Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2008-14. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

The constructive characterization of (k, ℓ)-edge-connected digraphs

Erika R. Kovács and László A. Végh

November 2008

The constructive characterization of (k, ℓ)-edge-connected digraphs ⋆

Erika R. Kovács and László A. Végh ⋆⋆

Abstract

We give a constructive characterization for (k, ℓ)-edge-connected digraphs, proving a conjecture of Frank and Szegő.

1 Introduction

By constructive characterization of a certain class $\mathcal{H}$ of graphs, we mean a set of construction steps and a set of basic instances in $\mathcal{H}$ satisfying the following. Every graph constructed by a sequence of such steps starting from one of the basic instances is in $\mathcal{H}$ and moreover all graphs in $\mathcal{H}$ can be obtained this way. For example, a graph is connected if and only if it can be obtained from a single vertex by adding new edges between old vertices or adding a new edge between an old and a new vertex. The well-known ear-decomposition gives a constructive characterization for 2-connected graphs. A survey on constructive characterizations can be found in [8].

A digraph is called **k-edge-connected** if deleting any $k-1$ edges leaves it strongly connected. By Menger's well-known theorem, this is equivalent to the property that there are k edge-disjoint paths from any vertex to any other. A classical constructive characterization is Mader's theorem for k-edge-connected digraphs.

Theorem 1.1 (Mader [9])**.** *A directed graph is k-edge-connected if and only if it can be obtained from a single vertex by the iterative application of the following two operations.*

(i) add a new edge (possibly a loop),

(ii) subdivide k existing edges and identify the subdividing vertices.

⋆Research supported by the Hungarian National Foundation for Scientific Research Grant, OTKA K60802 and TS049788. The first author was also supported by the Communication Networks Laboratory (CNL) at the Eötvös University, while the second author by the Siemens-ZIB Scholarship Program.

⋆⋆Department of Operations Research, Eötvös University, Pázmány P. s. 1/c, Budapest, Hungary, H-1117, MTA-ELTE Egerváry Research Group (EGRES). E-mail: {koverika,veghal}@cs.elte.hu

Operation (ii) will be called **pinching k edges with z.** By pinching 0 edges we simply mean the addition of a new vertex of degree 0.

A digraph $G = (V, E)$ is called **(k, ℓ)-edge-connected** for some integers $0 \le \ell \le k$, if G has a root vertex s and for each vertex $z \ne s$, there exist k edge disjoint sz paths and ℓ edge disjoint zs paths. Note that (k, k)-edge-connectivity with arbitrary root coincides with k-edge-connectivity and $(k, 0)$-edge-connectivity means by Edmonds' disjoint arborescences theorem [1] that there are k disjoint spanning arborescences rooted in s.

A related concept for undirected graphs is the following. An undirected graph is called **(k, ℓ)-partition-connected** if for any partition of the vertices into $t \ge 2$ classes, there are at least $k(t-1) + \ell$ edges connecting different classes. Note that $(k, 0)$-partition-connectivity is by Tutte's theorem [10] equivalent to having k disjoint spanning arborescences. The link between these concepts for directed and undirected graphs is established by the following theorem:

Theorem 1.2 (Frank, [2])**.** *For integers $0 \le \ell \le k$, an undirected graph G has a (k, ℓ)-edge-connected orientation if and only if G is (k, ℓ)-partition connected.*

The following theorem is the main result of the paper. It was conjectured by Frank and Szegő ([6], Conjecture 5.6.):

Theorem 1.3. *For $0 \le \ell \le k-1$, a directed graph is (k, ℓ)-edge-connected if and only if it can be built up from the single vertex s by the following two operations.*

(i) add a new edge,

(ii) for some i with $\ell \le i \le k-1$, pinch i existing edges with a new vertex z, and add $k-i$ new edges from old vertices to z.

Using Theorem 1.2, the following undirected counterpart is a straightforward consequence.

Corollary 1.4. *For $0 \le \ell \le k-1$, an undirected graph is (k, ℓ)-partition-connected if and only if it can be built up from the single vertex s by the following two operations.*

(i) add a new edge,

(ii) for some i with $\ell \le i \le k-1$, pinch i existing edges with a new vertex z, and add $k-i$ new edges between z and some old vertices.

The special case of Theorem 1.3 for $\ell = 0$ was shown by Frank [3] and for $\ell = 0$ by Frank and Szegő [6], while for $\ell = k-1$ it was proved by Frank and Király in [5]. In all cases of this theorem and as well in Theorem 1.1, it is straightforward that all graphs constructed by operations (i) and (ii) are (k, ℓ)-edge-connected, so the nontrivial part is the reverse direction. The reverse operation of (i) is deleting an edge, thus we may focus our attention to minimally (k, ℓ)-edge-connected graphs in the sense that deleting any edge would destroy (k, ℓ)-edge-connectivity.

An exceedingly important tool is Mader's splitting off theorem. In a digraph $G = (V, E)$, we mean by splitting off two edges $e = xz$, $f = zy$ the operation of deleting e

and f and adding the new edge xy. Let $\rho(X) = \rho_G(X) = \rho_E(X)$ and $\delta(X) = \delta_G(X) = \delta_E(X)$ denote the in- and out-degrees of the set X, respectively. Let $\rho(z)$ and $\delta(z)$ denote the in- and out degrees of the vertex z. If $\rho(z) = \delta(z)$, we mean by a **complete splitting at** z a sequence of splitting off operations of all edges incident to z and finally removing z. We say that a digraph $G = (U + z, E)$ is k**-edge-connected in** U if there exist k-edge-disjoint paths between every two vertices in U (the paths may possibly use z).

Theorem 1.5 (Mader)**.** *Let $G = (U + z, E)$ be a digraph k-edge-connected in U and $\rho(z) = \delta(z)$. Then there exists a complete splitting at z resulting in a k-edge-connected graph.*

Theorem 1.5 may be used to prove Theorem 1.1. For the nontrivial direction it is enough to find a vertex z in a minimally k-edge-connected graph having both in- and out-degree k. An easy consequence of this theorem can be used to derive the constructive characterization of $(k, 0)$-edge-connected graphs. However, for the cases $\ell = 1$ and $\ell = k - 1$ a nontrivial generalization of Mader's theorem is needed, which is due to Frank. We say that the digraph $G = (U + z, E)$ is (k, ℓ)**-edge-connected in** U for a root node $s \in U$, if for every vertex $x \in U$ there are k-edge-disjoint paths from s to x and ℓ paths from x to s.

