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A result on crossing families of odd sets

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Abstract

The following question is answered: given a crossing family $\mathcal{F}$ of odd subsets of an even-sized ground set V, what is the condition of the existence of a pairing M of the elements of V for which $d_M(X) = 1$ for every $X \in \mathcal{F}$? We show that the pairing exists if and only if $\mathcal{F}$ does not have a specific configuration of 4 sets. We present a consequence related to the conjecture of Woodall on dijoins.

1 Introduction

Let V be a ground set of even cardinality. Two sets $X \subseteq V$ and $Y \subseteq V$ are said to be *crossing* if the sets $X - Y, Y - X, X \cap Y, V - (X \cup Y)$ are all non-empty. A family $\mathcal{F}$ of subsets of V is called *crossing* if the intersection and union of any two crossing members of $\mathcal{F}$ are also in $\mathcal{F}$.

A *pairing* M of V is a set of unordered pairs of elements of V so that every element appears in exactly one pair. For a pairing M of V and $X \subseteq V$ let $d_M(X)$ denote the number of pairs in M with exactly 1 element in X. The problem addressed in this paper is to characterize families $\mathcal{F}$ of subsets of V for which there exists a pairing M such that $d_M(X) = 1$ for every $X \in \mathcal{F}$ (such a pairing is called a *feasible pairing*).

A set is called even (odd) if its cardinality is even (odd). If there is a feasible pairing for $\mathcal{F}$ then obviously every set in $\mathcal{F}$ is odd. We say that a family $\mathcal{F}$ of odd sets contains a *bad configuration* if it contains four sets with the following properties:

- The intersection of any 3 sets is empty,
- The union of any 3 sets is V,
- The intersection of any 2 sets is odd.

If $\mathcal{F}$ contains a bad configuration $\{X_1, X_2, X_3, X_4\}$, then there is no pairing M with $d_M(X) = 1$ for every $X \in \mathcal{F}$. To see this, consider $(u, v) \in M$ with $u \in X_1$ and $v \notin X_1$. There is an index $i \in \{2, 3, 4\}$ such that $u \in X_1 - X_i$ and $v \in X_i - X_1$, which implies that $d_M(X_1) + d_M(X_i) \geq 2 + d_M(X_1 \cap X_i) \geq 3$, where the last inequality holds because $X_1 \cap X_i$ is odd.

Our main result is that if $\mathcal{F}$ is a crossing family then this is the only possible obstacle for the existence of a pairing.

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Theorem 1.1. *Let $\mathcal{F}$ be a crossing family of odd subsets of V that contains no bad configuration. Then V has a pairing M such that $d_M(X) = 1$ for every $X \in \mathcal{F}$.*

This result is somewhat related to the odd-vertex pairing theorem of Nash-Williams [1], which can be stated in the following form. Let $G = (V, E)$ be an undirected graph, and let $p : 2^V \to \mathbb{Z}_+$ be a symmetric skew supermodular set function with even values, such that $p(X \cup Y) \leq \max\{p(X), p(Y)\}$ for every $X, Y \subseteq V$. If $p(X) \leq d_G(X)$ for every $X \subseteq V$, then there is a pairing M of the odd-degree nodes of G such that $d_M(X) \leq d_G(X) - p(X)$ for every $X \subseteq V$. It would be interesting to find a common generalization of these two results.

We prove Theorem 1.1 in Section 2. A corollary related to Woodall's conjecture on dijoins and a conjecture on a possible generalization are presented in Section 3. The theorem can also be stated in a more general (but essentially equivalent) form that includes a crossing family of even sets where $d_M(X) = 0$ is required.

Theorem 1.2. *Let $\mathcal{F}_0$ be a crossing family of even sets and let $\mathcal{F}_1$ be a crossing family of odd sets such that $\mathcal{F}_0 \cup \mathcal{F}_1$ is also crossing and $\mathcal{F}_1$ contains no bad configuration. Then V has a pairing M for which $d_M(X) = 0$ for every $X \in \mathcal{F}_0$ and $d_M(X) = 1$ for every $X \in \mathcal{F}_1$.*

