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An algorithm for weighted fractional matroid matching

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Abstract

Let M be a matroid on ground set E . A subset $l \subseteq E$ is called a *line* when $r(l) \in \{1, 2\}$. Given a set of lines $L = \{l_1, \dots, l_k\}$ in M , a vector $x \in \mathbb{R}_+^L$ is called a *fractional matching* when $\sum_{l \in L} x_l a(F)_l \leq r(F)$ for every flat F of M . Here $a(F)_l$ is equal to 0 when $l \cap F = \emptyset$, equal to 2 when $l \subseteq F$ and equal to 1 otherwise. We refer to $\sum_{l \in L} x_l$ as the *size* of x .

It was shown by Chang et al. that a maximum size fractional matching can be found in polynomial time. In this paper we give an efficient algorithm to find for given weight function $w : L \rightarrow \mathbb{Q}$ a *maximum weight* fractional matching.

Keywords: matroid, matroid matching, matroid parity, fractional matching, lattice polytope, algorithm.

1 Introduction

Let M be a matroid on the set E , and let $\{a_1, b_1\}, \dots, \{a_k, b_k\}$ be disjoint pairs of elements from E . A subset I of $\{1, \dots, k\}$ is called a *matching* if $\bigcup_{i \in I} \{a_i, b_i\}$ is an independent set in the matroid M . The problem of finding a maximum size matching is called the *matroid matching problem* and was proposed by Lawler [6] as a common generalization of non-bipartite matching and matroid intersection.

Lovász [7] and, independently, Korte and Jensen [5] showed that for general matroids given by an independence oracle, the matroid matching problem requires an exponential number of oracle calls, and hence is not solvable in polynomial time. Also, the problem of finding a maximum size clique in a graph can be formulated as a matroid matching problem, where the oracle is removed and independence can be read off from the input graph directly, showing that the matroid matching problem contains NP-hard problems (see [11]).

However, matroid matching has become a powerful tool in combinatorial optimization since Lovász [7] proved a min-max formula and constructed a polynomial time

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algorithm for matroid matching in representable matroids that are given by an explicit linear representation over a field. Already linear matroid matching has a wide range of applications, among which are a polynomial time algorithm for packing Mader paths [11], graph rigidity [8] and computing the maximum genus surface in which an input graph can be embedded [3]. More efficient algorithms for linear matroid matching have been developed by Gabow and Stallmann [4] and by Orlin and Vande Vate [9].

An outstanding problem concerning matroid matching is to construct an efficient algorithm for finding a *maximum weight* matching for linearly represented matroids.

To gain better understanding of the matroid matching polytope, Vande Vate [13] introduced a fractional relaxation of this polytope, called the *fractional matroid matching polytope*. The integer points of this polytope correspond exactly to the matroid matchings, and although the polytope itself is not integer, the vertices are half-integer. Furthermore, in the two extremes where the matroid matching is in fact a matroid intersection problem or a non-bipartite matching problem, the fractional matroid matching polytope coincides with the common independent set polytope and with the fractional matching polytope respectively.

In a series of two papers, Chang, Llewellyn and Vande Vate [2, 1] showed that there exists a polynomial time algorithm to optimize the all-one objective function over the fractional matroid matching polytope.

In the present paper, we consider optimizing arbitrary weight functions over the fractional matroid matching polytope. Extending the result from [13], we show that not only is the polytope half-integer, the system defining the polytope is in fact half-TDI. Our main result is the construction of a polynomial time algorithm for optimizing arbitrary weight functions over the fractional matroid matching polytope, showing that the polytope is algorithmically tractable.

2 Preliminaries

Let M be a matroid with groundset E and rankfunction r_M . Here we will assume that M does not have loops. The set E need not be finite, but we do require that M has finite rank. The span of $X \subseteq E$ is denoted by $\text{cl}_M(X)$, the smallest flat containing X . The set $\mathcal{L}(M)$ of flats of the matroid M form a lattice, with join $S \vee T := \text{cl}_M(S \cup T)$ and meet $S \wedge T := S \cap T$. The rank function r_M is submodular on $\mathcal{L}(M)$:

$$r_M(S) + r_M(T) \geq r_M(S \cup T) + r_M(S \cap T) = r_M(S \vee T) + r_M(S \wedge T) \quad \text{for all } S, T \in \mathcal{L}(M). \quad (1)$$

When the matroid is clear from the context, we will suppress the matroid in the notation. For further notation and preliminaries on matroids, we refer the reader to [10].

Let L be a finite set of *lines* of M , where a *line* is a subset of E of rank 1 or 2. We do not require the lines to be flats. For any subset $X \subseteq E$, we define its *degree vector*

$a(X) : L \rightarrow \{0, 1, 2\}$ by

$$a(X)_l := \begin{cases} 0 & \text{if } X \cap l = \emptyset, \\ 2 & \text{if } X \supseteq l, \\ 1 & \text{otherwise.} \end{cases} \quad (2)$$

It is important to observe that for every line $l \in L$ the function $a(\cdot)_l : \mathcal{L}(M) \rightarrow \{0, 1, 2\}$ is *supermodular*:

$$a(S)_l + a(T)_l \leq a(S \vee T)_l + a(S \wedge T)_l \quad \text{for all } l \in L \text{ and all } S, T \in \mathcal{L}(M). \quad (3)$$

This follows from simple case-checking. The only nontrivial case is when $a(S)_l = a(T)_l = 1$. In that case either $l \cap (S \cap T) \neq \emptyset$ so that $a(S \wedge T), a(S \vee T) \geq 1$ or $l \cap (S \setminus T), l \cap (T \setminus S) \neq \emptyset$ and hence $l \subseteq S \vee T$.

