{D-1}(q_e)$ be a basis for the orthogonal complement of $\langle S(q_e)\rangle$ in $\mathbb{R}^D$. Then the constraint that $S(u) - S(v) \in \langle S(q_e)\rangle$ is equivalent to the system of simultaneous equations $(S(u) - S(v)) \cdot R_i(q_e) = 0$ for all $1 \leq i \leq D - 1$. Combining these constraints for each edge $e \in E_H$ with the constraints that $(S(u) - S(v)) \cdot T(q_e) = 0$ for edge $e = uv \in E_B$, we obtain a system of $|E_B| + (D-1)|E_H|$ equations in the unknowns $S(v)$, $v \in V$. The matrix of coefficients of this system is a $(|E_B| + (D - 1)|E_H|) \times D|V|$ matrix $R(G, q)$ with the first $|E_B|$ rows indexed by $E_B$, sequences of $(D - 1)$ consecutive rows in the remaining rows indexed by $E_H$ and sequences of $D$ consecutive columns indexed by $V$. The entries in the row corresponding to an edge $e \in E_B$ and columns corresponding to a vertex $u \in V$ are given by the $1 \times D$ matrix $X_{e,u}$ where $X_{e,u} = T(q_e)$ if $e = uv$ is incident to $u$ and $u < v$ in the ordering on $V$ induced by the order of the column labels, $X_{e,u} = -T(q_e)$ if $e = uv$ is incident to $u$ and $u > v$, and $X_{e,u}$ is the zero matrix if $e$ is not incident to $u$. The entries in the rows corresponding to an edge $e \in E_H$ and columns corresponding to a vertex $u \in V$ are given by the $(D - 1) \times D$ matrix $X_{e,u}$ where $X_{e,u} = \begin{pmatrix} R_1(q_e) \\ R_2(q_e) \\ \vdots \end{pmatrix}$ if $e = uv$ is incident to $u$ and $u < v$, $X_{e,u} = -\begin{pmatrix} R_1(q_e) \\ R_2(q_e) \\ \vdots \end{pmatrix}$ if $e = uv$ is incident to $u$ and $u > v$, and $X_{e,u}$ is the zero matrix if $e$ is not incident to $u$. We refer to $R(G, q)$ as a *body-bar-and-hinge rigidity matrix* of $(G, q)$. By the above, a map $S : V \to \mathbb{R}^D$ is an infinitesimal motion of $(G, q)$ if and only if $S$ belongs to the null space $Z(G, q)$ of $R(G, q)$. We will refer to $Z(G, q)$ as the *space of infinitesimal motions* of $(G, q)$. Note that the entries in $R(G, q)$ are not uniquely determined by $(G, q)$ when $E_H \neq \emptyset$, since they depend on the choice of the basis for the orthogonal complement of $\langle S(q_e)\rangle$. On the other hand, the space of infinitesimal motions $Z(G, q)$, and hence the rank of $R(G, q)$, is uniquely determined by $(G, q)$. We will refer to the rank of $R(G, q)$ as the *rank* of $(G, q)$ and denote it by $r(G, q)$.

**Example:**

Consider the 2-edge-colored multigraph $G = (V, E_B, E_H)$ of Figure 1 with $V = \{X, Y, Z\}$, $E_B = \{d, e, f, g\}$, and $E_H = \{h\}$. Let $p_0 = (0, 0), p_1 = (1, 0), p_2 = (0, 1)$. Let the line segments associated with the bars and the 0-dimensional affine subspace associated with the hinge be defined by putting $q(d) = q(f) = [p_0, p_1]$, $q(e) = q(g) = [p_0, p_2]$, and $q(h) = \{p_2\}$. This gives rise to a 2-dimensional body-bar-and-hinge realization $(G, q)$ of $G$.

To define the rigidity matrix of $(G, q)$ we first calculate $M_{p_0,p_1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $M_{p_0,p_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, which yields $T(p_0, p_1) = (0, -1, 0)$ and $T(p_0, p_2) = (0, 0, -1)$. Next we observe that for the affine subspace $A = \{p_2\}$ associated with hinge $h$ we have $M_A = (0, 1, 1)$, which gives $S(A) = (1, -1, 0)$. We may choose $R_1(p_2) = (0, 0, 1)$

and $R_2(p_2) = (1,1,0)$ as a basis for the orthogonal complement of $\langle S(A)\rangle$ in $\mathbb{R}^3$. Thus we obtain:

$$R(G,q) = \begin{array}{c} \left[\begin{array}{ccccccccc} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \end{array}\right] \end{array} \begin{array}{c} d \\ e \\ f \\ g \\ h \\ h \end{array}$$

where the first, second and third sets of three consecutive columns correspond to $X$, $Y$ and $Z$, respectively.

It is easy to check that the rank of $R(G,q)$ is equal to five. Consider the multigraph $G^H$ obtained from $G$ by replacing the edge $h$ by two edges $h^1, h^2$. We may obtain a body-and-bar realization $(G^H, \tilde{q})$ of $G^H$ from $(G,q)$ by letting $\tilde{q}(h^1) = [p_0, p_2]$, $\tilde{q}(h^2) = [p_1, p_2]$ and $\tilde{q}(b) = q(b)$ for all $b \in \{d,e,f,g\}$. Note that the bar-lines of $h_1$ and $h_2$ intersect the affine subspace $\{p_2\}$ of hinge $h$. As above, we calculate $M_{p_1,p_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, which yields $T(p_1,p_2) = (1,1,-1)$. This implies that the rigidity matrix $R(G^H, \tilde{q})$ is obtained from $R(G,q)$ by replacing the last two rows by the rows

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

Note that the replacement of the hinge by two bars, as above, does not change the rank of the rigidity matrix, since the subspaces spanned by the last two rows of $R(G^H, \tilde{q})$ and $R(G,q)$ are the same.