We show that Theorem 1.2 follows from Theorem 1.1. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be families with the properties in Theorem 1.2. We use induction on $|\mathcal{F}_0|$; if $\mathcal{F}_0 \subseteq \{\emptyset, V\}$ then we are done, otherwise let X_0 be a nonempty proper subset of V in $\mathcal{F}_0$. We construct two new instances of the problem, $(V', \mathcal{F}'_0, \mathcal{F}'_1)$ and $(V'', \mathcal{F}''_0, \mathcal{F}''_1)$:

$$
\begin{aligned}
&V' := V \cap X_0,\\
&\mathcal{F}'_0 := \{X \in \mathcal{F}_0 : \ X \subsetneq X_0\} \cup \{X \subseteq X_0 : X \cup (V - X_0) \in \mathcal{F}_0\},\\
&\mathcal{F}'_1 := \{X \in \mathcal{F}_1 : \ X \subseteq X_0\} \cup \{X \subseteq X_0 : X \cup (V - X_0) \in \mathcal{F}_1\},\\
&V'' := V - X_0,\\
&\mathcal{F}''_0 := \{X \in \mathcal{F}_0 : \ X \subseteq V - X_0\} \cup \{\emptyset \neq X \subseteq V - X_0 : X \cup X_0 \in \mathcal{F}_0\},\\
&\mathcal{F}''_1 := \{X \in \mathcal{F}_1 : \ X \subseteq V - X_0\} \cup \{X \subseteq V - X_0 : X \cup X_0 \in \mathcal{F}_1\}.
\end{aligned}
$$

It is easy to see that $\mathcal{F}'_0, \mathcal{F}'_1, \mathcal{F}'_0 \cup \mathcal{F}'_1$, and $\mathcal{F}''_0, \mathcal{F}''_1, \mathcal{F}''_0 \cup \mathcal{F}''_1$ are all crossing families on V' and V'' respectively. Suppose that $\mathcal{F}'_1$ contains a bad configuration $\{X'_1, X'_2, X'_3, X'_4\}$, and the corresponding sets in $\mathcal{F}_1$ are $\{X_1, X_2, X_3, X_4\}$. If $X_i = X'_i$ for at least 3 sets, then the union of these 3 sets is an even set in $\mathcal{F}_1$, which is impossible. If $X_i \neq X'_i$ for at least 3 sets, then the intersection of these 3 sets is an even set in $\mathcal{F}_1$, again impossible. So $X_i = X'_i$ for exactly 2 sets, but this means that $\{X_1, X_2, X_3, X_4\}$ would be a bad configuration in $\mathcal{F}_1$. A similar argument shows that $\mathcal{F}''_1$ cannot contain a bad configuration.

Since $\max\{|\mathcal{F}'_0|, |\mathcal{F}''_0|\} \leq |\mathcal{F}_0|$, there is a feasible pairing M' on V' and a feasible pairing M'' on V''. Their union, $M = M' \cup M''$ is a pairing on V. If a set $X \in \mathcal{F}_0$ ($X \in \mathcal{F}_1$) does not cross X_0, then $d_M(X) = 0$ ($d_M(X) = 1$) by the construction. If $X \in \mathcal{F}_0$ crosses X_0, then $X \cap X_0 \in \mathcal{F}'_0$ and $X - X_0 \in \mathcal{F}''_0$, so $d_M(X) = d_{M'}(X \cap X_0) + d_{M''}(X - X_0) = 0$. If $X \in \mathcal{F}_1$ crosses X_0, and $X \cap X_0$ is even (odd), then $X \cap X_0 \in \mathcal{F}'_0$ and $X - X_0 \in \mathcal{F}''_1$ ($X \cap X_0 \in \mathcal{F}'_1$ and $X - X_0 \in \mathcal{F}''_0$), so $d_M(X) = d_{M'}(X \cap X_0) + d_{M''}(X - X_0) = 1$. It follows that M is a feasible pairing on V.

2 Proof of the main theorem

Let $\mathcal{F}$ be a crossing family of odd subsets of V that contains no bad configuration. The argument at the end of the previous section shows that a minimal counterexample for Theorem 1.2 is in fact a counterexample for Theorem 1.1. Therefore we may assume that Theorem 1.2 holds on any ground set smaller than V. The following lemma is the key ingredient of the proof.

Lemma 2.1. *For every $s \in V$ there exists $t \in V$ so that there are no crossing members X, Y of $\mathcal{F}$ with $s \in X \subseteq V - t$ and $t \in Y \subseteq V - s$.*

Proof. Let s be an arbitrary element of V. An ordered pair (X, Y) of crossing members of $\mathcal{F}$ is called a *relevant pair* if $s \in X$ and $s \notin Y$. The *forbidden set* corresponding to the pair (X, Y) is the set $Y - X$. The statement of the lemma is equivalent to the claim that $V - s$ cannot be covered by forbidden sets.