A vector $x \in \mathbb{R}^L$ is called a *fractional matching in (M, L)* when

$$\begin{aligned} x &\geq 0, \\ a(T) \cdot x &\leq r(T) \text{ for every flat } T \text{ of } M. \end{aligned} \quad (4)$$

The set of fractional matchings in (M, L) is called the *fractional matching polytope*. Although the number of flats may be infinite, the number of distinct degree-vectors $a(T)$ is finite and hence the fractional matching polytope is indeed a polytope.

We refer to $|x| := \sum_{l \in L} x_l$ as the *size* of x . The maximum size of a fractional matching in (M, L) is denoted by $\nu^*(M, L)$. The size of a fractional matching x is at most $\frac{1}{2}r(E)$, and when equality holds, x is called a *perfect* fractional matching.

In what follows, it will be convenient to work with finite formal sums of flats, that is, expressions of the form $y = \sum_{F \in \mathcal{L}(M)} \lambda_F F$, where $\lambda \in \mathbb{R}^{\mathcal{L}(M)}$ and $\lambda_F = 0$ for all but a finite number of flats F . We denote $y_F := \lambda_F$ and $\text{supp}(y) := \{F \mid y_F \neq 0\}$. We extend all functions on the $\mathcal{L}(M)$ linearly to finite formal sums and write $r(y) := \sum_F \lambda_F r(F)$ and $a(y) := \sum_F \lambda_F a(F)$.

Consider the program of optimizing a nonnegative weight function $w : L \rightarrow \mathbb{Z}_+$ over the fractional matching polytope:

$$\begin{aligned} &\text{maximize} && w \cdot x \\ &\text{subject to} && x \geq 0 \\ &&& a(T) \cdot x \leq r(T) \quad \text{for all } T \in \mathcal{L}(M). \end{aligned} \quad (5)$$

The dual linear program is given over vectors $y \in \mathbb{R}^{\mathcal{L}(M)}$ by

$$\begin{aligned} &\text{minimize} && r(y) \\ &\text{subject to} && y \geq 0 \\ &&& a(y) \geq w. \end{aligned} \quad (6)$$

It was shown in [13] that the fractional matching polytope is half-integer and that for $w = \mathbf{1}$, there is a half-integer optimal dual solution. Here we show that the system (5) is half-TDI. That is, for every $w : L \rightarrow \mathbb{Z}_+$, the dual has a half-integer optimal dual solution. This implies half-integrality of the fractional matching polytope (see [12]).

Theorem 2.1. *System (5), describing the fractional matching polytope, is half-TDI.*

Proof. We show that the dual linear program (6) has a half-integer solution. Let y be an optimal solution for which $f(y) := \sum_{F \in \mathcal{L}(M)} y_F r(F)^2$ is maximal. Such a y exists, because we may restrict to a finite number of flats, one representative for each degree-vector, and because the function $f(y)$ is bounded from above by $r(E) \cdot r(y)$.

The support $\mathcal{L}^+ := \text{supp}(y)$ of y is a chain. Indeed, suppose that $S, T \in \mathcal{L}^+$ and $S \not\subseteq T, T \not\subseteq S$. We may assume that $r(T) \geq r(S)$. Let $\epsilon := \min\{y_S, y_T\} > 0$, and define $y' := y + \epsilon(S \wedge T + S \vee T - S - T)$. Then y' is a feasible solution to (6) by supermodularity of the $a(\cdot)_l, l \in L$. By the submodularity of r , y' will be an optimal dual solution and $d := r(S \vee T) - r(T) = r(S) - r(S \wedge T) > 0$. It then follows that $f(y') - f(y) = 2\epsilon d \cdot (r(T) - r(S) + d) \geq 2\epsilon d \cdot d > 0$, contradicting the choice of y .

Denote $\mathcal{L}^+ = \{T_1, \dots, T_k\}$ where $T_1 \subset \dots \subset T_k$. Let $A \in \{0, 1, 2\}^{\{1, 2, \dots, k\} \times L}$ given by $A_{i,l} := a(T_i)_l$ be the matrix with the degree vectors of the T_i as rows. To prove that there exists a half-integral optimum y , it suffices to show that every non-singular square submatrix A' of A has a half-integer inverse. Since $A'^{-1} = (UA')^{-1}U$, where

$$U = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix},$$

it suffices to show that $D := UA'$ has a half-integer inverse. This follows from the following claim.