We will henceforth adopt the following conventions and notation for a body-bar-and-hinge framework $(G,q)$. Let $G^H$ be the multigraph obtained by replacing each edge $e \in E_H$ by $D-1$ parallel edges $e^1, e^2, \ldots, e^{D-1}$. We use $(D-1)e$ to denote the set of parallel edges of $G^H$ corresponding to $e$ and put $(D-1)E_H = \bigcup_{e \in E_H}(D-1)e$. Thus $E(G^H) = E_B \cup (D-1)E_H$. Given a rigidity matrix $R(G,q)$ for $(G,q)$ we assume that the rows of $R(G,q)$ are indexed by the edges of $G^H$, where for each $e = uv \in E_H$, $e^i$ is associated to the $i$'th constraint $(S(u) - S(v)) \cdot R_i(q_e) = 0$. In addition we adopt the convention that the vectors $x \in \mathbb{R}^D$ are given in the form $x = (x_{i,j})$ where $0 \le i < j \le d$ and the coordinates $x_{i,j}$ are listed in lexicographic order. We assume that the columns of the rigidity matrix $R(G,q)$ of a body-bar-and-hinge framework $(G,q)$ are indexed by the set $DV = \bigcup_{0 \le i < j \le d} V_{i,j}$ where $V_{i,j}$ is a copy of $V$ for each $(i,j)$, $0 \le i < j \le d$, and the elements of $V_{i,j}$ correspond to the $(i,j)$'th coordinate in $\mathbb{R}^D$. Let $F \subseteq E_B \cup (D-1)E_H$ and $Y \subseteq DV$. We use $R[F,Y]$ to denote the submatrix of $R(G,q)$ indexed by $F$ and $Y$. Given $U \subseteq V$, we use $F_U$ to denote the set of all edges of $F$ joining two vertices in $U$ and put $i_F(U) = |F_U|$.

**Lemma 3.1.** *Let $(G,q)$ be a $d$-dimensional body-bar-and-hinge framework with rigidity matrix $R(G,q)$ and $F \subseteq E_B \cup (D-1)E_H$. Suppose that the rows of $R(G,q)$ indexed by $F$ are linearly independent. Then $i_F(U) \le D(|U|-1)$ for all $\emptyset \ne U \subseteq V$.*

**Proof:** We proceed by contradiction. Suppose $i_F(U) > D(|U|-1)$ for some $U \subseteq V$. Let $M = R[F_U, DU]$ and let $A$ be the standard basis for $\mathbb{R}^D$. For each $a \in A$ define $S_a : U \to \mathbb{R}^D$ by putting $S_a(u) = a$ for all $u \in U$. It is easy to check that each $S_a$ belongs to the null space of $M$. Thus the null space of $M$ has dimension at least $D$ and hence rank $M \leq D|U| - D$. Since $|F_U| = i_F(U) > D(|U|-1)$, the rows of $M$ are linearly dependent. Since all non-zero entries in the rows of $R(G,q)$ indexed by $F_U$ occur in the columns indexed by $DU$, the rows of $R(G,q)$ indexed by $F_U$ are linearly dependent. This gives a contradiction since $F_U \subseteq F$. •

**Corollary 3.2.** *Let $(G,q)$ be a d-dimensional body-bar-and-hinge framework. Then $r(G,q) \leq D(|V|-1)$, with equality if and only if $(G,q)$ is infinitesimally rigid.*

**Proof:** The inequality follows from Lemma 3.1 by taking $F$ to be the edges indexing a maximum linearly independent set of rows in a rigidity matrix $R(G,q)$ for $(G,q)$, and $U = V$. The proof of Lemma 3.1 implies that equality holds if and only if the null space of $R(G,q)$ is spanned by the vectors $S_a$ where $a \in A$, each of which satisfies $S_a(u) = S_a(v)$ for all $u, v \in V$. Thus equality holds if and only if each $S$ in the null space of $R(G,q)$ satisfies $S(u) = S(v)$ for all $u, v \in V$. •

# 4 Edge-disjoint forests

In this section we relate the necessary condition for a set of rows in a rigidity matrix to be linearly independent given in Lemma 3.1 to structural results on forest covers of multigraphs. Let $G = (V, E)$ be a multigraph. For a family $\mathcal{F}$ of pairwise disjoint subsets of $V$ let $E_G(\mathcal{F})$ denote the set, and $e_G(\mathcal{F})$ the number, of edges of $G$ connecting distinct members of $\mathcal{F}$.

The following theorem is well-known, see for example [5, Chapter 51].

**Theorem 4.1.** *[3, 4, 9] Let $G = (V, E)$ be a multigraph and let $k$ be a positive integer. Then:*
*(a) the maximum size of the union of $k$ forests in $G$ is equal to the minimum value of*

$$e_G(\mathcal{P}) + k(|V| - |\mathcal{P}|)$$

*taken over all partitions $\mathcal{P}$ of $V$;*
*(b) $G$ contains $k$ edge-disjoint spanning trees if and only if*

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$$

*for all partitions $\mathcal{P}$ of $V$;*
*(c) the edge set of $G$ can be covered by $k$ forests if and only if*

$$|E(G[U])| \leq k(|U| - 1)$$

*for each nonempty subset $U$ of $V$.* •

For a partition $\mathcal{Q}$ of $V$ let

$$\text{def}_{G,k}(\mathcal{Q}) = k(|\mathcal{Q}| - 1) - e_G(\mathcal{Q})$$

denote the *k-deficiency* of $\mathcal{Q}$ in $G$ and let

$$\text{def}_k(G) = \max\{\text{def}_{G,k}(\mathcal{Q}) : \mathcal{Q} \text{ is a partition of } V\}.$$

Note that $\text{def}_k(G) \geq 0$ since $\text{def}_G(\{V\}) = 0$.

Let $(G, q)$ be a body-bar-and-hinge realization of a 2-edge-coloured multigraph $G$ in $\mathbb{R}^d$. Lemma 3.1 and Theorem 4.1(c) imply that the maximum number of linearly independent rows in a rigidity matrix $R(G, q)$ for $(G, q)$ is at most the maximum number of edges in the union of $D$ forests of $G^H$. Theorem 4.1(a) now implies that $D(|V| - 1) - \text{def}_D(G^H)$ is an upper bound on $r(G, q)$. We shall see in Sections 5 and 6 below that $G$ always has a realization $(G, q)$ for which this upper bound is attained. In particular, $G$ has an infinitesimally rigid realization if and only if $G^H$ has $D$ edge-disjoint spanning trees.