Theorem 1.6 (Frank, [4])**.** *Let $G = (U + z, E)$ be a digraph (k, ℓ)-edge-connected in U and $\rho(z) = \delta(z)$. Then there exists a complete splitting at z resulting in a (k, ℓ)-edge-connected graph.*

The reason why the proof is more difficult in case of $\ell = 1$ and $\ell = k - 1$ than for is $\ell = 0$ and $\ell = k$ is due to the following reason. In the latter cases it was enough to find a vertex satisfying certain conditions for the in- and outdegrees, and one could always perform a complete splitting at such a vertex. However, for $\ell = 1$ and $\ell = k - 1$ the conditions for the degrees do not suffice and a more thorough analysis of the structure of minimal (k, ℓ)-edge-connected graphs is needed.

To give some motivation of our proof, we sketch the proof for $\ell = k - 1$ by Frank and Király [5]. Consider a minimally $(k, k - 1)$-edge-connected graph. The reverse operation of (ii) may be applied at a vertex z with in-degree k and out-degree $k - 1$. We call such vertices **special**. If for a special vertex z we manage to find an edge uz so that $G - uz$ is $(k, k - 1)$-edge-connected in $U = V - z$, then Theorem 1.6 may be applied to $G' = (U + z, E - uz)$ giving a $(k, k - 1)$-edge-connected graph G'' on U. Then we can get G from G'' by by applying step (ii) with pinching those $k - 1$ edges with z which were resulted by the splitting off and finally adding the edge uz.

However, not every special vertex z admits an edge uz as above (and it is already nontrivial to find a special vertex). The proof uses an indirect argument: assume that every edge in $xy \in E$ satisfies one of the following conditions. On the one hand, if y is special, then we assume that $G - xy$ is not $(k, k - 1)$-edge-connected in $V - y$. On the other hand, if y is not special, we use that G is minimally $(k, k - 1)$-edge-connected, thus $G - xy$ is not $(k, k - 1)$-edge-connected. One can define a notion of tight sets so that each edge will be "blocked" by a tight set. Then the uncrossing method may be used for these tight sets to derive a final contradiction.

Our proof is motivated by this argument, but for general ℓ, severe new difficulties arise. Starting from a minimally (k,ℓ)-edge-connected digraph, we call a vertex z **special** if according to its in- and out-degree it might be the result of operation (ii) in Theorem 1.3, that is, if $\ell \leq \delta(z) \leq k-1$ and $\rho(z) = k$. We say that a subset F of edges entering a special vertex z is **locally admissible** at z, if $G - F$ is (k,ℓ)-edge-connected in $V - z$. F is called **sufficient** at z if $|F| = k - \delta(z)$. Once a sufficient locally admissible F is found, Theorem 1.6 may be applied to $G - F$ and z and the proof finishes as for $\ell = k-1$.

Thus our aim is to find a special vertex z and a sufficient locally admissible set F at z. It is easy to characterize the maximal size of a locally admissible set for a given special z, however, this size may be strictly smaller than $k - \delta(z)$. The main difficulty is how to handle together the locally admissible sets belonging to different special vertices. The notion of **globally admissible** edge sets in Definition 2.3 is introduced for this purpose.[1] For a globally admissible edge set and an arbitrary special vertex z, the subset $F_z \subseteq F$ of edges entering z is locally admissible at z. However, the converse is not true in the sense that the union of locally admissible edge sets belonging to different special vertices will not be necessarily globally admissible. We say that a globally admissible edge set F is **sufficient**, if for some special z, F_z is sufficient; otherwise it is called **insufficient**. What we prove is the existence of a sufficient globally admissible edge set. Unfortunately, it is not true that every maximal globally admissible set is sufficient, as it will be shown by an example in Section 6.

Among other methods, splitting off techniques will be used also in the proof of the existence of a sufficient globally admissible set. However, Theorem 1.6 turns out to be too weak for our aims. Actually, Theorem 1.6 is a special case of an abstract theorem of Frank [4] for covering positively crossing supermodular functions by a digraph. This theorem is presented in Section 4 where we formulate a generalization which will enable us the usage of a complete splitting operation preserving a property stronger than the (k,ℓ)-edge-connectivity.

The way we handle tight sets also differs from the standard uncrossing methods. A set is called **tight** with respect to a globally admissible set F if the inequality concerning this set in the definition of global admissibility holds with equality. As in the proof for $\ell = k-1$, for a maximal F there is a tight set "blocking" each edge in $E - F$. However, it is not possible to apply the uncrossing method to arbitrary tight sets for an arbitrary globally admissible F. The intersection and union of two tight sets will be tight only under the assumption that F is maximal and insufficient. It turns out interestingly that under this assumption, some basic types of tight sets do not occur at all. This will be discussed in Section 5.

The paper is organized as follows. In Section 2, the precise definitions are given and some basic properties are exhibited. We also give the proof of Theorem 1.3 here based on the main technical tool Theorem 2.1. This is a special case of the stronger Theorem 2.7 also stated in this section, and proved in Section 3 relying on three basic lemmata. Section 4 describes the general splitting-off theorem and the proof of the first basic lemma, while Section 5 contains a sequence of technical claims and the

[1] This highly technical definition is omitted at this point.

proof of the other two basic lemmata. Finally, in Section 6 we describe the structure of locally admissible sets and present a polynomial algorithm for finding a sufficient locally admissible set F at a special vertex z. We also show an example of an insufficient maximal globally admissible edge set.