Claim 2.2. *Let $D \in \{0, 1, 2\}^{n \times n}$ be a nonsingular matrix with column sums at most 2, and let $b \in \mathbb{Z}^n$. Then the unique solution x of $Dx = b$ is half-integer.*

Proof of claim. The proof is by induction on n . If some entry x_i of x is integer, then by deleting column i and some row j of D , we obtain a nonsingular matrix D' , and we can apply induction. Hence we may assume that the coefficients of x are all non-integer. This implies that all column sums of D are equal to 2. If some entry $D_{i,j}$ of D equals 2, we can apply induction by deleting row i and column j of D . Hence we may assume that all columns of D contain two entries that equal 1. Thus D is the edge-node incidence matrix of a graph $G = (V, E)$, and x is the unique perfect fractional b -matching in G . If $d_G(v) = 1$ for some $v \in V$, say $vz \in E$, then $x(vz) = b(v) \in \mathbb{Z}$, contradicting our assumption. If all degrees are at least 2, then since $|V| = |E|$, G is the union of (odd) cycles. The half-integrality of x is then obvious. \square

\square

Let $S \subseteq T$ be flats in M . The sum $\frac{1}{2}(S+T)$ is called a *cover* of (M, L) if $a(S+T)_l \geq 2$ for every line $l \in L$. The cover is called *mimimum* if $r(\frac{1}{2}(S+T)) = \nu^*(M, L)$. So the minimum covers are the optimal solutions y with $\text{supp}(y)$ a chain, of the dual program (6) for $w = \mathbf{1}$.

When $\frac{1}{2}(S + T)$ and $\frac{1}{2}(S' + T')$ are two minimum covers, also $\frac{1}{2}(S \wedge S' + T \vee T')$ is a minimum cover (see [2]). It follows that there is a ‘canonical’ minimum cover $\frac{1}{2}(S^* + T^*)$ with the property that $S^* \subseteq S \subseteq T \subseteq T^*$ for every minimum cover $\frac{1}{2}(S + T)$. This cover is called the *dominant cover* in [2], where a characterization of the dominant cover was given in terms of the maximum size fractional matchings. The *closure* of a fractional matching x , denoted by $cl(x)$, is defined to be the smallest flat containing the lines in the support of x .

Theorem 2.3. [2] *Let $\frac{1}{2}(S^* + T^*)$ be the dominant cover of (M, L) . Then $T^* = \bigcap_x cl(x)$, where x runs over all maximum size fractional matchings in (M, L) .*

Theorem 2.4. [2] *If x is a vertex of the fractional matroid matching polytope, then the closure $cl(x)$ is a tight flat with respect to x , i.e. $|x| = r(cl(x))$ holds.*

The characterization of T^* in the Theorem 2.3 implies that

$$S^* = cl\left(\bigcup_{l \notin T^*} (T^* \cap l)\right), \quad (7)$$

since $S^* \subseteq T^*$ and $\frac{1}{2}(S^* + T^*)$ must cover each line $l \in L$.

Chang et al. [1] presented an algorithm that given the matroid M and the set of lines L , computes the dominant cover and a maximum size (extreme) fractional matching x in time polynomial in $|L|$ and $r(E)$. More precisely, the matroid M is given by a rank-oracle. To accomodate infinite matroids (of finite rank), and in particular full linear matroids, the groundset E and the lines L are not given explicitly. Rather, it suffices that there is an oracle that takes as input a line l and a flat F represented by a base of F , and outputs a base for $l \cap F$.

In this paper, we use this algorithm to construct an algorithm that finds, for given weights on the lines, a maximum weight fractional (perfect) matching and an optimal dual solution.

Our algorithm uses direct sums of minors of the matroid M . To make the above oracle available in those matroids, it will suffice that the oracle in addition to a base of $l \cap F$ also provides a base of $l \setminus F$.

3 A matroid operation

Given a chain $\mathcal{F} = \{F_1, \dots, F_k\}$ of flats with $F_1 \subset F_2 \cdots \subset F_k$, we define the matroid $M \star \mathcal{F}$ on the same ground set E by:

$$M \star \mathcal{F} := \bigoplus_{i=0}^k (M|_{F_{i+1}}/F_i), \quad (8)$$

where we set $F_0 := \emptyset$ and $F_{k+1} := E$. Observe that when M has no loops (as will always be assumed), also $M \star \mathcal{F}$ has no loops. The flats of $M \star \mathcal{F}$ are precisely the sets $S_0 \cup \dots \cup S_k$, with $S_i \subseteq F_{i+1} \setminus F_i$ and $S_i \cup F_i$ is a flat of M for every $i = 0, \dots, k$.

For $X \subseteq E$, we can write $X = X_0 \cup \dots \cup X_k$, where $X_i := X \cap (F_{i+1} \setminus F_i)$ and we take $F_0 := \emptyset$ and $F_{k+1} := E$. Using the submodularity of the rank function, we see that

$$\begin{aligned}
r_{M \star \mathcal{F}}(X) &= \sum_{i=0}^k r_{M \star \mathcal{F}}(X_i) \\
&= \sum_{i=0}^k (r(X_i \cup F_i) - r(F_i)) \\
&\leq \sum_{i=0}^k (r(X \cap F_{i+1}) - r(X \cap F_i)) \\
&= r(X).
\end{aligned} \tag{9}$$

We denote the *closure* of $X \subseteq E$ in the matroid $M \star \mathcal{F}$ by $X \star \mathcal{F}$:

$$X \star \mathcal{F} := \bigcup_{i=0}^k (\text{cl}((X \cap F_{i+1}) \cup F_i) \setminus F_i). \tag{10}$$

It is useful to observe that when $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, the equality $M \star \mathcal{F} = (M \star \mathcal{F}_1) \star \mathcal{F}_2$ holds. In general, for a subset $X \subseteq E$ we only have the inclusion $X \star \mathcal{F} \subseteq (X \star \mathcal{F}_1) \star \mathcal{F}_2$.