# 5 Infinitesimal rigidity of body-and-bar frameworks

A *body-and-bar framework* is a body-bar-and-hinge framework $(G, q)$ in which each edge of $G$ is a bar i.e. $E_H = \emptyset$. Let $p_0$ be the zero vector in $\mathbb{R}^d$, and $p_i$ be the vector with a one in the $i$'th position and zeros elsewhere, for all $1 \leq i \leq d$.

**Lemma 5.1.** *Let $G = (V, E)$ be a multigraph and $F \subseteq E$. Suppose that $F$ can be partitioned into $D$ forests $F_{i,j}$, $0 \leq i < j \leq d$. Let $(G, q)$ be a body-and-bar realization of $G$ in $\mathbb{R}^d$ with the property that $q_e = [p_i, p_j]$ when $e \in F_{i,j}$, for all $0 \leq i < j \leq d$. Then the rows of $R(G, q)$ indexed by $F$ are linearly independent.*

**Proof:** We may use the definition of $q_e$ for $e \in F$ to deduce that the vector $T(q_e) = (t_{i,j})$, $0 \leq i < j \leq d$ is as follows.
(a) If $e \in F_{0,k}$ for some $1 \leq k \leq d$ then

$$t_{i,j} = \begin{cases} -1 & \text{when } (i,j) = (k-1, d), \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $e \in F_{h,k}$ for some $1 \leq h < k \leq d$ then

$$t_{i,j} = \begin{cases} 1 & \text{when } (i,j) = (h-1, k-1) \text{ or } (i,j) = (h-1, d), \\ -1 & \text{when } (i,j) = (k-1, d), \\ 0 & \text{otherwise.} \end{cases}$$

Let $R = R(G, q)$ and let $R_e$ be the row of $R$ indexed by $e$ for each $e \in F$. Suppose that $\sum_{e \in F} \lambda_e R_e = 0$ for some scalars $\lambda_e$. Consider $R[F, V_{i-1,j-1}]$ for some fixed $(i, j)$, $1 \leq i < j \leq d$. Since all non-zero entries in $R[F, V_{i-1,j-1}]$ occur in the rows indexed by $F_{i,j}$ and since $R[F_{i,j}, V_{i-1,j-1}]$ is the directed incidence matrix of the forest $F_{i,j}$ we must have $\lambda_e = 0$ for all $e \in F_{i,j}$, for all $1 \leq i < j \leq d$. We now let $F' = \bigcup_{1 \leq k \leq d} F_{0,k}$. Since $R[F', V_{k,d}]$, is the directed incidence matrix of the forest $F_{0,k}$, for each fixed $k$, $1 \leq k \leq d$, we must have $\lambda_e = 0$ for all $e \in F'$. Thus the rows of $R(G, q)$ indexed by $F$ are linearly independent. •

**Remark** Whiteley [14] has pointed out that Lemma 5.1 follows from his [11, Theorem 8]. He assigns a vector $T(q_e)$ for each edge $e$ of $G$ by choosing a suitable vector from the standard basis of $\mathbb{R}^D$ (depending on the forest in the cover which contains $e$). This corresponds to mapping the edges of $G$ onto the edges of the projective $(d+1)$-simplex with one vertex at the origin and the other vertices at the points at infinity on the ends of the coordinate axes. He shows that the resulting projective body-and-bar framework is independent. One can use the fact that projective transformations preserve independence to deduce our Lemma 5.1.

Let $(G, q)$ be a body-and-bar realization of a multigraph $G$ in $\mathbb{R}^d$ and $R(G, q)$ be its rigidity matrix. By an *edge-induced* submatrix of $R(G, q)$ we will mean a submatrix obtained by deleting some of the rows of $R(G, q)$. The body-and-bar realization $(G, q)$ is said to be *generic* if $R(G, q)$ and all of its edge-induced submatrices have maximum rank, taken over all $d$-dimensional body-and-bar realizations of $G$. It can be seen that if the (multi)set of coordinates of the endpoints of all the line segments $q_e$, $e \in E$, is algebraically independent over $\mathbb{Q}$ then $(G, q)$ will be generic. Thus 'almost all' body-and-bar realizations of $G$ in $\mathbb{R}^d$ are generic.

**Theorem 5.2.** *[6] Let $G = (V, E)$ be a multigraph and $(G, q)$ be a generic body-and-bar realization of $G$ in $\mathbb{R}^d$. Then $r(G, q) = D(|V| - 1) - def_D(G)$.*

**Proof:** Lemma 3.1 and Theorem 4.1(a),(c) imply that $r(G, q) \leq D(|V|-1) - \mathrm{def}_D(G)$. Equality holds since $G$ has a particular realization $(G, q_0)$ for which rank $R(G, q_0) = D(|V| - 1) - \mathrm{def}_D(G)$ by Lemma 5.1.

**Corollary 5.3.** *[6] A multigraph $G$ can be realized as an infinitesimally rigid body-and-bar framework in $\mathbb{R}^d$ if and only if $G$ has $D$ edge-disjoint spanning trees.*

We can also deduce the following result which implies, in particular, that if $G$ has an infinitesimally rigid body-and-bar realization, then it has one with at most $D$ different bar-lines.

**Theorem 5.4.** *Every multigraph $G = (V, E)$ has a maximum rank body-and-bar realization $(G, q)$ in $\mathbb{R}^d$ with*

$$q_e \in \{[p_i, p_j] \;:\; 0 \leq i < j \leq d\}$$

*for all $e \in E$.*

# 6 Infinitesimal rigidity of body-bar-and-hinge frameworks

We use Lemma 5.1 to determine the maximum rank of a body-bar-and-hinge realization of a 2-edge-coloured multigraph in $\mathbb{R}^d$. Let $p_0, p_1, \ldots, p_d$ be as defined at the beginning of Section 5 and put $C = \{p_0, p_1, \ldots, p_d\}$.

**Theorem 6.1.** *Let $G = (V, E_B, E_H)$ be a 2-edge-coloured multigraph. Then the maximum rank of a body-bar-and-hinge realization of $G$ in $\mathbb{R}^d$ is $D(|V| - 1) - def_D(G^H)$.*

**Proof:** Lemma 3.1 implies that the rank of any body-bar-and-hinge realization of $G$ in $\mathbb{R}^d$ is at most $D(|V| - 1) - \mathrm{def}_D(G^H)$. It remains to show that $G$ has a realization with this rank.