2 Basic concepts and the Proof of Theorem 1.3

Let $G = (V, E)$ be a (k, ℓ)-edge connected directed graph with root $s \in V$. For $X \subseteq V$, let $\gamma(X) = k$ if $s \notin X$ and $\gamma(X) = \ell$ if $s \in X$. For two sets $X, Y \subseteq V$ and an edge set H, by $\delta_H(X, Y)$ we mean the number of directed edges in the edge set H from $X - Y$ to $Y - X$. Let $d_H(X, Y) = \delta_H(X, Y) + \delta_H(Y, X)$ and $\bar{d}_H(X, Y) = d_H(X, V - Y)$. Let $\Delta_H^{in}(X)$ and $\Delta_H^{out}(X)$ denote the set of edges in H entering and leaving the set X, respectively. Whenever the index H is omitted in these concepts then it corresponds to the edge set E. X and Y are said to be **crossing**, if $X \cap Y \neq \emptyset$, $X \cup Y \neq V$ and neither of X and Y contains the other. Sometimes we will use x in the sense of $\{x\}$; $A + x$ (resp. $A - x$) will denote $A \cup \{x\}$ (resp. $A - \{x\}$). For a real number α, let $\alpha^+ = \max(0, \alpha)$ denote its positive part.

A vertex $v \in V$ is called **special**, if $\rho(v) = k$, $\ell \leq \delta(v) \leq k - 1$. Let S denote the set of special vertices ($S \neq \emptyset$ is not assumed). If $X \subseteq S$ then we say that X is a **special set.** Observe that $s \notin S$ as $\delta(s) \geq k$. For a $z \in S$, a subset F of edges entering z is **locally admissible at** z, if $G - F$ is (k, ℓ)-edge-connected in $V - z$ and $|F| \leq k - \delta(z)$. A locally admissible F will be called **sufficient** if $|F| = k - \delta(z)$. Theorem 1.3 will be an easy consequence of the following.

Theorem 2.1. *In a minimally (k, ℓ)-edge-connected digraph $G = (V, E)$ there exists a special vertex z with a sufficient locally admissible set at z.*

Let us see how Theorem 1.3 follows from this.

Proof of Theorem 1.3. First let us show that the operations (i) and (ii) preserve (k, ℓ)-edge-connectivity. This is straightforward in case of (i). For (ii), let $G' = (V + z, E')$ denote the digraph resulting from the (k, ℓ)-edge-connected digraph $G = (V, E)$ by applying (ii). For every $v \in V - s$, the k edge-disjoint paths from s to v and the ℓ edge-disjoint paths from v to s in G naturally give the same number of paths in G'. Thus the only problem could be if there are too few paths from s to z or from z to s.

In this case, by Menger's theorem we have a subset X of $V + z$ with $s \notin X$, $z \in X$, and either $\rho(X) < k$ or $\delta(X) < \ell$. Since G' is (k, ℓ)-edge-connected in V, the only possibility is $X = \{z\}$. However, $\rho(z) = k$ and $\delta(z) \geq \ell$ gives a contradiction.

For the other direction, if G is not minimally (k, ℓ)-edge-connected, then an edge can be deleted preserving (k, ℓ)-edge-connectivity. Otherwise, Theorem 2.1 is applicable. Consider the special vertex z and the sufficient locally admissible F. $G - F$ is (k, ℓ)-edge-connected in $V - z$ and $\rho(z) = \delta(z)$, satisfying the conditions of Theorem 1.6. For the graph G' resulting by a complete splitting at z, operation (ii) can be applyied to get G. □

The locally admissible edge sets are characterized by the following claim.

Claim 2.2. *$F \subseteq \Delta^{in}(z)$ is locally admissible at z if and only if $|F| \le k - \delta(z)$ and for each $\emptyset \ne X \subsetneq V$, $X \ne \{z\}$,*

$$\rho_{E-F}(X) \ge \gamma(X) \tag{1}$$

Proof. If F is locally admissible then for $X \ne V - z$, (1) is the necessary cut condition as $G - F$ is (k, ℓ)-edge-connected in $V - z$. If $X = V - z$ then it is equivalent to $\delta_{E-F}(z) \ge \ell$, which follows since $\delta_F(z) = 0$. The converse direction follows by Menger's theorem. □

It is easy to check in polynomial time whether a set of edges entering z is locally admissible. Furthermore these edge sets have a nice structure: they form a matroid. A consequence is that a building sequence can be found in polynomial time for a (k, ℓ)-edge-connected graph G. These will be shown in Section 6.

Consider now an arbitrary edge set $F \subseteq E$. Let F_v denote the subset of F entering vertex v. Let $\mu(X) = \delta_F(V - S - X, X)$, and let $t(X) = \min\{\delta_F(V - S - X, v) : v \in X\}$. A v giving the minimum value in the definition of $t(X)$ is called a **seed** of X. Let $T(X) = \max\{\rho_{F_v}(X) : v \in X\}$, and a v giving the maximum value is called a **sprout** of X. Note that a set can have multiple seeds and sprouts.

Definition 2.3. In a digraph $G = (V, E)$ with special vertices $S \subseteq V$, we say that $F \subseteq E$ is **globally admissible**, if

$$\rho(X) \ge \gamma(X) + \rho_F(X), \qquad \text{if } X - S \ne \emptyset, X \subsetneq V, \tag{2}$$

$$\rho(X) \ge k + T(X), \qquad \text{if } X \text{ is special}, |X| \ge 2, \tag{3}$$

$$\rho(X) \ge \gamma(X) + \mu(X) - t(X), \qquad \text{for every } \emptyset \ne X \subsetneq V, \tag{4}$$

$$|F_v| \le \rho(v) - \delta(v), \qquad \text{for every special vertex } v \text{ and,} \tag{5}$$

$$F_v = \emptyset, \qquad \text{if } v \notin S. \tag{6}$$

Note that if X is not special then all vertices in $X - S$ are seeds of X and $t(X) = 0$, thus (2) implies (4). For a special set X, we have two conditions. In the right hand side of (4), we consider only the edges coming from vertices not in S, however, not all such edges are taken into account. The importance of (3) is clear from the following claim.