Since no loops are created when constructing $M \star \mathcal{F}$ from M it follows from (9) that every line in M is a line in $M \star \mathcal{F}$ as well. We should stress that the degree vector $a(X)$ of $X \subseteq E$ does not depend on the matroid and is the same for M as for $M \star \mathcal{F}$.

In the following lemma's we investigate the relations between fractional matchings in the matroids M and $M \star \mathcal{F}$.

Lemma 3.1. *Let $S = S_0 \cup \dots \cup S_k$ be a flat in $M \star \mathcal{F}$, with $S_i \subseteq F_{i+1} \setminus F_i$. Then*

$$a(S) = \sum_{i=0}^k [a(S_i \cup F_i) - a(F_i)]. \tag{11}$$

Proof. For $i = 0, \dots, k$ let the flat T_i be the flat in M defined by $T_i := S_i \cup F_i$. Since $a(S) = \sum_i a(S_i)$, we have to show that for every $l \in L$ and $i = 0, \dots, k$ the equality $a(S_i)_l + a(F_i)_l = a(T_i)_l$ holds. Since S_i and F_i are disjoint, the cases where $a(S_i)_l \in \{0, 2\}$ or $a(F_i)_l \in \{0, 2\}$ are clear. In the remaining case $l \cap S_i \neq \emptyset$ and $l \cap F_i \neq \emptyset$. Since F_i is a flat in M , this implies that $r(l \cap T_i) > 1$. This implies $l \subseteq T_i$, since $r(l) \leq 2$ and T_i is a flat in M . \square

The following proposition shows that a fractional matching in $M \star \mathcal{F}$ is a fractional matching in M as well. Conversely, when we have a fractional matching in M , it is also a fractional matching in $M \star \mathcal{F}$, provided that the flats in \mathcal{F} are tight with respect to the fractional matching.

Proposition 3.2. *If x is a fractional matching in $(M \star \mathcal{F}, L)$, then x is a fractional matching in (M, L) . If x is a fractional matching in (M, L) , and $a(F)x = r_M(F)$ for every $F \in \mathcal{F}$, then x is a fractional matching in $(M \star \mathcal{F}, L)$.*

Proof. To prove the first statement, let T be a flat of M . Then

$$a(T)x \leq a(T \star \mathcal{F})x \leq r_{M \star \mathcal{F}}(T \star \mathcal{F}) \leq r_M(T). \quad (12)$$

To prove the second statement, let $S = S_0 \cup \dots \cup S_k$ be a flat in $M \star \mathcal{F}$. Then

$$\begin{aligned} a(S)x &= \sum_{i=0}^k [a(S_i \cup F_i) - a(F_i)]x \\ &= \sum_{i=0}^k [a(S_i \cup F_i)x - r_M(F_i)] \\ &\leq \sum_{i=0}^k [r_M(S_i \cup F_i) - r_M(F_i)] \\ &= r_{M \star \mathcal{F}}(S). \end{aligned} \quad (13)$$

Here the first equality follows from Lemma 3.1 and the second equality is the assumption made in the proposition. \square

4 Algorithm for maximum weight perfect fractional matching

Let M be a matroid with ground set E and let L be a finite set of lines in M . Let $w : L \rightarrow \mathbb{Q}_+$ be nonnegative rational weights on the lines. We will assume that (M, L) has a perfect fractional matching.

Throughout the algorithm we keep a chain $\mathcal{F} = \{F_1, \dots, F_k\}$ of flats in M , with $\emptyset \subset F_1 \subset \dots \subset F_k \subset E$. For convenience we always define $F_0 := \emptyset$ and $F_{k+1} := E$. We also keep a dual feasible solution y with support \mathcal{F} or $\mathcal{F} \cup \{E\}$. That is $y = \lambda_1 F_1 + \dots + \lambda_k F_k + \lambda E$ satisfies $a(y) \geq w$, $\lambda_1, \dots, \lambda_k \in \mathbb{Q}_{>0}$ and $\lambda \in \mathbb{Q}$.