We denote by U the set of elements of V that are covered by some forbidden set. Let $\mathcal{P}$ be a family of relevant pairs whose forbidden sets cover U, it is of minimal size with this property, and it has the following two additional properties:

- every forbidden set corresponding to a pair in $\mathcal{P}$ is maximal (it is not a proper subset of another forbidden set),
- if $(X, Y) \in \mathcal{P}$, $X' \subseteq X$, $Y' \supseteq Y$, and one of the inclusions is proper, then (X', Y') is not a relevant pair.

Clearly there is a family $\mathcal{P}$ with these properties, and we fix such a family. Our aim is to analyze the structure of $\mathcal{P}$ to show that $U \neq V - s$.

In describing the structure we will frequently use a *symmetry argument* that goes as follows. An equivalent instance of the problem can be defined by taking the complement of each set in $\mathcal{F}$. The obtained family, which we now denote by $\mathcal{F}^*$, is crossing, it is composed of odd sets, and it contains no bad configuration since the complement of a bad configuration is also a bad configuration. Moreover, the complement of each relevant pair in $\mathcal{F}$ is a relevant pair in $\mathcal{F}^*$ (where the complement of (X, Y) is $(V - Y, V - X)$), and they define the same forbidden set, so we may assume that $\mathcal{P}^*$ consists of the complements of the pairs in $\mathcal{P}$. This means that for any possible configuration of pairs in $\mathcal{P}$ the configuration obtained by complementing the pairs is also possible.

Most of the proof is concerned with unordered pairs of relevant pairs in $\mathcal{P}$. A pair of pairs in $\mathcal{P}$ (or *popp* for short) is called *nice* if the forbidden sets of the two pairs are disjoint and it is called *ugly* otherwise (we assume that the two pairs are not identical in a popp). The following claim is a characterization of the possible structures a popp can have.

Claim 2.2. *If $((X_1, Y_1), (X_2, Y_2))$ is a nice popp, then one of the following holds:*

1. *$X_1 = X_2$, $Y_1 \cap Y_2 = \emptyset$,*
2. *$X_1 \supsetneq X_2$, $Y_1 \cap (X_2 \cup Y_2) = \emptyset$ (or same with reversed index),*

3. *X_1 and X_2 are crossing, $Y_1 \cap (X_2 \cup Y_2) = \emptyset$, and $Y_2 \cap (X_1 \cup Y_1) = \emptyset$,*

4. *$X_1 \cup X_2 = V$, $Y_1 \cap Y_2 = \emptyset$,*

5. *$Y_1 = Y_2$, $X_1 \cup X_2 = V$,*

6. *$Y_1 \supsetneq Y_2$, $X_2 \cup (X_1 \cap Y_1) = V$ (or same with reversed index),*

7. *Y_1 and Y_2 are crossing, $X_1 \cup (X_2 \cap Y_2) = V$, and $X_2 \cup (X_1 \cap Y_1) = V$.*

If $((X_1, Y_1), (X_2, Y_2))$ is an ugly popp, then one of the following holds:

$(u1)$ *X_1 and X_2 are crossing, $Y_1 = Y_2 =: Y$, $X_1 \cup X_2 \cup Y = V$, $X_1 \cap X_2 \cap Y = \emptyset$,*

$(u2)$ *Y_1 and Y_2 are crossing, $X_1 = X_2 =: X$, $X \cup Y_1 \cup Y_2 = V$, $X \cap Y_1 \cap Y_2 = \emptyset$.*

Proof. First, let us consider the case $Y_1 \cap Y_2 = \emptyset$. The case when $X_1 = X_2$ or $X_1 \cup X_2 = V$ appears in the claim, so we assume that these do not hold.

- If $X_1 \supsetneq X_2$, then $Y_1 \cap X_2 = \emptyset$, otherwise (X_1, Y_1) should be replaced by (X_2, Y_1) in $\mathcal{P}$.

- If X_1 and X_2 are crossing, then $Y_1 \cap (X_1 \cap X_2) = \emptyset$, otherwise (X_1, Y_1) should be replaced by $(X_1 \cap X_2, Y_1)$ in $\mathcal{P}$. Suppose that $Y_1 \cap X_2 \neq \emptyset$. Then $Y_1 \cap X_2$ is odd, so $Y_1 \cap (X_1 \cup X_2)$ is even, hence $Y_1 \cup (X_1 \cup X_2) = V$; similarly, $Y_2 \cup (X_1 \cup X_2) = V$, but this is impossible since $X_1 \cup X_2 \neq V$ and $Y_1 \cap Y_2 = \emptyset$.