Claim 2.4. *If F is globally admissible, then for each $v \in S$, F_v is locally admissible.*

Proof. We have to verify (1). If X is not special, then $\rho_{E-F_v}(X) \ge \rho_{E-F}(X) \ge \gamma(X)$, by (2). If X is special and $|X| \ge 2$, then by (3), $\rho_{E-F_v}(X) \ge \rho(X) - T(X) \ge k$. □

Claim 2.5. *If F is globally admissible in G and $F' \subseteq F$, then F' is also globally admissible in G.*

Proof. When removing an edge from F, the right hand sides in (2), (3) and (4) cannot increase. □

$F = \emptyset$ is globally admissible if and only if G is (k, ℓ)-edge-connected. By the above claim, any graph G that admits a globally admissible F is automatically (k, ℓ)-edge connected.

We say that a globally admissible set F is **maximal**, if there is no edge $uv \in E - F$ so that $F + uv$ is also globally admissible. A globally admissible F is called **sufficient** if (5) holds with equality for at least one special v, otherwise it is called **insufficient**.

Let us introduce now the various types of tight sets. We say that a set X is **tight** with respect to the globally admissible F, if at least one of (2), (3) or (4) holds with equality for X. A tight set with $X - S \neq \emptyset$ is called **normal tight**. A special tight X with $|X| \geq 2$ satisfying (3) with equality is called T-tight and for a **special tight** set for which (4) holds with equality we use the term μ-tight. If $s \notin X$ then X is called **in-tight** and if $s \in X$ then $V - X$ is called **out-tight**. Note that according to these definitions, an out-tight set is not necessarily tight.

Claim 2.6. *If F is globally admissible and for $uv \in E - F$, $v \in S$, $F + uv$ is not globally admissible, then uv enters a tight set X satisfying one of the followings: (a) X is a normal tight set, or (b) X is a T-tight set with sprout v, or (c) X is μ-tight, $u \in V - S$ and X has a seed t with $t \neq v$.*

Proof. By the maximality of F, $F + uv$ should violate one of (2), (3) or (4). If none of them holds with equality for F, then it cannot happen as the right hand sides may increase by at most 1. Thus uv must enter a tight set X. If X is T-tight and v is not a sprout of v, then $T(X)$ does not increase by adding uv to F thus (3) will not be violated for X. Similarly, if X is μ-tight and $u \in S$, then (4) remains unchanged for $F + uv$. If $u \notin S$ but the unique seed of X is v, then for $F + uv$, both $\mu(X)$ and $t(X)$ increase by 1. □

Note that if F is maximal globally admissible, this claim applies for every edge $uv \in E - F$, $v \in S$.

For technical reasons, we do not prove Theorem 2.1 directly, but a slight generalization. In order to state this form, the following notion is needed. A globally admissible edge set F **saturates** the graph G if every edge $e = uv \in E - F$ with $v \notin S$ enters a normal tight set. We are going to prove the following:

Theorem 2.7. *Let $F_0 \subseteq \Delta^{out}(s)$ be an arbitrary globally admissible set of edges in $G = (V, E)$ so that F_0 saturates G. Then there exists a sufficient globally admissible $F \supseteq F_0$.*

The (k, ℓ)-edge-connectivity of G follows by the existence of F_0. However, G is not assumed to be minimal subject to this property. Nevertheless $F_0 = \emptyset$ is a globally admissible edge set saturating G if and only if G is a minimally (k, ℓ)-edge-connected digraph. Thus Theorem 2.1 is a direct consequence of Theorem 2.7. Unfortunately, it is not true that *every* maximal globally admissible $F \supseteq F_0$ is sufficient, as shown by a counterexample in Section 6.

Let uv be an edge entering the tight set X. If $v \in S$ and X and uv satisfy one of the conditions in Claim 2.6 or $v \notin S$ and X is normal tight, then we say that X **blocks** uv.

We conclude this section with some elementary propositions.

Claim 2.8. *If $X, Y \subseteq V$, then*

$$\rho(X) + \rho(Y) = \rho(X \cap Y) + \rho(X \cup Y) + d(X, Y), \text{ and} \tag{7}$$
$$\rho(X) + \rho(Y) = \rho(X - Y) + \rho(Y - X) + (\rho(X \cap Y) - \delta(X \cap Y)) + \bar{d}(X, Y). \tag{8}$$

Claim 2.9. *For any $X, Y \subseteq V$,*

$$\gamma(X) + \gamma(Y) = \gamma(X \cup Y) + \gamma(X \cap Y), \text{ and} \tag{9}$$
$$\gamma(X) + \gamma(Y) \leq \gamma(X - Y) + \gamma(Y - X). \tag{10}$$

Claim 2.10. *For any $X \subseteq V$, $\rho(X) - \delta(X) = \sum_{v \in X} \rho(v) - \delta(v)$.*

Claim 2.11. *Assume F is insufficient globally admissible, and $Z \neq \emptyset$ is special. Then $\delta(Z) < \rho_{E-F}(Z)$.*

Proof. For each $v \in Z$, $\rho(v) - \delta(v) > |F_v|$, thus by summing for all $v \in Z$, $\rho(Z) - \delta(Z) = \sum_{v \in Z} \rho(v) - \delta(v) > \sum_{v \in Z} |F_v|$. The right hand side is at least $\rho_F(Z)$, thus the claim follows. □

Claim 2.12. *For $G = (U + u, E)$ with $\rho(u) = \delta(u)$, let G_u denote the result of an (arbitrary) complete splitting at u. Then for an $X \subsetneq U + u$, $\rho_{G_u}(X - u) \leq \rho_G(X)$.*

Proof. If $u \notin X$, then the claim follows since splitting off a pair of edges incident to u cannot increase the degree of $X = X - u$. In the case of $u \in X$, $\rho_{G_u}(X - u) \leq \delta_G(U - X, u) + \delta_G(U - X, X - u) = \rho_G(X)$. □

3 Proof of Theorem 2.7

The proof relies on three basic lemmata. First:

Lemma 3.1. *Let $F_0 \subseteq \Delta^{out}(s)$ be an insufficient globally admissible set of edges, and $\rho(u) = \delta(u)$ for some $s \neq u \in V$. There exists a complete splitting at u so that F_0 is globally admissible in the resulting graph.*

This will be proved in Section 4, the next two in Section 5.