By Theorem 4.1(a), there exists $F \subseteq E_B \cup (D-1)E_H$ such that $|F| = D(|V| - 1) - \mathrm{def}_D(G^H)$ and such that $F$ can be partitioned into $D$ forests $F_{i,j}$, $0 \le i < j \le d$. Let $(G^*, q^*)$ be the body-and-bar framework obtained by putting $G^* = (V, F)$ and $q^*_e = [p_i, p_j]$ when $e \in F_{i,j}$. By Lemma 5.1, $r(G^*, q^*) = |F| = D(|V| - 1) - \mathrm{def}_D(G^H)$. Note that, for each $e \in E_H$, there exists at least one pair $(i, j)$, $0 \le i < j \le d$, such that $(D-1)e \cap F_{i,j} = \emptyset$.

Let $(G, q)$ be a body-bar-and-hinge realization of $G$ in $\mathbb{R}^d$ with the properties that if $e \in E_B \cap F$ then $q_e = q^*_e$, and if $e \in E_H$ then $q_e$ is the affine subspace of $\mathbb{R}^d$ spanned by $C - \{p_i, p_j\}$ for some $0 \le i < j \le d$ with $(D-1)e \cap F_{i,j} = \emptyset$. We shall show every infinitesimal motion of $(G, q)$ is an infinitesimal motion of $(G^*, q^*)$. Let $S : V \to \mathbb{R}^D$ be an infinitesimal motion of $(G, q)$.

Suppose $e = uv \in E_B$. Since $S$ is an infinitesimal motion of $(G, q)$, we have $(S(u) - S(v)) \cdot T(q_e) = 0$. Since $q^*_e = q_e$, we have $T(q_e) = T(q^*_e)$ and hence $(S(u) - S(v)) \cdot T(q^*_e) = 0$.

Suppose $e = uv \in E_H$ and $f \in (D-1)e \cap F$. Since $S$ is an infinitesimal motion of $(G, q)$, we have $(S(u) - S(v)) \in \langle S(q_e) \rangle$. We also have $q^*_f = [p_h, p_k]$ for some $p_h, p_k \in C$ with $\{p_h, p_k\} \cap q_e \neq \emptyset$. Hence by Lemma 2.3, we have $(S(u) - S(v)) \cdot T(q^*_f) = 0$.

Thus $S$ satisfies all the 'bar-constraints' for $(G^*, q^*)$ and hence is an infinitesimal motion of $(G^*, q^*)$. It follows that $Z(G, q) \subseteq Z(G^*, q^*)$ and hence $r(G, q) \ge r(G^*, q^*) = D(|V| - 1) - \mathrm{def}_D(G^H)$. •

**Remark** The ideas of replacing a hinge by an equivalent set of $D-1$ bars and mapping the edges in $E_H$ to suitably chosen $(d-2)$-facets of a $(d+1)$-simplex are taken from Whiteley [11, Theorem 10].

**Example continued:**

We illustrate the proof of Theorem 6.1 by considering the multigraph of Figure 1. Observe that $\mathrm{def}_3(G^H) = 1$ and hence $3(|V| - 1) - \mathrm{def}_3(G^H) = 5$. It follows that we may choose a set of five edges $F = \{d, e, f, h^1, h^2\}$ of $G^H$ which can be partitioned into three forests, say $F_{0,1} = \{d, f\}$, $F_{0,2} = \{e, h^1\}$, and $F_{1,2} = \{h^2\}$. First we construct a maximum rank body-and-bar realization of $G^* = (V, F)$ by putting $q^*(d) = q^*(f) =$

$\{p_0, p_1\}$, $q^*(e) = q^*(h^1) = \{p_0, p_2\}$, and $q^*(h^2) = \{p_1, p_2\}$. The body-bar-and-hinge realization $(G, q)$ defined in the Example in Subsection 3.4 satisfies the requirements given in the proof of the Theorem. Hence rank $R(G, q) =$ rank $R(G^*, q^*) = 5$ and $(G, q)$ has maximum rank.

Since $\mathrm{def}_D(G^H) = 0$ if and only if $G^H$ has $D$ edge-disjoint spanning trees, Theorem 6.1 immediately implies:

**Corollary 6.2.** *A 2-edge-coloured multigraph $G$ can be realized as an infinitesimally rigid body-bar-and-hinge framework in $\mathbb{R}^d$ if and only if $G^H$ has $D$ edge-disjoint spanning trees.*

Given a multigraph $G$ and a positive integer $k$, let $kG$ be the multigraph obtained by replacing each edge of $G$ by $k$ parallel edges.

**Corollary 6.3.** *[7, 11] A multigraph $G$ can be realized as an infinitesimally rigid body-and-hinge framework in $\mathbb{R}^d$ if and only if $(D-1)G$ has $D$ edge-disjoint spanning trees.*

Let $(G, q)$ be a body-and-hinge realization of a multigraph $G$ in $\mathbb{R}^d$ and $R(G, q)$ be a rigidity matrix for $(G, q)$. By an *edge-induced* submatrix of $R(G, q)$ we will mean a submatrix obtained by deleting some $(D-1)$-tuples of rows corresponding to a subset of the edges of $G$ from $R(G, q)$. The body-and-hinge realization $(G, q)$ is said to be *generic* if $R(G, q)$ and all of its edge-induced submatrices have maximum rank, taken over all $d$-dimensional body-and-hinge realizations of $G$.

The proof of Theorem 6.1 gives the following result on maximum rank realizations. It implies, in particular, that if $G$ has an infinitesimally rigid body-and-hinge realization, then it has one with at most $D$ different hinge-subspaces.