Next, we consider the case when $X_1 \cup X_2 = V$. By the symmetry argument, the possible popps in this case are exactly the complements of the possible popps when $Y_1 \cap Y_2 = \emptyset$.

Finally, we consider the case when $X_1 \cup X_2 \neq V$ and $Y_1 \cap Y_2 \neq \emptyset$. We show that either $(u1)$ or $(u2)$ holds. Since $(X_1 \cap X_2, Y_1 \cup Y_2)$ is not a relevant pair (otherwise it would replace either (X_1, Y_1) or (X_2, Y_2) in $\mathcal{P}$), we have $(X_1 \cap X_2) \cap (Y_1 \cup Y_2) = \emptyset$ or $(X_1 \cap X_2) \cup (Y_1 \cup Y_2) = V$.

- First suppose that $(X_1 \cap X_2) \cap (Y_1 \cup Y_2) = \emptyset$ and $(X_1 \cap X_2) \cup (Y_1 \cup Y_2) \neq V$. We may assume that $X_1 \cup (Y_1 \cup Y_2) \neq V$. Since $X_1 \cup (Y_1 \cup Y_2)$ is odd and $(X_1 \cap X_2) \cup (Y_1 \cup Y_2)$ is even, we have $X_2 \cup (Y_1 \cup Y_2) \neq V$. Thus $(X_i, Y_1 \cup Y_2)$ is a relevant pair $(i = 1, 2)$, so $Y_1 = Y_2$ by the choice of $\mathcal{P}$, and this is case $(u1)$.

- Suppose that $(X_1 \cap X_2) \cap (Y_1 \cup Y_2) \neq \emptyset$ and $(X_1 \cap X_2) \cup (Y_1 \cup Y_2) = V$. By the symmetry argument, this corresponds to case $(u2)$.

- Finally, let us consider the case when $(X_1 \cap X_2) \cap (Y_1 \cup Y_2) = \emptyset$ and $(X_1 \cap X_2) \cup (Y_1 \cup Y_2) = V$. Suppose that $Y_2 - (X_1 \cup Y_1 \cup X_2) \neq \emptyset$. Then $(X_1 \cup Y_1) \cap X_2$ is odd, so $Y_1 \cap X_2$ is even, which means that $Y_1 \cap X_2$ must be empty since $Y_1 \cup X_2 \neq V$. But then either X_2 or $X_2 \cap Y_2$ is even, which is impossible. We obtained that $Y_2 - (X_1 \cup Y_1 \cup X_2) = \emptyset$. A similar argument shows that $Y_1 - (X_2 \cup Y_2 \cup X_1)$, $(X_1 \cap Y_1 \cap Y_2) - X_2$, and $(X_2 \cap Y_2 \cap Y_1) - X_1$ are all empty. However, in this case $\{X_1, Y_1, X_2, Y_2\}$ would be a bad configuration in $\mathcal{F}$.

We have verified all possible cases, so the claim is proven. We remark that the last case is the only part of the proof of Theorem 1.1 where we use the fact that there is no bad configuration in $\mathcal{F}$. □

The following three claims deal with the relation between two ugly popps.

Claim 2.3. *If $((X_1, Y_1), (X_2, Y_2))$ is an ugly popp, then (X_1, Y_1) does not appear in any other ugly popp.*

Proof. Suppose that $((X_1, Y_1), (X_2, Y_2))$ is a $(u1)$-popp, so $Y_1 = Y_2 =: Y$ (the proof for $(u2)$ follows by the symmetry argument). The pair (X_1, Y) does not appear in another $(u1)$-popp because the forbidden set of the other pair in the popp would be a subset of Y, and Y is already covered by $Y - X_1$ and $Y - X_2$. Suppose that (X_1, Y) forms a $(u2)$-popp with (X_1, Y'). Then the popp $((X_2, Y), (X_1, Y'))$ cannot belong to any of the classes in Claim 2.2 since X_1 and X_2 are crossing and Y and Y' are crossing. □

Claim 2.4. *If $((X_1, Y), (X_2, Y))$ and $((X'_1, Y'), (X'_2, Y'))$ are $(u1)$-popps, then $Y \cap Y' = \emptyset$. If $((X, Y_1), (X, Y_2))$ and $((X', Y'_1), (X', Y'_2))$ are $(u2)$-popps, then $X \cup X' = V$.*