Observe that for a perfect fractional matching x in (M, L) we have:

$$\begin{aligned} wx = \sum_{l \in L} w_l x_l &\leq \sum_{l \in L} a(y)_l x_l \\ &= \sum_{i=1}^k \lambda_i a(F_i)x + \lambda a(E)x \\ &\leq \sum_{i=1}^k \lambda_i r(F_i) + \lambda r(E) = r(y). \end{aligned} \quad (14)$$

Here the equality $a(E)x = r(E)$ is used in the last line. It follows that x has maximum weight if and only if

$$w_l = a(y)_l \quad \text{for all } l \in \text{supp}(x), \quad (15)$$

$$a(F)x = r(F) \quad \text{for all } F \in \text{supp}(y). \quad (16)$$

Denote $L_y := \{l \in L \mid w_l = a(y)_l\}$. The above two conditions will be met once we find a perfect fractional matching x in $(M \star \mathcal{F}, L_y)$. Indeed, in that case x is also a perfect fractional matching in (M, L) by Proposition 3.2. Equalities (15) will be satisfied by definition of L_y , and equalities (16) will be satisfied because $a(E)x = r(E)$ implies that for every i

$$\begin{aligned} r(E) = a(E)x &= a(F_i)x + a(E \setminus F_i)x \\ &\leq r_{M \star \mathcal{F}}(F_i) + r_{M \star \mathcal{F}}(E \setminus F_i) \\ &= r(F_i) + (r(E) - r(F_i)) \\ &= r(E), \end{aligned} \tag{17}$$

and hence $a(F_i) = r(F_i)$. The description of the algorithm is as follows.

Initially, we set $\mathcal{F} := \emptyset$, $y := \lambda E$, where $\lambda := \frac{1}{2} \max\{w_l \mid l \in L\}$. The iteration of the algorithm is as follows.

Find a maximum size fractional matching x in $(M \star \mathcal{F}, L_y)$, and find the dominant cover $\frac{1}{2}(S^* + T^*)$, of $(M \star \mathcal{F}, L_y)$. This can be done by the algorithm from [1]. Now consider two cases.

Case 1: If x is a *perfect* fractional matching, x will be a maximum weight perfect fractional matching in (M, L) and y is an optimal dual solution. We output x and y and stop.

Case 2: If x is not perfect, write $S^* = S_0^* \cup \dots \cup S_k^*$ and $T^* = T_0^* \cup \dots \cup T_k^*$, where $S_i^*, T_i^* \subseteq F_{i+1} \setminus F_i$ for $i = 0, \dots, k$. We define

$$z := \sum_{i=0}^k (S_i^* \cup F_i - F_i) + \sum_{i=0}^k (T_i^* \cup F_i - F_i) - E. \tag{18}$$

Observe that $a(z)_l = a(S^* + T^*)_l - 2 \in \{-2, -1, 0, 1, 2\}$ for every $l \in L$. In particular, $a(z)_l \geq 0$ for $l \in L_y$ and $a(z)_l = 0$ for $l \in \text{supp}(x)$.

Let

$$\begin{aligned} \epsilon_1 &:= \max\{t \geq 0 \mid (y + tz)_F \geq 0 \text{ for all } F \neq E\}, \\ \epsilon_2 &:= \max\{t \geq 0 \mid a(y + tz) \geq w\}, \end{aligned} \tag{19}$$

and let ϵ be the minimum of ϵ_1 and ϵ_2 . Let $y' := y + \epsilon z$ and $\mathcal{F}' := \text{supp}(y') \setminus \{E\}$. By definition of ϵ , y' will be a dual feasible solution. We reset $y := y'$ and $\mathcal{F} := \mathcal{F}'$ and iterate.

Finding a maximum weight fractional matching

The algorithm for finding a maximum weight *perfect* fractional matching described above, can be used to find a maximum weight fractional matching as follows.

Let the matroid M , a set of lines L and nonnegative weights $w : L \rightarrow \mathbb{Z}_+$ be given. Let B be a base of M . By introducing parallel copies if necessary, we may assume that $\{b\} \notin L$ for every $b \in B$. We define $L' := L \cup \{\{b\}, b \in B\}$ and $w' : L' \rightarrow \mathbb{Z}_+$ by $w'(l) := w(l)$ for $l \in L$ and $w'(\{b\}) := 0$ for $b \in B$.

Let x be a vertex of the fractional matroid matching polytope for (M, L) , and $B' \subset B$ be a base of $M/\text{cl}(x)$. Define $x' : L' \rightarrow \{0, \frac{1}{2}, 1\}$ by $x'_l := x_l$ for $l \in L$, $x'_{\{b\}} := \frac{1}{2}$ for $b \in B'$ and $x'_{\{b\}} := 0$ for $b \in B \setminus B'$. It follows from Theorem 2.4 that x' is a perfect fractional matching in (M, L') .

It follows that the fractional matchings in (M, L) are precisely the restrictions to L , of the perfect fractional matchings in (M, L') . Using the above algorithm we find a maximum w' -weight perfect fractional matching x' in (M, L') and an optimal dual solution y' . Restricting x' to L gives the required maximum w -weight fractional matching in (M, L) . Since $a(y')_{\{b\}} \geq w'(\{b\}) = 0$ for every $b \in B$, it follows from the fact that $\text{supp}(y')$ is a chain, that $y'_E \geq 0$ and hence is an optimal dual solution for the problem of finding a maximum weight fractional matching in (M, L) .

5 Running time of the algorithm

In this section, we will show that the algorithm for maximum weight perfect matching described in the previous section terminates after $O(r^3)$ iterations. To this end, we will define a parameter $\psi(\mathcal{F}, S^*, T^*)$ that measures the progress made.

Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a chain of flats in M , where $\emptyset =: F_0 \subset F_1 \subset \dots \subset F_k \subset F_{k+1} := E$. For a subset $X = X_0 \cup \dots \cup X_k$ of E where $X_i \subseteq F_{i+1} \setminus F_i$ ($i = 0, \dots, k$), we define

$$\phi(\mathcal{F}, X) := \sum_{i=0}^k [r(F_i \cup X_i)^2 - r(F_i)^2]. \quad (20)$$

Observe that $\phi(\mathcal{F}, X)$ is a nonnegative integer no larger than $r(E)^2$. Clearly, for $X \subseteq X'$ we have $\phi(\mathcal{F}, X) \leq \phi(\mathcal{F}, X')$ and $\phi(\mathcal{F}, X) = \phi(\mathcal{F}, X \star \mathcal{F})$.

For flats S and T of M , we define

$$\psi(\mathcal{F}, S, T) := \phi(\mathcal{F}, S) + \phi(\mathcal{F}, T) + 2r(E)r_{M \star \mathcal{F}}(T). \quad (21)$$

So $\psi(\mathcal{F}, S, T)$ is a nonnegative integer no larger than $4r(E)^2$. The following two lemmas will display some useful properties of ϕ (and hence of ψ).

Lemma 5.1. *Let $X \subseteq X'$ be subsets of E . Then*

$$\phi(\mathcal{F}, X') - \phi(\mathcal{F}, X) \leq (r_{M \star \mathcal{F}}(X') - r_{M \star \mathcal{F}}(X)) \cdot 2r(E), \quad (22)$$

and equality holds if and only if $r_{M \star \mathcal{F}}(X') = r_{M \star \mathcal{F}}(X)$.

Proof. Write $X = X_0 \cup \dots \cup X_k$ and $X' = X'_0 \cup \dots \cup X'_k$ where $X_i \subseteq X'_i \subseteq F_{i+1} \setminus F_i$.

We have

$$\begin{aligned}
\phi(\mathcal{F}, X') - \phi(\mathcal{F}, X) &= \sum_{i=0}^k (r(F_i \cup X'_i)^2 - r(F_i \cup X_i)^2) \\
&= \sum_{i=0}^k (r(F_i \cup X'_i) - r(F_i \cup X_i))(r(F_i \cup X'_i) + r(F_i \cup X_i)) \\
&\leq \sum_{i=0}^k (r(F_i \cup X'_i) - r(F_i \cup X_i)) \cdot 2r(E) \\
&= (r_{M \star \mathcal{F}}(X') - r_{M \star \mathcal{F}}(X)) \cdot 2r(E). \tag{23}
\end{aligned}$$

Equality holds if and only if for every i we have $r(F_i \cup X'_i) = r(F_i \cup X_i)$. \square

Lemma 5.2. *Let $X = X_0 \cup \dots \cup X_k$ with $X_i \subseteq F_{i+1} \setminus F_i$, and let $\mathcal{F}' \supset \mathcal{F}$ be a longer chain of flats in M . Suppose that $r_{M \star \mathcal{F}}(X) = r_{M \star \mathcal{F}'}(X)$. Then $\phi(\mathcal{F}, X) \leq \phi(\mathcal{F}', X)$, and equality holds if and only if $\mathcal{F}' \cup \{F_i \vee X_i, i = 0, \dots, k\}$ is a chain.*

Proof. Consider a fixed $i \in \{0, \dots, k\}$. Let $H_0 \subset \dots \subset H_{l+1}$ be the flats $\{F \in \mathcal{F}' \mid F_i \subseteq F \subseteq F_{i+1}\}$. Write $X_i := Y_0 \cup \dots \cup Y_l$ where $Y_j \subseteq H_{j+1} \setminus H_j$ for $j = 0, \dots, l$. Denote $r := r(F_i \cup X_i) - r(F_i)$ and $r_j := r(H_j \cup Y_j) - r(H_j) \leq r(H_{j+1}) - r(H_j)$ for $j = 0, \dots, l$.

The assumption $r_{M \star \mathcal{F}}(X) = r_{M \star \mathcal{F}'}(X)$ implies that $r = r_0 + \dots + r_l$. We have the following inequalities.

$$\begin{aligned}
r(F_i \cup X_i)^2 - r(F_i)^2 &= (r(F_i) + r)^2 - r(F_i)^2 \\
&= \sum_{j=0}^l [(r(F_i) + r_0 + \dots + r_{j-1} + r_j)^2 - \\
&\quad - (r(F_i) + r_0 + \dots + r_{j-1})^2] \\
&\leq \sum_{j=0}^l [(r(H_j) + r_j)^2 - r(H_j)^2] \\
&= \sum_{j=0}^l (r(H_j \cup Y_j)^2 - r(H_j)^2). \tag{24}
\end{aligned}$$

The inequality follows from the fact that $r(H_j) \geq r(F_i) + r_0 + \dots + r_{j-1}$ for $j = 0, \dots, l$, and equality holds if and only if for every j either $r(H_j) = r(F_i) + r_0 + \dots + r_{j-1}$ or $r_j = 0$. That is, if j is the largest index for which $r_j > 0$, then we have equality if and only if $H_j \subseteq F_i \vee X_i \subseteq H_{j+1}$. Summing over all i proves the lemma. \square

In the remainder of this section, we will prove that the algorithm described in the previous section is correct, and that the number of iterations is at most $O(r(E)^3)$.

