Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2008-10. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

A matroid intersection algorithm

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May 2008

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Abstract

We propose an algorithm for Rado's min-max formula, which, by a polynomial reduction, also implies a polynomial time algorithm for matroid intersection. The algorithm is different from Edmonds', since here we do not use alternating paths, and instead, we exploit contractions of the matroid. We remark that Iwata, Takazawa [2] constructed an independent even factor algorithm that, as a special case, implies a similar matroid intersection algorithm. Further related algorithms have been proposed by the author for matching problems, see [3].

In the matroid intersection problem we are given as an input two matroids on the same ground set, and the goal is to find a maximum cardinality common independent set. Computationally, we assume matroids are given by an independence testing oracle. Edmonds constructed a polynomial time algorithm to find a maximum common independent set and proved the following min-max formula. In this paper we propose a different algorithmic proof.

Theorem 0.1 (Edmonds, [1]). *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank-functions r_1, r_2. Then the maximum cardinality of a common independent set is equal to*

$$\min_{U \subseteq S} \left(r_1(U) + r_2(S - U) \right). \tag{1}$$

A notable special case of Edmonds' result is the following due to Rado. Consider a set S partitioned into a family of disjoint sets $\mathcal{S} = \{S_1, \cdots, S_n\}$. A **partial system of representatives** (with respect to $\mathcal{S}$) – **psr**, for short – is a set containing at most one element of any S_i. The family of psr's is the family of independent sets of a matroid, thus Rado's result is a special case of Edmonds'.

Theorem 0.2 (Rado, [4]). *Let $M = (S, \mathcal{I})$ be a matroid with rank-function r, and let $\mathcal{S}$ be a partition of the groundset S. The maximum cardinality of an independent psr is equal to*

$$\min_{\mathcal{Z} \subseteq \mathcal{S}} \left(r(\bigcup \mathcal{Z}) + |\mathcal{S}| - |\mathcal{Z}| \right). \tag{2}$$

⋆Research is supported by OTKA grant K60802 and by the MTA-ELTE Egerváry Research Group of the Hungarian Academy of Sciences.

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We solve the matroid intersection problem by first reducing it to that of finding a maximum independent psr, and then solving the maximum independent psr problem.

To show this reduction, consider an instance $M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$ of the matroid intersection problem. Let $S' := \{s' : s \in S\}$ be a disjoint copy of S, and let M_2' be the copy of M_2. Consider, as a new ground set, $S \cup S'$. On the new ground set, consider the direct sum $M := M_1^* \oplus M_2'$, and the partition $\mathcal{S} := \{\{s, s'\} : s \in S\}$. Let X' be a maximum independent psr X' with respect to $M, \mathcal{S}$. By Rado's Theorem $|X'| = |S - U| + r_1^*(U) + r_2(U)$ for some $U \subseteq S$. Here X' partitions into $X' = \{x : x \in X_1\} \cup \{x' : x \in X_2\}$ such that $X_1 \in \mathcal{I}_1^*$ and $X_2 \in \mathcal{I}_2$. Consider a base B of M_1 disjoint from X_1. Then $B \cap X_2$ is a common independent set of the two matroids, and the following calculation shows that it is a maximum common independent set: $|B \cap X_2| \geq |X_1| + |X_2| + |B| - |S| = \gamma + r_1(S) - |S| = |S - U| + r_1^*(U) + r_2(U) + r_1(S) - |S| = |S - U| + (|U| + r_1(S - U) - r_1(S)) + r_2(U) + r_1(S) - |S| = r_1(S - U) + r_2(U)$. Note that independence oracles for M_1 and M_2 imply an independence oracle for M, thus – assuming a polynomial time algorithm to find a maximum independent psr – the above reasoning implies a polynomial time algorithm to find a maximum common independent set.

To solve the maximum independent psr problem, consider a matroid $M = (S, \mathcal{I})$ with its groundset S partitioned into a family $\mathcal{S} = \{S_1, \cdots, S_n\}$ of disjoint sets. For technical reasons, we allow that some S_i's are empty, i.e. a couple of emptysets may be listed in $\mathcal{S}$. To solve the maximum independent psr problem, we construct an augmentation procedure, that is, given $M, \mathcal{S}$ as above, and an independent psr $T \in \mathcal{I}$, find an independent psr larger than T, or find a **certificate** $\mathcal{Z} \subseteq \mathcal{S}$ such that $|T| = r(\bigcup \mathcal{Z}) + |\mathcal{S}| - |\mathcal{Z}|$. To do this, we first check if T is a maximum psr, and if not, find an element s such that $T + s$ is a psr. If T is a maximum psr, then $\mathcal{Z} := \{S_i \in \mathcal{S} : S_i = \emptyset\}$ is a certificate, and we are done. Otherwise there is an element $s \notin T$ such that $T + s$ is a psr. If $T + s$ is independent, then we are done. Otherwise let $C = C(T, s)$ be the unique circuit in $T + s$. Let $S' := S - C$, $S_C := \bigcup \{S_i : S_i \cap C \neq \emptyset\} - C$, and $\mathcal{S}' := \{S_i : S_i \cap C = \emptyset\} \cup \{S_C\}$. Define $M' := M/C$, say with rank-function r'. Clearly, $T' := T - C \subseteq S'$ is an independent psr with respect to $\mathcal{S}', M'$. By applying the above procedure recursively for $M', \mathcal{S}'$ we find a larger independent psr, or a certificate $\mathcal{Z}'$. First, assume the former, say T'' is an independent psr with respect to $M', \mathcal{S}'$ that is larger than T'. By definition of $\mathcal{S}'$, there is an element $y \in C$ such that $T''' := T'' \cup (C - y)$ is a psr of $\mathcal{S}$. By the definition of M', T''' is an independent psr with respect to $M, \mathcal{S}$ that is larger than T. (The construction of T''' from T'' is called expansion.) Second, assume the latter, say $|T'| = r'(\bigcup \mathcal{Z}') + |\mathcal{S}'| - |\mathcal{Z}'|$ for some $\mathcal{Z}' \subseteq \mathcal{S}'$. Then $S_C \in \mathcal{Z}'$, since T' is disjoint from S_C. Define $\mathcal{Z} := \mathcal{Z}' - S_C \cup \{S_i : S_i \cap C \neq \emptyset\}$. Then$\bigcup \mathcal{Z} = \bigcup \mathcal{Z}' \cup C$. We get that $|T| = |T'| + |C| - 1 = r'(\bigcup \mathcal{Z}') + |\mathcal{S}'| - |\mathcal{Z}'| + |C| - 1 = (r(\bigcup \mathcal{Z}) - |C| + 1) + (|\mathcal{S}| - |C|) - (|\mathcal{Z}'| + |C|) + |C| - 1 = r(\bigcup \mathcal{Z}) + |\mathcal{S}| - |\mathcal{Z}|$. This proves that the algorithm returns a maximum indepentent psr T with a certificate $\mathcal{Z}$.

Let us now summarize the augmentation procedure described above to find a larger independent psr, or a certificate. We maintain a triple $M, \mathcal{S}, T$, where T is an independent psr with respect to $M, \mathcal{S}$. We repeat the following steps recursively. Check if

T is a maximum independent psr, and in that case, terminate saying T is maximum. Otherwise find s such that $T + s$ is a psr, and if it is independent, terminate. Otherwise construct $M', \mathcal{S}', T'$, and recursively apply the augmentation procedure with respect to the triple $M', \mathcal{S}', T'$. In either case, we obtain a solution for the original triple $M, \mathcal{S}, T$.

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