Proof. We prove the claim for $(u1)$; it follows for $(u2)$ by the symmetry argument. By Claim 2.3, $((X_i, Y), (X'_j, Y'))$ is a nice popp for every $(i, j) \in \{1, 2\}^2$. Thus either $Y \cap Y' = \emptyset$ or $X_i \cup X'_j = V$ for every $(i, j) \in \{1, 2\}^2$, but in the latter case $Y \subseteq V - (X_1 \cap X_2) \subseteq X'_1 \cap X'_2 \subseteq V - Y'$, so $Y \cap Y' = \emptyset$ anyway. □

Claim 2.5. *If $((X_1, Y), (X_2, Y))$ is a $(u1)$-popp and $((X, Y_1), (X, Y_2))$ is a $(u2)$-popp, then $V - X \subseteq X_1 \cap X_2$ or $Y \subseteq V - (Y_1 \cup Y_2)$.*

Proof. By Claim 2.3, $((X, Y_i), (X_j, Y))$ is a nice popp for every $(i, j) \in \{1, 2\}^2$. Therefore $X \cup X_j = V$ or $Y_i \cap Y = \emptyset$ holds for every $(i, j) \in \{1, 2\}^2$. This means that if $X \cup (X_1 \cap X_2) \neq V$, then $Y \cap (Y_1 \cup Y_2) = \emptyset$. □

To prove the lemma, we have to show that the forbidden sets of the pairs in $\mathcal{P}$ cannot cover $V - s$. This follows from a simple parity argument if every popp is nice: in that case, the forbidden sets of the pairs in $\mathcal{P}$ are pairwise disjoint and they are all even, so their union is an even set, while $V - s$ is odd.

The situation is more complicated if there are ugly popps. However, Claims 2.4 and 2.5 imply that the union of the two forbidden sets of an ugly popp is disjoint from any other forbidden set of a member of $\mathcal{P}$.

We choose a *special ugly popp* which is minimal in the following sense. For each $(u1)$-popp $((X_1, Y), (X_2, Y))$ we consider the set $X_1 - X_2$ and for each $(u2)$-popp $((X, Y_1), (X, Y_2))$ we consider the set $Y_1 - Y_2$. We choose an ugly popp for which this set is minimal, i.e. no strict subset of it corresponds to another ugly popp.

Suppose first that the special popp is a $(u1)$-popp $((X_1, Y), (X_2, Y))$.

Claim 2.6. *The set $V - (X_2 \cup Y)$ is not covered by the forbidden sets of pairs in $\mathcal{P}$.*

Proof. We first show that any forbidden set $Y' - X'$ covers an even number of elements of $V - (X_2 \cup Y)$. By Claim 2.3, $((X_i, Y), (X', Y'))$ is a nice popp $(i = 1, 2)$, so either $X' \cup (X_1 \cap X_2) = V$ (in which case we are done) or $Y \cap Y' = \emptyset$. We may assume that $Y' \cap X_1 \neq \emptyset$ and $X' \cup (X_2 \cup Y) \neq V$. There are 2 possibilities for (X_1, Y_1) and (X', Y') according to Claim 2.2: $X' \cup X_1 = V$ or $X' \subseteq X_1$. In the first case $X' \supseteq X_2$ since they cannot be crossing by Claim 2.2, so $(Y' - X') - (X_2 \cup Y) = Y' - X'$ which is an even set. In the second case $(Y' - X') - (X_2 \cup Y) = Y' - (X' \cup X_2)$, again an even set.

We proved that each forbidden set covers an even number of elements from $V - (X_2 \cup Y)$ (an odd set). To complete the proof, we show that the union of the two forbidden sets corresponding to the two members of an ugly popp covers an even number of elements from $V - (X_2 \cup Y)$.

- First, consider a $(u1)$-popp $((X'_1, Y'), (X'_2, Y'))$. Claim 2.4 implies that $Y \cap Y' = \emptyset$. Suppose that $Y' - X_2 \neq \emptyset$; we show that $Y' - X_2$ is even.

  If $Y' \cap X_2 \neq \emptyset$, then $Y' - (X_2 \cup Y) = Y' - X_2$ is even, so we can assume that $Y' \cap X_2 = \emptyset$. By Claim 2.2 either $X'_i \subseteq X_1$ or $X'_i \cup X_1 = V$ $(i = 1, 2)$. It is not possible that $X'_1 \cup X'_2 \subseteq X_1$, since then $X'_1 \cup X'_2 \cup Y' \neq V$. First suppose that $(X'_1 \cap X'_2) \cup X_1 = V$. Then $X_2 \subseteq X'_1 \cap X'_2$ since X'_1 and X'_2 cannot cross X_2 by Claim 2.2, so $X'_1 - X'_2 \subsetneq X_1 - X_2$ which contradicts the choice of the special ugly popp. Now suppose that $X'_1 \subseteq X_1$ and $X'_2 \cup X_1 = V$. Then $X_2 \subseteq X'_2$ since X'_2 cannot cross X_2 by Claim 2.2, so $X'_1 - X'_2 \subsetneq X_1 - X_2$, which again contradicts the choice of the special ugly popp.