Lemma 3.2. *Assume F' is a globally admissible edge set and X is a tight set with $|X| \geq 2$, $s \notin X$, $|X - S| \leq 1$. Then for any maximal globally admissible $F \supseteq F'$, F is sufficient.*

Lemma 3.3. *If F is maximal globally admissible with $u \in S + s$ for each $uv \in F$, then F is sufficient.*

Let us now turn to the proof of Theorem 2.7. Consider a counterexample $G = (V, E)$ and F_0 so that $|V|$ is minimal, and subject to this, $|F_0|$ is maximal. Consider a maximal globally admissible $F \supseteq F_0$. By the assumption, F is insufficient.

Case I

Assume there is a $u \in V$ with $\rho(u) = \delta(u) = k$. By Lemma 3.1, there is a complete splitting at u so that F_0 is globally admissible in the resulting graph $G_u = (V - u, E')$.

Claim 3.4. *F_0 saturates G_u.*

Proof. The set of special vertices is the same S in G and G_u. Consider an edge $e = yz$ in G_u with $z \notin S$. Assume first that e was an edge in G as well. There is a normal tight set $X \subseteq V$ blocking e in G, since F_0 saturated G. Claim 2.12 implies $\rho_{G_u}(X - u) \leq \rho_G(X)$. $X - u$ is also normal and as the subset of F_0 entering $X - u$ in G_u is the same as the subset in G entering X, it follows that $X - u$ blocks e in G_u.

If $e = yz$ is a new edge then take a set X that blocked uz in G. X is again a normal tight set in G_u. Note that $y \notin X$ otherwise the in-degree of X would be smaller in G_u than in G while the value of $\rho_{F_0}(X)$ does not change. Hence X blocks e in G_u, completing the proof. □

As G_u has less vertices than G, by the minimality of $|V|$ there exists a special vertex w and a sufficient locally admissible edge set F_w so that $F' = F_w \cup F_0$ is globally admissible. Note that w is special in G as well.

From G_u we can get to G by pinching the k splitted edges with u. By abuse of notation, we will denote by F_w the edge set in G corresponding to F_w in G_u in the sense that if an edge $xw \in F_w$ has been divided by u, then we replace xw by uw in F_w. We will also use F' in this sense in G. Unfortunately, it might happen that F' is not globally admissible in G. Consider a globally admissible F_1 maximal subject to the condition $F_0 \subseteq F_1 \subseteq F'$ with $|F_1|$ as large as possible. If $F_1 = F_0 \cup F_w$ then F_1 is sufficient as $\delta_G(w) = \delta_{G_u}(w)$ Otherwise, we are going to prove that there is a tight set Z for F_1 with $|Z - S| \leq 1$ so Lemma 3.2 is applicable giving a sufficient globally admissible superset of F_1.

Assume $F_w - F_1 \neq \emptyset$, and consider an edge $zw \in F_w - F_1$. By Claim 2.6, zw is blocked by some tight set Z with respect to F_1.

Claim 3.5. *$Z \subseteq S \cup \{u\}$*

Proof. $Z = V - u$ is impossible as $\delta_{F_1}(u) < |F_w| \leq k - \ell$, thus $\rho_{E-F_1}(V - u) > \ell$. Assume we have $V - Z - u \neq \emptyset$ and $Z - S - u \neq \emptyset$. As F' is admissible in G_u and $Z - u$ is not special, $\rho_{G_u,E'-F'}(Z - u) \geq \gamma(Z)$ follows. Claim 2.12 for $(V, E - F')$ implies $\rho_{G,E-F'}(Z) \geq \rho_{G_u,E'-F'}(Z - u)$. However, $\rho_{E-F_1}(Z) > \rho_{E-F'}(Z) \geq \gamma(Z)$ as $zw \in F_1 - F$ enters Z, showing that Z cannot be tight. This implies the claim. □

Case II

Assume Case I does not hold and there is an edge $uv \in F$ with $u \in V - S - s$. Let $G_1 = (V, E - uv + sv)$, $F_1 = F_0 + sv$.

Claim 3.6. *F_1 is globally admissible in G_1 and saturates it.*

Proof. If $|\{u, s\} \cap X| \neq 1$ then no term changes in the conditions (2), (3) and (4) for X. This is in fact always the case for (3). If $u \in X$, $s \notin X$, then in (2) and (4), both sides increase by one, while if $u \notin X$, $s \in X$, both sides decrease by one. (Note that $t(X) = 0$ in both cases as $X - S \neq \emptyset$.)

This implies the admissibility and that all tight sets remain the same. Thus if an edge $uv \in E - F$ with $v \notin S$ was blocked by a normal tight set for F_0 in G, then the same blocks it in G_1, proving the saturation. □

By the choice of G and F_0, there is a sufficient edge set $F' \supseteq F_1$ in G_1 with $|F'_w| = k - \delta_{G_1}(w)$ for some w special vertex in G_1. All vertices but u and s have the same in- and out-degrees in G and G_1, thus w is special in G unless $w = u$ and $\rho(w) = \delta(w) = k$. Having excluded Case I, this cannot happen.

Let $F'' = F' - sv + uv$. As in the previous claim, it is straightforward to show that F'' is globally admissible in G containing F_0.

Case III.

For all edges in $uv \in F$, $u \in S + s$. Now the conditions of Lemma 3.3 are satisfied, showing that F is sufficient.