Proof of the running time bound. We will show that the pair of nonnegative integers $(r(E) - 2\nu^*(M \star \mathcal{F}, L_y), \psi(\mathcal{F}, S^*, T^*))$ decreases lexicographically in each iteration.

Consider an iteration. Let y be the current dual solution, \mathcal{F} the current chain of flats in M and let $\frac{1}{2}(S^* + T^*)$ be the dominant cover of $(M \star \mathcal{F}, L_y)$. Denote by y' , \mathcal{F}' and $\frac{1}{2}(S^{*'} + T^{*'})$ the corresponding objects in the next iteration.

We define the subset \bar{L} of lines by

$$\bar{L} := \{l \in L_y \mid l \in \text{supp}(x)\} \quad \text{for some maximum size fractional matching } x \text{ in } (M \star \mathcal{F}, L_y), \quad (25)$$

and the chain $\bar{\mathcal{F}}$ of flats in M by

$$\bar{\mathcal{F}} := \mathcal{F} \cup \{S_i^* \cup F_i, i = 0, \dots, k\} \cup \{T_i^* \cup F_i, i = 0, \dots, k\}, \quad (26)$$

where $S_i^* := S^* \cap (F_{i+1} \setminus F_i)$ and $T_i^* := T^* \cap (F_{i+1} \setminus F_i)$ for $i = 0, \dots, k$. Note that $\bar{\mathcal{F}} = \mathcal{F} \cup \mathcal{F}'$, and hence $M \star \bar{\mathcal{F}} = (M \star \mathcal{F}) \star \mathcal{F}' = (M \star \mathcal{F}') \star \mathcal{F}$.

By complementary slackness, $a(\frac{1}{2}(S^* + T^*))_l = 2$ holds for every $l \in \bar{L}$. Therefore we have $a(z)_l = 0$ for every $l \in \bar{L}$ so that

$$\bar{L} \subseteq L_{y'}. \quad (27)$$

Again by complementary slackness, $a(S^*)x = r_{M \star \mathcal{F}}(S^*)$ and $a(T^*)x = r_{M \star \mathcal{F}}(T^*)$ holds for any maximum size fractional matching x in $(M \star \mathcal{F}, L_y)$. Hence it follows by Proposition 3.2 that x is also a fractional matching in $(M \star \bar{\mathcal{F}}, \bar{L})$ and hence in $(M \star \mathcal{F}', L_{y'})$. We can conclude that

$$\nu^*(M \star \mathcal{F}, L_y) = \nu^*(M \star \bar{\mathcal{F}}, \bar{L}) \leq \nu^*(M \star \mathcal{F}', L_{y'}). \quad (28)$$

When strict inequality holds in (28), we are done. Therefore, in the remainder of the proof, we may assume that

$$\nu^*(M \star \mathcal{F}, L_y) = \nu^*(M \star \mathcal{F}', L_{y'}). \quad (29)$$

Denote

$$\bar{S} := S^{*' \star} \bar{\mathcal{F}}, \quad \bar{T} := T^{*' \star} \bar{\mathcal{F}}. \quad (30)$$

Since

$$\nu^*(M \star \bar{\mathcal{F}}, \bar{L}) \leq \nu^*(M \star \bar{\mathcal{F}}, L_{y'}) \quad (31)$$

$$\leq \frac{1}{2} r_{M \star \bar{\mathcal{F}}}(\bar{S} + \bar{T}) \quad (32)$$

$$\leq \frac{1}{2} r_{M \star \mathcal{F}'}(S^{*' \star} + T^{*' \star}) \quad (33)$$

$$= \nu^*(M \star \mathcal{F}', L_{y'}) = \nu^*(M \star \bar{\mathcal{F}}, \bar{L}), \quad (34)$$

we find that $\frac{1}{2}(\bar{S} + \bar{T})$ is a minimum cover in $(M \star \bar{\mathcal{F}}, L_{y'})$, and hence also in $(M \star \bar{\mathcal{F}}, \bar{L})$.

Clearly, $\frac{1}{2}(S^* + T^*)$ is also a minimum cover of $(M \star \bar{\mathcal{F}}, \bar{L})$. Furthermore, since x is a maximum size fractional matching in $(M \star \mathcal{F}, L_y)$ if and only if x is a maximum size fractional matching in $(M \star \bar{\mathcal{F}}, \bar{L})$, it follows that $\frac{1}{2}(S^* + T^*)$ is in fact the dominant cover of $(M \star \bar{\mathcal{F}}, \bar{L})$. Indeed, let $\frac{1}{2}(S + T)$, $S \subseteq T$ be the dominant cover of $(M \star \bar{\mathcal{F}}, \bar{L})$.