- Finally, consider a $(u2)$-popp $((X, Y_1), (X, Y_2))$. We claim that $V - X$ is disjoint from $V - (X_2 \cup Y)$. Clearly this is true if $V - X \subseteq X_1 \cap X_2$, so by Claim 2.5 we may assume that $Y \subseteq V - (Y_1 \cup Y_2)$. Then $Y \subseteq X$ since $X \cup Y_1 \cup Y_2 = V$. By Claim 2.3 $((X_2, Y), (X, Y_i))$ is a nice popp $(i = 1, 2)$, so by Claim 2.2 either $X_2 \cup X = V$ (in which case we are done) or $X_2 \subseteq X$. In the latter case Claim 2.2 states that $Y_i \cap X_2 = \emptyset$ and therefore $Y_i \subseteq X_1 - X_2$ $(i = 1, 2)$, but this contradicts the choice of the special ugly popp.

Since an odd set cannot be covered by disjoint even sets, the proof of the claim is complete. □

If the special popp is a $(u2)$-popp $((X, Y_1), (X, Y_2))$, then by the symmetry argument Claim 2.6 implies that the set $X \cap Y_1$ is not covered by the forbidden sets of the pairs in $\mathcal{P}$. We proved that there is always an element of $V - s$ that is not covered by forbidden sets. This completes the proof of the lemma. □

By Lemma 2.1 there is a pair of elements (s, t) for which there are no crossing members X, Y of $\mathcal{F}$ with $s \in X \subseteq V - t$ and $t \in Y \subseteq V - s$. We show that we can fix (s, t) to be in the pairing and reduce the size of the problem. We define the following instance of the problem in Theorem 1.2:

$$V' := V - \{s,t\},$$
$$\mathcal{F}'_0 := \{X - s : \ X \in \mathcal{F}, s \in X \subseteq V - t\} \cup \{X - t : \ X \in \mathcal{F}, t \in X \subseteq V - s\},$$
$$\mathcal{F}'_1 := \{X \in \mathcal{F} : \ X \subseteq V - \{s,t\}\} \cup \{X - \{s,t\} : \ X \in \mathcal{F}, \{s,t\} \subseteq X\}.$$

It is easy to see that $\mathcal{F}'_0$ consists of even sets and $\mathcal{F}'_1$ consists of odd sets. It is also straightforward that $\mathcal{F}'_1$ and $\mathcal{F}'_1 \cup \mathcal{F}'_2$ are crossing families: if X' and Y' are crossing in $\mathcal{F}'_1$ ($\mathcal{F}'_1 \cup \mathcal{F}'_2$), then the corresponding sets X and Y in $\mathcal{F}$ are crossing, and both $X \cap Y$ and $X \cup Y$ have corresponding sets in $\mathcal{F}'_1$ ($\mathcal{F}'_1 \cup \mathcal{F}'_2$).

Let X' and Y' be crossing sets in $\mathcal{F}'_0$, and let X and Y be the corresponding sets in $\mathcal{F}$. The property that determined the choice of s and t implies that either $s \in X \cap Y \subseteq X \cup Y \subseteq V - t$, or $t \in X \cap Y \subseteq X \cup Y \subseteq V - s$. In both cases, $X' \cap Y'$ and $X' \cup Y'$ are both in $\mathcal{F}'_0$.

The fact that $\mathcal{F}'_1$ does not contain a bad configuration follows the same way as it was proven at the end of Section 1. Since Theorem 1.2 holds on any ground set smaller than V, there is a pairing M' of V' so that $d_{M'}(X) = 0$ for every $X \in \mathcal{F}'_0$ and $d_{M'}(X) = 1$ for every $X \in \mathcal{F}'_1$. Let $M := M' + (s,t)$; then M is a pairing of V and the construction guarantees that $d_M(X) = 1$ for every $X \in \mathcal{F}$. This completes the proof of Theorem 1.2.

3 A corollary related to Woodall's conjecture

Let $D = (V,E)$ be a directed graph. An edge set $F \subseteq E$ is a *directed cut cover* if it contains at least one edge from every directed cut of D; it is a *directed cut k-cover* if it contains at least k edges from every directed cut.