**Theorem 6.4.** *Every multigraph $G = (V, E)$ has a maximum rank body-and-hinge realization $(G, q)$ in $\mathbb{R}^d$ in which, for each $e \in E$, $q_e$ is equal to the affine subspace of $\mathbb{R}^d$ spanned by $C - \{p_i, p_j\}$ for some $0 \leq i < j \leq d$.*

Let $G = (V, E)$ be a multigraph and $(G, q)$ be a body-and-hinge realization of $G$ in $\mathbb{R}^d$. For each $v \in V$ let $E_v$ be the set of edges incident to $v$. We say that $(G, q)$ is a *hinge-coplanar realization* if the hinges $q_e$, $e \in E_v$, are all contained in a common hyperplane for each $v \in V$. By Corollary 6.3, Conjecture 1.1 is equivalent to the following statement: a multigraph $G$ has an infinitesimally rigid hinge-coplanar body-and-hinge realization in $\mathbb{R}^d$ if and only if $DG$ has $D-1$ edge-disjoint spanning trees. Our proof technique for Theorem 6.1 gives a related result.

**Theorem 6.5.** *Let $G = (V, E)$ be a multigraph. Suppose $(d-1)G$ has $d$ edge-disjoint spanning trees. Then $G$ has an infinitesimally rigid body-and-hinge realization $(G, q)$ in $\mathbb{R}^d$, in which every hinge $q_e$ is contained in a common hyperplane.*

**Proof:** Let $\{T_{i,d} \ : \ 0 \leq i \leq d-1\}$, be a set of $d$ edge-disjoint spanning trees in $(d-1)G$. Let $T$ be an arbitrary spanning tree of $G$ and put $T_{i,j} = T$ for all

$0 \leq i < j \leq d-1$. Then $\{T_{i,j} : 0 \leq i < j \leq d\}$ is a set of $D$ edge-disjoint spanning trees in $(D-1)G$, with the property that, for each $e \in E$, we have $T_{i,d} \cap (D-1)e = \emptyset$ for some $0 \leq i \leq d-1$.

Let $(G,q)$ be a body-and-hinge realization of $G$ in $\mathbb{R}^d$ with the property that, if $e \in E$, then $q_e$ is the affine subspace of $\mathbb{R}^d$ spanned by $C - \{q_i, q_d\}$ for some $0 \leq i \leq d-1$ with $(D-1)e \cap T_{i,d} = \emptyset$. We may show that $(G,q)$ is infinitesimally rigid as in the proof of Theorem 6.1. Furthermore, for each $e \in E$, $q_e$ is contained in the hyperplane of $\mathbb{R}^d$ spanned by $p_0, p_1, \ldots, p_{d-1}$. •

# 7 Panel-and-hinge frameworks

A hinge-coplanar body-and-hinge realization $(G,q)$ of a multigraph $G$ is said to be *non-degenerate* if there is a unique hyperplane $\Pi_{q,v}$ containing the hinges $q_e$, $e \in E_v$, for each $v \in V$, and $\Pi_{q,u} \cap \Pi_{q,v} = q_e$ for all $e = uv \in E$. We will refer to the hyperplanes $\Pi_{q,v}$ in a non-degenerate hinge-coplanar realization of $G$ as *vertex hyperplanes*. Note that the infinitesimally rigid hinge-coplanar realization given by Theorem 6.5 is far from being non-degenerate since all its hinges lie in the same hyperplane. We shall show, however, that if $G$ has minimum degree at least two and has no multiple edges, then we can perturb the body-and-hinge framework given by Theorem 6.5 in such a way that it becomes an infinitesimally rigid non-degenerate hinge-coplanar realization.

**Lemma 7.1.** *Let $G = (V,E)$ be a graph. Suppose $G$ has an infinitesimally rigid hinge-coplanar body-and-hinge realization $(G,q)$ in $\mathbb{R}^d$. Then $G$ has an infinitesimally rigid non-degenerate hinge-coplanar body-and-hinge realization in $\mathbb{R}^d$.*

**Proof:** Since $(G,q)$ is infinitesimally rigid, for each $v \in V$, the hinges $q_e$, $e \in E_v$ must span the hyperplane which contains them. Let $\Pi_{q,v}$ be the hyperplane which contains the hinges $q_e$, $e \in E_v$. We may choose a coordinate system for $\mathbb{R}^d$ such that no hyperplane $\Pi_{q,v}$ contains the origin. Then each hyperplane $\Pi_{q,v}$ can be uniquely expressed as $\Pi_{q,v} = \{x \in \mathbb{R}^d : x \cdot c_{q,v} = 1\}$ for some $c_{q,v} \in \mathbb{R}^d$. It will suffice to show that that there exists an infinitesimally rigid hinge-coplanar body-and-hinge framework $(G,\hat{q})$ in $\mathbb{R}^d$ such that the vectors $c_{\hat{q},v}$ are distinct for all $v \in V$. Suppose, recursively, that for some subset $U \subset V$, we have $c_{q,u} \neq c_{q,w}$ for all $uw \in E$ with $u,w \in U$. Choose $v \in V - U$. We may assume that $c_{q,v} = c_{q,u}$ for some $f = uv \in E$ with $u \in U$, otherwise we replace $U$ by $U \cup \{v\}$. We can construct a new body-and-hinge framework $(G,\hat{q})$ by rotating the hyperplane $\Pi_{q,v}$ about the hinge $q_f$ to become a new hyperplane $\Pi$ which is not parallel to any of the hyperplanes $\Pi_{q,w}$ for $wv \in E$, and then putting $\hat{q}_e = q_e$ for all $e \in E$ which are not incident to $v$, and $\hat{q}_e = \Pi_{q,w} \cap \Pi$ for all edges $e = wv \in E$ incident to $v$. For each $w \in V$, the hinges $\hat{q}_e$, $e \in E_w$ are all contained in a unique hyperplane, $\Pi_{\hat{q},w}$. Indeed, $\Pi_{\hat{q},w} = \Pi_{q,w}$ for $w \neq v$ and $\Pi_{\hat{q},v} = \Pi$. Furthermore, the body-and-hinge framework $(G,\hat{q})$ will be infinitesimally rigid as long as the rotation of $\Pi_{q,v}$ about the hinge $q_f$ is sufficiently small. [1] We can now iterate, replacing $(G,q)$ and $U$ by $(G,\hat{q})$ and

[1]This can be seen by replacing $c_{q,v}$ by a vector of indeterminates $x_{q,v}$. For each edge $e \in E_v$,