It suffices to show that $T^* = T$. For every maximum size fractional matching x in $(M \star \mathcal{F}, L_y)$, we have $\text{cl}_{M \star \mathcal{F}}(x) \supseteq T^*$. This implies that $\text{cl}_{M \star \mathcal{F}}(x)$ is also a flat in $M \star \overline{\mathcal{F}}$, and hence $\text{cl}_{M \star \mathcal{F}}(x) = \text{cl}_{M \star \overline{\mathcal{F}}}(x)$. By the characterization of the dominant cover, Theorem 2.3, it now follows that

$$T = \bigcap_x \text{cl}_{M \star \overline{\mathcal{F}}}(x) = \bigcap_x \text{cl}_{M \star \mathcal{F}}(x) = T^*, \quad (35)$$

where the intersection runs over all maximum size fractional matchings x in $(M \star \mathcal{F}, L_y)$.

By comparing the two covers of $(M \star \overline{\mathcal{F}}, \overline{L})$, we obtain

$$S^* \subseteq \overline{S}, \quad \overline{T} \subseteq T^*. \quad (36)$$

Note that $r_{M \star \mathcal{F}'}(S^{*'}) = r_{M \star \overline{\mathcal{F}}}(S^{*'})$, $r_{M \star \mathcal{F}'}(T^{*'}) = r_{M \star \overline{\mathcal{F}}}(T^{*'})$, and $r_{M \star \overline{\mathcal{F}}}(\overline{S} - S^* + \overline{T} - T^*) = 0$, thus by Lemma 5.2 and Lemma 5.1 we get that

$$\begin{aligned} \psi(\mathcal{F}', S^{*'}, T^{*'}) &\leq \psi(\overline{\mathcal{F}}, \overline{S}, \overline{T}) \\ &= \phi(\overline{\mathcal{F}}, \overline{S}) + \phi(\overline{\mathcal{F}}, \overline{T}) + 2r(E)r_{M \star \overline{\mathcal{F}}}(\overline{T}) \\ &\leq \phi(\overline{\mathcal{F}}, S^*) + \phi(\overline{\mathcal{F}}, T^*) + 2r(E)r_{M \star \overline{\mathcal{F}}}(\overline{T} + \overline{S} - S^*) \\ &= \phi(\overline{\mathcal{F}}, S^*) + \phi(\overline{\mathcal{F}}, T^*) + 2r(E)r_{M \star \overline{\mathcal{F}}}(T^*) \\ &= \psi(\overline{\mathcal{F}}, S^*, T^*) \\ &= \psi(\mathcal{F}, S^*, T^*). \end{aligned} \quad (37)$$

We need to show that $\psi(\mathcal{F}', S^{*'}, T^{*'}) < \psi(\mathcal{F}, S^*, T^*)$ holds. Suppose that equality holds in the second inequality in (37). Then we must have equality in (36), by Lemma 5.1. This means that we are in the case $\epsilon_1 < \epsilon_2$. Indeed, if $\epsilon_1 \geq \epsilon_2$, there exists an $l \in L_{y'} \setminus L_y$ with $a(S^* + T^*)_l < 2$. However, $a(\overline{S} + \overline{T})_l = 2$, which would imply that we do not have equality in (36).

Since $\epsilon_1 < \epsilon_2$, there is an $F_i \in \mathcal{F} \setminus \mathcal{F}'$. Since $z_{F_i} < 0$, we have either $(S_i^* \neq \emptyset$ and $S_{i-1}^* \not\supseteq F_i \setminus F_{i-1})$ or $(T_i^* \neq \emptyset$ and $T_{i-1}^* \not\supseteq F_i \setminus F_{i-1})$.

Now let $\mathcal{F}' = \{F'_1, F'_2, \dots, F'_m\}$, and choose j such that $F'_j \subsetneq F_i \subsetneq F'_{j+1}$.

First, assume that $S_i^* \neq \emptyset$ and $S_{i-1}^* \not\supseteq F_i \setminus F_{i-1}$. Then we apply Lemma 5.2 with respect to \mathcal{F}' and $\overline{\mathcal{F}}$, and $X := S^* = \overline{S}$. Note that $F'_j \vee X_j = F'_j \vee \overline{S}_i$. Since $\{F'_j \vee \overline{S}_i\} \cup \overline{\mathcal{F}}$ is not a chain, we get that $\phi(\mathcal{F}', S^{*'}) < \phi(\overline{\mathcal{F}}, \overline{S})$, and thus, $\psi(\mathcal{F}', S^{*'}, T^{*'}) < \psi(\mathcal{F}, S^*, T^*)$.

Second, assume that $T_i^* \neq \emptyset$ and $T_{i-1}^* \not\supseteq F_i \setminus F_{i-1}$. Here, if $S_i^* \neq \emptyset$, then the first case applies, thus we may also assume that $S_i^* = \emptyset$. Apply Lemma 5.2 with respect to \mathcal{F}' and $\overline{\mathcal{F}}$, and $X := T^* = \overline{T}$. Note that $F'_j \vee X_j = F'_j \vee \overline{T}_i$. Since $\{F'_j \vee \overline{T}_i\} \cup \overline{\mathcal{F}}$ is not a chain, we get that $\phi(\mathcal{F}', T^{*'}) < \phi(\overline{\mathcal{F}}, \overline{T})$, and thus, $\psi(\mathcal{F}', S^{*'}, T^{*'}) < \psi(\mathcal{F}, S^*, T^*)$. \square

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