Woodall [3, 4] conjectured that the maximum number of edge-disjoint directed cut covers in a digraph equals the minimum size of a directed cut. This conjecture is wide open, there are only a few classes of graphs for which it is known to hold (source-sink connected graphs, series-parallel graphs, transitive closure of directed trees).

A possible generalization would be that a directed cut k-cover always contains k edge-disjoint directed cut covers. However, Schrijver [2] showed that this is not true in general. His counterexample also shows that there is a directed graph $D = (V,E)$ and a directed cut 2-cover $F \subseteq E$ such that for any $F' \subseteq F$, there is a directed cut that contains at most 3 edges from F and is disjoint from either F' or $F - F'$.

In this section we show that for every $k \geq 2$, if $D = (V,E)$ is a directed graph and $F \subseteq E$ is a directed cut $2k$-cover, then there is exists $F' \subseteq F$ such that both F' and $F - F'$ are k-covers of the directed cuts that contain at most $2k+1$ edges from F. We conjecture that the following is also true: if $k \geq 2$, then any directed cut $2k$-cover can be partitioned into two directed cut k-covers.

Let $D = (V,E)$ be a directed graph and $F \subseteq E$ a subset of edges. We use the notation

$$\begin{aligned}
d_F^{in}(X) &:= |\{uv \in F: \ u \in V-X, v \in X\}, \\
d_F^{out}(X) &:= |\{uv \in F: \ u \in X, v \in V-X\}, \\
d_F(X) &:= d_F^{in}(X) + d_F^{out}(X).
\end{aligned}$$

Let us define the following families of sets:

$$\begin{aligned}
\mathcal{I} &:= \{\emptyset \neq X \subsetneq V: \ d_E^{out}(X) = 0\}, \\
\mathcal{I}_j &:= \{X \in \mathcal{I}: \ d_F(X) = j\}.
\end{aligned}$$

If X and Y are in $\mathcal{I}$, then $X \cap Y \in \mathcal{I} \cup \{\emptyset\}$, $X \cup Y \in \mathcal{I} \cup \{V\}$, and $d_F(X) + d_F(Y) = d_F(X \cap Y) + d_F(X \cup Y)$.

Theorem 3.1. *Let $D = (V, E)$ be a digraph, $k \geq 2$ an integer, and $F \subseteq E$ a directed cut $2k$-cover. Then there is a pairing M of the nodes with $d_F(v)$ odd such that*

$$\begin{aligned}
d_M(X) &= 0 \quad \text{if } X \in \mathcal{I}_{2k}, \\
d_M(X) &= 1 \quad \text{if } X \in \mathcal{I}_{2k+1}.
\end{aligned}$$

Proof. We prove the theorem by induction on $|V|$. Let T denote the set of nodes where $d_F(v)$ is odd. First, suppose that there is a set $X_0 \in \mathcal{I}_{2k}$ with $2 \leq |X_0| \leq |V| - 2$. Let $D' = (V', E')$ be the graph obtained by contracting $V - X_0$ to one node, let $D'' = (V'', E'')$ be the graph obtained by contracting X_0 to one node, and let F' and F'' be the edge sets obtained from F by the contraction. Then F' is a directed cut $2k$-cover of D' and F'' is a directed cut $2k$-cover of D''. By induction there is a pairing M' of $T \cap X_0$ and a pairing M'' of $T - X_0$ that satisfy the conditions of the theorem.

We can define a pairing M of T by $M := M' \cup M''$. If $X \in \mathcal{I}_{2k}$, then $d_M(X) = d_{M'}(X \cap X_0) + d_{M''}(X - X_0) = 0$. If $X \in \mathcal{I}_{2k+1}$, then there are four possibilities: (*i*) $X \cap X_0 = \emptyset$; (*ii*) $X \cup X_0 = V$; (*iii*) $X \cap X_0 \in \mathcal{I}_{2k+1}$ and $X \cup X_0 \in \mathcal{I}_{2k}$; (*iv*) $X \cap X_0 \in \mathcal{I}_{2k}$ and $X \cup X_0 \in \mathcal{I}_{2k+1}$. In all of these cases, $d_M(X) = d_{M'}(X \cap X_0) + d_{M''}(X - X_0) = 1$, so M satisfies the conditions of the theorem.