$U \cup \{v\}$, respectively. •

A $d$-dimensional *panel-and-hinge framework* $(G,p)$ is a graph $G=(V,E)$ of minimum degree at least two, together with a map $p : V \to \mathbb{R}^d$ such that $p_u$ and $p_v$ are linearly independent for all $uv \in E$. We define the *hinge-hyperplane* of $v$ to be $\Pi_{p,v} = \{x \in \mathbb{R}^d \ : \ x \cdot p_v = 1\}$ for all $v \in V$. The map $p$ induces a map $\tilde{p}$ which associates a $(d-2)$-dimensional affine subspace $\tilde{p}_e$ with each edge $e = uv \in E$ given by $\tilde{p}_e = \Pi_{p,u} \cap \Pi_{p,v}$. The map $\tilde{p}$ gives rise to a non-degenerate hinge-coplanar body-and-hinge framework $(G,\tilde{p})$ which we refer to as the *body-and-hinge framework associated to* $(G,p)$. We say that the panel-and-hinge framework $(G,p)$ is *infinitesimally rigid* if its associated body-and-hinge framework $(G,\tilde{p})$ is infinitesimally rigid.

Let $(G,p)$ be a panel-and-hinge realization of a multigraph $G$ in $\mathbb{R}^d$ and $R(G,\tilde{p})$ be a rigidity matrix of its associated body-and-hinge realization. The panel-and-hinge framework $(G,p)$ is said to be *generic* if $R(G,\tilde{p})$ and all of its edge-induced submatrices have maximum rank, taken over all $d$-dimensional panel-and-hinge realizations of $G$. It can be seen that if the (multi)set of coordinates of the vectors $p_v$, $v \in V$, is algebraically independent over $\mathbb{Q}$ then $(G,p)$ will be generic. Thus 'almost all' panel-and-hinge realizations of $G$ in $\mathbb{R}^d$ are generic. Note that since not all body-and-hinge frameworks are associated to panel-and-hinge frameworks, it is conceivable that the body-and-hinge framework associated to a generic panel-and-hinge framework may not be generic when viewed as a body-and-hinge framework.

Theorem 6.5 and Lemma 7.1 give the following sufficient condition for a graph to have an infinitesimally rigid panel-and-hinge realization.

**Theorem 7.2.** *Let $G=(V,E)$ be a graph. Suppose $(d-1)G$ has $d$ edge-disjoint spanning trees. Then every generic panel-and-hinge realization of $G$ in $\mathbb{R}^d$ is infinitesimally rigid.*

**Proof:** It follows from Theorem 6.5 and Lemma 7.1 that $G$ has an infinitesimally rigid realization as a non-degenerate hinge-coplanar body-and-hinge framework $(G,q)$. We may choose the coordinate system such that no vertex hyperplane of $(G,q)$ contains the origin. Define $p : V \to \mathbb{R}^d$ by letting $p_v$ be the unique vector such that $\Pi_{q,v} = \{x \in \mathbb{R}^d \ : \ x \cdot p_v = 1\}$. Then $(G,p)$ is a panel-and-hinge framework and $(G,q)$ is the body-and-hinge framework associated to $(G,p)$. Since $(G,q)$ is infinitesimally rigid, $(G,p)$ is also infinitesimally rigid. Since $G$ has an infinitesimally rigid realization as a panel-and-hinge framework, all generic panel-and-hinge realization of $G$ are infinitesimally rigid. •

we can define a set of $d-1$ points spanning $q_e$ whose components are rational functions of the components of $x_{q,v}$. Thus we may construct a rigidity matrix in which the entries are rational, and hence continuous, functions of the components of $x_{q,v}$. This matrix will have maximum rank when $x_{q,v} = c_{q,v}$, and hence the rank will remain constant for all $x_{q,v}$ sufficiently close to $c_{q,v}$.

# 8 Molecular frameworks

A *molecular framework* $(G, p)$ is a graph $G = (V, E)$ together with a map $p : V \to \mathbb{R}^3$ such that $p_u$ and $p_v$ are linearly independent for all $uv \in E$. The map $p$ induces a map $\hat{p}$ which associates the line $\hat{p}_e$ through $p_u, p_v$ with each $e = uv \in E$. The map $\tilde{p}$ gives rise to a 3-dimensional body-and-hinge framework $(G, \hat{p})$ which we refer to as the *body-and-hinge framework associated to* $(G, p)$. This body-and-hinge framework will have the property that, for each $v \in V$, the hinges $\hat{p}_e$, $e \in E_v$, will all be incident with the same point $p(v)$. We say that such a framework is *hinge-concurrent.* Since projective duality in $\mathbb{R}^3$ takes planes to points, lines to lines, and points to planes, a 3-dimensional hinge-concurrent body-and-hinge framework $(G, p)$ is the projective dual of the 3-dimensional hinge-coplanar body-and-hinge framework $(G, -p)$. The molecular framework $(G, p)$ is *infinitesimally rigid* if its associated body-and-hinge framework $(G, \hat{p})$ is infinitesimally rigid.

Let $(G, p)$ be a realization of a multigraph $G$ as a molecular framework in $\mathbb{R}^d$ and $R(G, \hat{p})$ be a rigidity matrix of its associated body-and-hinge realization. The molecular framework $(G, p)$ is said to be *generic* if $R(G, \hat{p})$ and all of its edge-induced submatrices have maximum rank, taken over all $d$-dimensional molecular realizations of $G$. It can be seen that if the (multi)set of coordinates of the points $p_v$, $v \in V$, is algebraically independent over $\mathbb{Q}$ then $(G, p)$ will be generic.