4 Splitting off

A set function $X : 2^V \to \mathbb{R}_+$ is called positively crossing supermodular, if for crossing X and Y with $p(X) > 0$, $p(Y) > 0$,

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y).$$

Frank has formulated the following theorem as an abstract generalization of Mader's splitting off theorem:

Theorem 4.1 (Frank, [4])**.** *Let U be a ground-set, m_i and m_o two non-negative integer valued functions on U with $m_i(U) = m_o(U)$. Let p be a non-negative, integer valued positively crossing supermodular set function on U with $p(\emptyset) = p(U) = 0$. Then there exists a digraph $H = (U, A)$ for which*

$$\rho_H(X) \geq p(X) \text{ for every } X \subseteq V \tag{11}$$

and

$$\rho_H(v) = m_i(v), \delta_H(v) = m_o(v) \text{ for every } v \in V \tag{12}$$

if and only if

$$m_i(X) \geq p(X) \quad \text{for every} \quad X \subseteq U \text{ and} \tag{13}$$

$$m_o(U - X) \geq p(X) \quad \text{for every} \quad X \subseteq U. \tag{14}$$

Theorem 1.6 is a straightforward consequence: for a graph $G = (U + z, E)$ which is (k, ℓ)-edge connected in U with root node $s \in U$, let E' denote the set of edges induced by U. For $v \in U$, let $m_o(v) = \delta_E(v, z)$ and $m_i(v) = \delta_E(z, v)$. Let $p(\emptyset) = p(V) = 0$ and let $p(X) = \max(0, \gamma(X) - \rho_{E'}(X))^+$ otherwise. It is easy to check that this function is positively crossing supermodular and that the conditions of the theorem hold due to the (k, ℓ)-connectedness in U. A set H of edges ensured by the theorem corresponds to the splitted edges.

Now we present a generalization of this theorem. The only difference will be that we require a property slightly weaker than positively crossing supermodularity. We remark that it is still only a special case of a theorem in the master thesis of T. Király [7, Theorem 2.8]. Our proof follows the same lines as the proof given in [5] for Theorem 4.1.

Theorem 4.2. *Let U be a ground-set, m_i and m_o two non-negative integer valued functions on U with $m_i(U) = m_o(U)$. Let p be a non-negative, integer valued set function on U with $p(\emptyset) = p(U) = 0$ satisfying the following property. For crossing sets $X, Y \in U$, with $p(X), p(Y) > 0$, either*

$$p(X) + p(Y) \;\leq\; p(X \cap Y) + p(X \cup Y) \text{ or} \tag{15}$$
$$p(X) + p(Y) \;<\; p(X - Y) + p(Y - X) + m_i(X \cap Y) - m_o(X \cap Y). \tag{16}$$

Then there exists a digraph $H = (U, A)$ satisfying (11) and (12) if and only if (13) and (14) hold.

Proof. Necessity is obvious as $p(X) \leq \rho_H(X) \leq \min\{m_i(X), m_0(U - X)\}$. For sufficiency, assume for a contradiction that no such H exists. An H satisfying (12) might be found easily. Let $q_H(X) = p(X) - \rho_H(X)$ denote the violation of (11) for X and let $\nu_H = \max_{X \subseteq U} q_H(X)$ denote the maximum violation. Let $\mathcal{F}_H := \{X \subset U : q_H(X) = \nu_H\}$ the set of maximally violating sets. As (11) does not hold, $\nu_H > 0$, thus $p(X) > 0$ for every $X \in \mathcal{F}_H$.

Claim 4.3. *Let $X, Y \in \mathcal{F}_H$ crossing. Then both $X \cap Y$ and $X \cup Y$ belong to $\mathcal{F}_H$.*

Proof. If (15) holds for X and Y then $2\nu_H = p(X) + p(Y) - \rho_H(X) - \rho_H(Y) \leq p(X \cup Y) + p(X \cap Y) - \rho_H(X \cup Y) - \rho_H(X \cap Y) \leq 2\nu_H$, thus the claim follows. Assume now (16) holds. Observe that $m_i(X \cap Y) - m_0(X \cap Y) = \rho_H(X \cap Y) - \delta_H(X \cap Y)$. Using this,

$$\begin{aligned} 2\nu_H = p(X) + p(Y) - \rho_H(X) - \rho_H(Y) < \\ < p(X - Y) + p(Y - X) + m_i(X \cap Y) - m_o(X \cap Y) - \rho_H(X) - \rho_H(Y) \leq \\ \leq 2\nu_H + \rho_H(X - Y) + \rho_H(Y - X) + (\rho_H(X \cap Y) - \delta_H(X \cap Y)) - \rho_H(X) - \rho_H(Y). \end{aligned}$$

Finally we get

$$\rho_H(X) + \rho_H(Y) < \rho_H(X - Y) + \rho_H(Y - X) + (\rho_H(X \cap Y) - \delta_H(X \cap Y)),$$

a contradiction to (8). □

Assume H is chosen so that (*) ν_H is as small as possible, and subject to this, (**) $|\mathcal{F}_\mathcal{H}|$ is as small as possible. Let K be a minimal member of $\mathcal{F}$ and $L \supseteq K$ be a maximal member. There is an edge $e = uv$ of H with $u, v \in K$ and an $f = xy$ with $x, y \in U - L$ otherwise K or L would violate (13) or (14). Let us replace e and f by uy and xv and let H' denote the resulting digraph.

Now $\rho_{H'}(X) \geq \rho_H(X) - 1$ for every $X \subseteq V$ and equality may hold only if $X \cap \{x, y, u, v\}$ is either $\{x, v\}$ or $\{u, y\}$. This condition cannot hold for an $X \in \mathcal{F}$ as it would imply that X and K are crossing.

$K \notin \mathcal{F}_{H'}$ as $\rho_{H'}(K) = \rho_H(K) + 2$. So by (**), there is an $X \in \mathcal{F}_{H'} - \mathcal{F}_H$ with $q_H(X) = \nu_H - 1$. By symmetry we may assume $X \cap \{x, y, u, v\} = \{x, v\}$. $p(X), p(K) > 0$. Again (15) gives a contradiction easily, and if (16) holds, then

$$\begin{aligned} 2\nu_H - 1 = p(X) + p(K) - \rho_H(X) - \rho_H(K) < \\ < p(X - K) + p(K - X) + m_i(X \cap K) - m_o(K \cap X) - \rho_H(X) - \rho_H(K) \leq \\ \leq 2\nu_H - 1 + \rho_H(X - K) + \rho_H(K - X) + \rho_H(X \cap K) - \delta_H(X \cap K) - \rho_H(X) - \rho_H(K). \end{aligned}$$

In the last equation we have used that by the minimal choice of X and $X - K \neq \emptyset$, $q_H(X - K) \leq \nu_H - 1$. This is again a contradiction to (8). □

Now we can derive Lemma 3.1 as an easy consequence.