We may thus assume that $|X| = 1$ or $|V - X| = 1$ for every $X \in \mathcal{I}_{2k}$. Note that these sets are disjoint or co-disjoint from T. We define a family $\mathcal{F}$ of sets on the ground set T by

$$\mathcal{F} := \{X \subseteq T: \ \exists Z \subseteq V - T: \ X \cup Z \in \mathcal{I}_{2k+1}\}.$$

The family $\mathcal{F}$ consists of odd sets, since $d_F(X) \equiv \sum_{v \in X} d_F(v) \mod 2$ for every $X \subseteq V$. Moreover, $\mathcal{F}$ is crossing: if X and Y are crossing sets in $\mathcal{F}$, then there are sets $X' \in \mathcal{I}_{2k+1}$ and $Y' \in \mathcal{I}_{2k+1}$ such that $X' \cap T = X$ and $Y' \cap T = Y$. The sets $X' \cap Y'$ and $X' \cup Y'$ are not in $\mathcal{I}_{2k}$ since $X' \cap Y' \cap T \neq \emptyset$ and $X' \cup Y' \cup (V - T) \neq V$, so they are in $\mathcal{I}_{2k+1}$. This means that $X \cap Y$ and $X \cup Y$ are in $\mathcal{F}$.

Suppose that $\mathcal{F}$ contains a bad configuration $\{X_1, X_2, X_3, X_4\}$. There are sets $X_i' \in \mathcal{I}_{2k+1}$ $(i = 1, \ldots, 4)$ with $X_i' \cap T = X_i$. By the definition of a bad configuration,

$X'_i \cap X'_j \cap T \neq \emptyset$ and $X'_i \cup X'_j \cup (V - T) \neq V$, so $X'_i \cap X'_j$ and $X'_i \cup X'_j$ are in $\mathcal{I}_{2k+1}$. Again by the definition of a bad configuration and by the modularity of d_F on $\mathcal{I}$, the intersection of 3 sets X'_i must be in $\mathcal{I}_{2k} \cup \{\emptyset\}$ and the union of 3 sets X'_i must be in $\mathcal{I}_{2k} \cup \{V\}$. This means that, by the modularity of d_F on $\mathcal{I}$, $d_F(X'_1) \equiv d_F(X'_1 \cap X'_2) + d_F(X'_1 \cap X'_3) + d_F(X'_1 \cap X'_4) \mod 2k$. But this is impossible if $k \geq 2$ because these sets are all in $\mathcal{I}_{2k+1}$ (note that there is no contradiction if $k = 1$).

We showed that $\mathcal{F}$ satisfies the conditions of Theorem 1.1, so there is a pairing M of T with $d_M(X) = 1$ for every $X \in \mathcal{F}$. This means that M satisfies the conditions of the theorem. □

Corollary 3.2. *Let $D = (V, E)$ be a digraph, $k \geq 2$ an integer, and $F \subseteq E$ a directed cut $2k$-cover. Then there is an edge set $F' \subseteq F$ so that*

$$\min\{d_{F'}(X), d_{F-F'}(X)\} \geq k \quad \textit{for every } X \in \mathcal{I}_{2k} \cup \mathcal{I}_{2k+1}.$$

Proof. Let M be the pairing that exists according to Theorem 3.1, and let G be the Eulerian graph obtained by taking the union of M and the edges of F without orientation. We select an arbitrary Eulerian orientation $\vec{G}$ of G, and denote by F' the set of edges in F that have the same orientation in D and in $\vec{G}$. If $X \in \mathcal{I}_{2k}$, then $d_M(X) = 0$, so $d_{F'}(X) = d^{in}_{\vec{G}}(X) = k$. If $X \in \mathcal{I}_{2k+1}$, then $d_M(X) = 1$, so $d_{F'}(X) \leq d^{in}_{\vec{G}}(X) = k + 1$ and $d_{F'}(X) \geq d^{in}_{\vec{G}}(X) - d_M(X) = k$. □

Note that Corollary 3.2 is not true for $k = 1$, as it is shown by the counterexample of Schrijver [2]. It is an interesting open question whether Theorem 3.1 can be generalized by requiring $d_M(X) \leq d_F(X) - 2k$ for every $X \in \mathcal{I}$.

Conjecture 3.3. *Let $D = (V, E)$ be a digraph, $k \geq 2$ an integer, and $F \subseteq E$ a directed cut $2k$-cover. Then there is a pairing M of the nodes with $d_F(v)$ odd such that $d_M(X) \leq d_F(X) - 2k$ for every $X \subseteq V$ for which $d^{out}_E(X) = 0$.*

If true, this would imply the following relaxation of the capacitated version of Woodall's conjecture.

Conjecture 3.4. *Let $D = (V, E)$ be a digraph, $k \geq 2$ an integer, and $F \subseteq E$ a directed cut $2k$-cover. Then F can be partitioned into two directed cut k-covers.*

References

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