Crapo and Whiteley [1] have shown that the projective duality between hinge-coplanar and hinge-concurrent body-and-hinge frameworks in $\mathbb{R}^3$ preserves infinitesimal rigidity. Combining this result with Theorem 7.2, we obtain:

**Theorem 8.1.** *Let $G$ be a graph. If $2G$ has three edge-disjoint spanning trees then every generic molecular realization of $G$ in $\mathbb{R}^3$ is infinitesimally rigid.*

As noted in Section 1, molecular frameworks are used as a model to study the flexibility of molecules. An alternative, but equivalent, model is to consider 'bar-and-joint' realizations of squares of graphs in $\mathbb{R}^3$. The *square* of a graph $G$ is the graph $G^2$ with the same vertex set as $G$ in which all vertices of distance at most two in $G$ are joined by an edge of $G^2$, see Figure 2. We refer the reader to [12] for the definition of a (generic) bar-and-joint framework. Whiteley [13] has shown that if $G = (V, E)$ is a graph of minimum degree at least two and $p : V \to \mathbb{R}^3$ is such that the points $p_v$, $v \in V$ are in general position in $\mathbb{R}^3$, then $(G, p)$ is an infinitesimally rigid molecular framework if and only if $(G^2, p)$ is an infinitesimally rigid bar-and-joint framework. Combining this result with Theorem 8.1 we obtain:

**Theorem 8.2.** *Let $G$ be a graph. If $2G$ has three edge-disjoint spanning trees then every generic bar-and-joint realization of $G^2$ in $\mathbb{R}^3$ is infinitesimally rigid.*

Note that Corollary 6.3 and Conjecture 1.1 would imply that every generic bar-and-joint realization of $G^2$ in $\mathbb{R}^3$ is infinitesimally rigid whenever $5G$ has six edge-disjoint spanning trees.

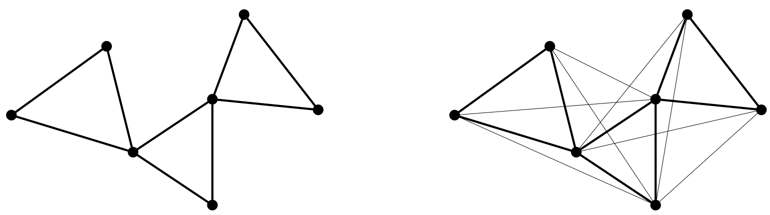

Figure 2: A graph $G$ and its square. Since $2G$ has three edge-disjoint spanning trees, Theorem 8.2 implies that every generic bar-and-joint realization of $G^2$ in $\mathbb{R}^3$ is infinitesimally rigid.

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# A Appendix

We present proofs of the lemmas contained in Section 2.

## Infinitesimal rotations

We need the following elementary result.

**Lemma A.1.** *Let $A$ be an affine subspace of $\mathbb{R}^d$ spanned by points $p_1, p_2, \dots, p_m$ and $x \in \mathbb{R}^d$. Then $x \in A$ if and only if $(x, 1)$ is in the subspace of $\mathbb{R}^{d+1}$ spanned by $(p_1, 1), (p_2, 1), \dots, (p_m, 1)$.*

Henceforth we assume that $A$ is a $(d-2)$-dimensional affine subspace of $\mathbb{R}^d$ spanned by points $p_1, p_2, \dots, p_{d-1}$. For $p \in \mathbb{R}^d$, let $S(A), M_{A,p}, v_{A,p}, v^*_{x,p}$ be as defined in Subsection 2.1. For $p' \in \mathbb{R}^d$ let $M_{A,p,p'}$ be the $(d+1) \times (d+1)$-matrix obtained from $M_{A,p}$ by adding $(p', 1)$ as a new row.

**Lemma A.2.** *(a)* $\det M_{A,p,p'} = (p', 1) \cdot v_{A,p}$.
*(b)* $v_{A,p} = (v^*_{A,p}, -v^*_{A,p} \cdot p)$.
*(c) Suppose $p \notin A$ and let $H$ be the hyperplane in $\mathbb{R}^d$ spanned by $A$ and $p$. Then $p' \in H$ if and only if $(p' - p) \cdot v^*_{A,p} = 0$.*

**Proof:** (a) This follows by expanding $\det M_{A,p,p'}$ along its last row.
(b) Clearly $\det M_{A,p,p} = 0$. By (a), $(p, 1) \cdot v_{A,p} = 0$. Thus $v_{d+1} = -v^*_{A,p} \cdot p$.
(c) Using Lemma A.1 and (a) we have

$$p' \in H \Leftrightarrow \det M_{A,p,p'} = 0 \Leftrightarrow (p', 1) \cdot v_{A,p} = 0.$$

By (b),

$$(p', 1) \cdot v_{A,p} = (p', 1) \cdot (v^*_{A,p}, -v^*_{A,p} \cdot p) = p' \cdot v^*_{A,p} - v^*_{A,p} \cdot p = (p' - p) \cdot v^*_{A,p}.$$

Hence $p' \in H$ if and only if $(p' - p) \cdot v^*_{A,p} = 0$. •

Lemma A.2(b) gives the first part of Lemma 2.1. Lemma A.2(c) implies that the vector $v^*_{A,p}$ is normal to the hyperplane $H$ spanned by $A$ and $p$ when $p \notin A$. We next show that the magnitude of $v^*_{A,p}$, $\|v^*_{A,p}\|$, is proportional to the Euclidean distance from $p$ to $A$, $dist(p, A)$.

**Lemma A.3.** *There exists a constant $\lambda \in \mathbb{R}$ such that $||v^*_{A,p}|| = \lambda\, dist(p, A)$, for all $p \in \mathbb{R}^d$.*

**Proof:** Let $u_i = p_i - p_1$ for $2 \le i \le d-1$ and $w = p - p_1$. Put $U = \langle\{u_2, u_3, \ldots, u_{d-1}\}\rangle$. Then $w$ can be uniquely expressed as $w = u + u^\perp$ where $u \in U$ and $u^\perp \in U^\perp$. Then $dist(p, A) = \|u^\perp\|$. Let $M^*_{A,p}$ be the matrix obtained from $M_{A,p}$ by adding $(v^*_{A,p}, 0)$ as a new row. By Lemma A.2(a),

$$|\det M^*_{A,p}| = v^*_{A,p} \cdot v^*_{A,p} = \|v^*_{A,p}\|^2. \tag{4}$$

On the other hand

$$\det M^*_{A,p} = \det \begin{pmatrix} p_1 & 1 \\ u_2 & 0 \\ u_3 & 0 \\ \vdots & \vdots \\ u_{d-1} & 0 \\ w & 0 \\ v^*_{A,p} & 0 \end{pmatrix} = \pm \det \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ u_{d-1} \\ u^\perp \\ v^*_{A,p} \end{pmatrix} \tag{5}$$