Proof of Lemma 3.1. As $F_0 \subseteq \Delta^{out}(s)$, $\mu(X) = \rho_F(X) = \delta_F(s, X)$ for every X. Observe that in this case we only have to guarantee (4) as it implies both (2) and (3).

Let $U = V - u$, and let $G' = (U, E')$ denote the deletion of u from G. Let us define $p(X)$ the following way. Let $p(\emptyset) = p(V) = 0$, otherwise let

$$p(X) := (\gamma(X) - \rho_{E'}(X) + \mu(X) - t(X))^+ = (\gamma(X) - \rho_{E'-F}(X) - t(X))^+$$

Let $m_o(z) = \delta_G(z, u)$ and $m_i(z) = \delta_G(u, z)$.

Claim 4.4. *The conditions of Theorem 4.2 are satisfied.*

Using this claim Lemma 3.1 follows immediately. Let us split off the edges incident to u according to the edge set H given by the theorem. As u was not special, the edges in F_0 are left unchanged. Let $G_u = (U, E' + H)$ denote the graph after the splitting. We have to prove that F is globally admissible in G_u. Again it is enough to verify (4), which is a direct consequence of $\rho_H(X) \geq p(X)$.

Proof of Claim 4.4. Consider crossing sets $X, Y \subseteq U$ with $p(X), p(Y) > 0$. $t(X) \geq t(X \cup Y)$ and if X has a seed in $X \cap Y$, then $t(X) = t(X \cap Y)$ and the same holds for exchanging X and Y. So if $X \cap Y - S \neq \emptyset$ or $X \cap Y$ is special but it contains a seed of X or Y, then $t(X) + t(Y) \geq t(X \cap Y) + t(X \cup Y)$ follows. In this case

$$\begin{aligned} p(X) + p(Y) = \gamma(X) + \gamma(Y) - t(X) - t(Y) - \rho_{E'-F}(X) - \rho_{E'-F}(Y) \leq \\ \leq \gamma(X \cup Y) + \gamma(X \cap Y) - t(X \cup Y) - t(X \cap Y) - \\ -\rho_{E'-F}(X \cup Y) - \rho_{E'-F}(X \cap Y) \leq p(X \cup Y) + p(X \cap Y), \end{aligned}$$

thus (15) holds. Assume now $X \cap Y$ is special and X has a seed $x \in X - Y$, Y has a seed $y \in Y - X$.

$$p(X) + p(Y) = \gamma(X) + \gamma(Y) - t(X) - t(Y) - \rho_{E'-F}(X) - \rho_{E'-F}(Y) \leq$$
$$\leq \gamma(X - Y) + \gamma(Y - X) - t(X) - t(Y) -$$
$$-\rho_{E'-F}(X - Y) - \rho_{E'-F}(Y - X) - (\rho_{E'-F}(X \cap Y) - \delta_{E'-F}(X \cap Y))$$

As F was insufficient, $|F_t| < \rho_E(t) - \delta_E(t)$ in the original graph G for every $t \in X \cap Y$, which implies $|F_t| < \rho_{E'}(t) + m_i(t) - \delta_{E'}(t) - m_o(t)$. This gives $m_o(t) - m_i(t) < \rho_{E'-F}(t) - \delta_{E'-F}(t)$, thus $m_o(X \cap Y) - m_i(X \cap Y) < \rho_{E'-F}(X \cap Y) - \delta_{E'-F}(X \cap Y)$. Now $t(X) = t(X - Y)$ and $t(Y) = t(Y - X)$ because of the seeds x and y, so we get

$$p(X) + p(Y) < \gamma(X - Y) + \gamma(Y - X) - t(X - Y) - t(Y - X) -$$
$$-\rho_{E'-F}(X - Y) - \rho_{E'-F}(Y - X) + (m_i(X \cap Y) - m_o(X \cap Y)) \leq$$
$$\leq p(X - Y) + p(Y - X) + m_i(X \cap Y) - m_o(X \cap Y).$$

It is left to verify (13) and (14). Let $X \subseteq U$. As F was globally admissible in G, $\rho_{E-F}(X) \geq \gamma(X) - t(X)$. $\rho_E(X) = m_i(X) + \rho_{E'}(X)$, giving (13). On the other hand, $\rho_{E-F}(X + u) \geq \gamma(X + u) - t(X + u) = \gamma(X)$ as $u \notin S$. $\rho_{E-F}(X + u) = m_o(U - X) + \rho_{E'-F}(X)$ thus $m_o(U - X) \geq \gamma(X) - \rho_{E'-F}(X)$, giving (14). □

□

5 Lemmata

Claim 5.1. *Assume $\emptyset \neq Z \subsetneq X \subsetneq V$, $X - Z \subseteq S$ and $\delta_{E-F}(Z, X - Z) = \emptyset$. Then $\rho(Z) < \rho(X) - \delta_F(V - X, X - Z)$ and $\rho_{E-F}(Z) < \rho_{E-F}(X)$.*

Proof. For the first part, $\delta(X - Z) < \rho_{E-F}(X - Z)$ by Claim 2.11 as $X - Z$ is special. $\rho(Z) = \rho(X) + \delta(X - Z, Z) - \delta_F(V - X, X - Z) - \delta_{E-F}(V - X, X - Z) < \rho(X) - \delta_F(V - X, X - Z)$ since $\delta(X - Z, Z) - \delta_{E-F}(V - X, X - Z) = \delta(X - Z, Z) - \rho_{E-F}(X - Z) \leq \delta(X - Z) - \rho_{E-F}(X - Z) < 0$ by the previous remark. The second part follows from this using $\rho_F(Z) + \delta_F(V - X, X - Z) \geq \rho_F(X)$. □

The next lemma describes strongly connectivity properties of various tight sets.