The vector $u^\perp$ is orthogonal to all vectors in $U$ by definition. The fact that $v^*_{A,p}$ is orthogonal to the hyperplane spanned by $A$ and $p$ by Lemma A.2(b) implies that $v^*_{A,p}$ is also orthogonal to $u^\perp$ and to all vectors in $U$. We may use the Gram-Schmidt orthogonalization process to construct an orthogonal basis for $U$, $u'_2, u'_3, \ldots, u'_{d-1}$ in such a way that $u'_i = u_i + \sum_{j<i} \lambda_{i,j} u_j$ for some scalars $\lambda_{i,j}$ and all $2 \le i \le d-1$. Let $M$ be the matrix with rows $u'_2, u'_3, \ldots, u'_{d-1}, u^\perp, v^*_{A,p}$. Then

$$\det \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ u_{d-1} \\ u^\perp \\ v^*_{A,p} \end{pmatrix} = \det M \tag{6}$$

Since the rows of $M$ are pairwise orthogonal,

$$(\det M)^2 = \det MM^T = \|u'_2\|^2 \|u'_3\|^2 \ldots \|u'_{d-1}\|^2 \|u^\perp\|^2 \|v^*_{A,p}\|^2.$$

Now (4), (5) and (6), imply that

$$\|v^*_{A,p}\|^4 = |\det M^*_{A,p}|^2 = |\det M|^2 = \|u'_2\|^2 \|u'_3\|^2 \ldots \|u'_{d-1}\|^2 \|u^\perp\|^2 \|v^*_{A,p}\|^2.$$

Since our choice of $u'_2, u'_3, \ldots, u'_{d-1}$ is independent of the choice of $p$, we have $\|v^*_{A,p}\| = \lambda \|u^\perp\| = \lambda\, dist(p, A)$, where $\lambda = \|u'_2\| \|u'_3\| \ldots \|u'_{d-1}\|$ is independent of the choice of $p$.

•

Lemmas A.2 and A.3 imply that $v^*_{A,p}$ has direction orthogonal to the hyperplane spanned by $A$ and $p$, and magnitude proportional to the distance of $p$ from $A$. This, and the fact $v^*_{A,p}$ varies continuously with $p$ gives the second part of Lemma 2.1.

Now suppose that $p, p' \in \mathbb{R}^d$ and that $M_{p,p'}$ and $T(p,p')$ are as defined at the end of Subsection 2.1. Expanding $\det M_{A,p,p'}$ along its last row and using the fact that $v_{A,p} = (v^*_{A,p}, -v^*_{A,p} \cdot p)$ as in the proof of Lemma A.2(b), we obtain

$$\det M_{A,p,p'} = v^*_{A,p} \cdot p' - v^*_{A,p} \cdot p. \tag{7}$$

We can also evaluate $\det M_{A,p,p'}$ by a Laplace expansion on its last two rows using $M_{p,p'}$. This gives

$$\det M_{A,p,p'} = \sum_{1 \le i < j \le D} s_{i,j} t_{i,j} = S(A) \cdot T(p,p'). \tag{8}$$

Combining (7) and (8) we obtain Lemma 2.2.

Finally, to prove Lemma 2.3, we suppose that the line spanned by $p, p'$ has a non-empty intersection with $A$. Choose $S \in \langle S(A) \rangle$. Then $S = \lambda S(A)$ for some $\lambda \in \mathbb{R}$. Now (8) implies that $S \cdot T(p,p') = \lambda \det M_{A,p,p'}$. Since the line spanned by $p, p'$ has a non-empty intersection with $A$, the affine subspace of $\mathbb{R}^d$ spanned by $A \cup \{p\}$ contains $p'$. This implies that $\det M_{A,p,p'} = 0$ by Lemma A.1 and hence $S \cdot T(p,p') = 0$. Thus Lemma 2.3 holds.

## Infinitesimal translations

We assume that $x \in \mathbb{R}^d$ and that $x_1, x_2, \ldots, x_d$ is a basis for $\langle x \rangle^\perp$. For $p \in \mathbb{R}^d$, let $S(x), M_{x,p}, v_{x,p}, v^*_{x,p}$ be as defined in Subsection 2.2. For $p' \in \mathbb{R}^d$ let $M_{A,p,p'}$, and $M^*_{A,p,p'}$, be the $(d+1) \times (d+1)$-matrices obtained from $M_{A,p}$ by adding $(p', 1)$, and $(p', 0)$, respectively, as a new row. It is easy to see that $\det M^*_{x,p,p'} = v^*_{x,p} \cdot p'$. Since $\det M^*_{x,p,x_i} = 0$ for all $1 \le i \le d-1$, $v^*_{x,p}$ is orthogonal to every vector in the orthogonal complement of $\langle x \rangle$, and hence $v^*_{x,p} = \lambda_p x$ for some scalar $\lambda_p$. Since $\det M_{x,p,p} = 0$, the last coordinate of $v_{x,p}$ is $-v^*_{x,p} \cdot p$. By considering $\det M^*_{x,p,v^*_{x,p}} = \|v^*_{x,p}\|^2$ and using a similar argument to that in the proof of Lemma A.3, we may deduce that $\|v^*_{x,p}\|$ is independent of the choice of $p$. Since $v^*_{x,p}$ changes continuously with $p$ we have $v^*_{x,p} = \lambda x$ for some constant $\lambda$ and hence $v_{x,p} = \lambda(x, -x \cdot p)$. This gives Lemma 2.4.

Expanding $\det M_{x,p,p'}$ along the last row and using the fact that $v_{x,p} = \lambda(x, -x \cdot p)$ by Lemma 2.4, we obtain

$$\det M_{x,p,p'} = v^*_{x,p} \cdot p' - v^*_{x,p} \cdot p, \tag{9}$$

where $v^*_{x,p} = \lambda x$. We can also evaluate $\det M_{x,p,p'}$ by a Laplace expansion on its last two rows. This gives

$$\det M_{x,p,p'} = \sum_{1 \le i < j \le \binom{d}{2}} s_{i,j} t_{i,j} = S(x) \cdot T(p,p'), \tag{10}$$

where $t_{i,j}$ and $T(p,p')$ are as defined at the end of Subsection 2.1

Combining (9) and (10) we obtain Lemma 2.5.