Lemma 5.2. *(i) Assume F is insufficient globally admissible, and X is an out-tight set. If for some $Z \subseteq X$, $\delta_{E-F}(Z, X - Z) = 0$, then Z is out-tight and $\Delta^{out}_{E-F}(Z) = \Delta^{out}_{E-F}(X)$. (ii) If X is normal in-tight, $Z \subseteq X$, then $\delta_{E-F}(Z, X - Z) = 0$ implies that $X - Z$ is also normal in-tight and $\Delta^{in}_{E-F}(X) = \Delta^{in}_{E-F}(X - Z)$. (iii) If X is μ-tight, and u is a seed of X, then there is an edge $uv \in E - F$ with $v \in X$. (iv) If X is T-tight and v is a sprout of X, then there is an edge $uv \in E - F$ with $u \in X$.*

Proof. (i) $\delta_{E-F}(X) = \ell$ and $\delta_{E-F}(Z) \geq \ell$. Thus if $\delta_{E-F}(Z, X - Z) = 0$ then all edges leaving Z must leave X as well, and this is what we wanted to prove.

(ii) Assume first $X - Z - S \neq \emptyset$. $\rho_{E-F}(X) = k$, $\rho_{E-F}(X - Z) \geq k$, and the claim follows as in the first part.

Assume now $X - Z$ is special. By Claim 5.1, $\rho_{E-F}(Z) < \rho_{E-F}(X) = k$, a contradiction as X was not special, thus neither is Z.

(iii) $\rho(X) = k+\delta_F(V-X-S, X-u)$. If all edges in X from u are in F, then we can use Claim 5.1 for $Z = \{u\}$, thus $k = \rho(u) < k+\delta_F(V-X-S, X-u)-\delta_F(V-X, X-u) \leq k$, a contradiction.

(iv) $\rho(X) = k + T(X) = k + \delta_F(V - X, v)$. If all edges in X entering v are in F, then Claim 5.1 can be applied for $Z = X-v$. Thus $k \leq \rho(X-v) < k+T(X)-\delta_F(V-X, v) = k$, a contradiction again. □

Claim 5.3. *For sets $\emptyset \neq Z \subseteq X$, $X - Z \subseteq S$, if X has a seed $u \in Z$ then $t(X) = t(Z)$.*

Proof. As $X - Z \in S$, for any $x \in Z$, $\delta_F(V - Z - S, x) = \delta_F(V - X - S, x)$. u is the vertex in X minimizing $\delta(V - X - S, x)$, thus the claim follows. □

In the next lemma, we show some configurations of tight sets which may not exist for an insufficient globally admissible F.

Lemma 5.4. *Assume F is insufficient globally admissible. There exists no $X \subseteq V$ with the following properties: $|X| \geq 2$, X is in-tight and (i) $X - S \neq \emptyset$ and there is a subpartition $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ of X so that $X - S \subseteq \cup\mathcal{Y}$, each Y_i is out-tight and proper subset of X or (ii) X is μ-tight and there is an out-tight $Y \subsetneq X$ containing a seed u of X; (iii) X is T-tight and there is an out-tight $Y \subsetneq X$ not containing a sprout z of X.*

Proof. (i) We may assume that there is no special Y_i as leaving out such members from $\mathcal{Y}$ the conditions still hold. Thus $\rho_{E-F}(Y_i) \geq k$ for each i and $\delta_{E-F}(Y_i) = \ell$ as they are out-tight sets. Let $X_0 = X - \cup\mathcal{Y}$. As X_0 is special, Claim 2.11 implies $\rho_{E-F}(X_0) - \delta_{E-F}(X_0) > \delta_F(X_0)$ if $X_0 \neq \emptyset$. Now $\rho_{E-F}(X) = k$, $\delta_{E-F}(X) \geq \ell$, thus

$$k - \ell \geq \rho_{E-F}(X) - \delta_{E-F}(X) =$$
$$= (\rho_{E-F}(X_0) - \delta_{E-F}(X_0)) + \sum_{i=1}^{m}(\rho_{E-F}(Y_i) - \delta_{E-F}(Y_i)) \geq \delta_F(X_0) + \sum_{i=1}^{m}(k - \ell),$$

a contradiction, since either $X_0 \neq \emptyset$ and thus the last inequality is strict, or $m \geq 2$ as we did not allow $\mathcal{Y} = \{X\}$.

(ii) Let u denote a seed of X as in the conditions. $t(X) = t(Y)$ by Claim 5.3. $\delta(Y) = \ell+\delta_F(Y)$ as Y is out-tight. Claim 2.11 gives $\rho(X-Y)-\delta(X-Y) > \rho_F(X-Y)$. Similarly to the previous case,

$$k + \mu(X) - t(X) - \ell - \delta_F(X) \geq \rho(X) - \delta(X) = \rho(X - Y) - \delta(X - Y) +$$
$$+\rho(Y) - \delta(Y) > \rho_F(X - Y) + k + \mu(Y) - t(Y) - \ell - \delta_F(Y).$$

This gives $\delta_F(Y) - \mu(Y) > \delta_F(X) + \rho_F(X - Y) - \mu(X)$. Using $\mu(X) = \mu(Y) + \delta_F(V - X - S, X - Y)$ and $\delta_F(Y) \leq \delta_F(X) + \delta_F(Y, X - Y)$, one gets $\delta_F(Y, X - Y) + \delta_F(V - X - S, X - Y) > \rho_F(X - Y)$, clearly a contradiction.

(iii) As in the previous two cases,

$$k + T(X) - \ell - \delta_F(X) \geq \rho(X) - \delta(X) = \rho(X - Y) - \delta(X - Y) + \\ + \rho(Y) - \delta(Y) > \rho_F(X - Y) + k - \ell - \delta_F(Y)$$

Thus $\delta_F(Y) + T(X) > \delta_F(X) + \rho_F(X - Y)$. As $\delta_F(Y) \leq \delta_F(X) + \delta_F(Y, X - Y)$ and $T(X) = \delta_F(V-X, z)$, we have $\delta_F(Y, X-Y) + \delta_F(V-X, z) > \rho_F(X-Y)$, a contradiction again. □

Claim 5.5. *(a) Assume F is insufficient globally admissible and $X \cap Y$ is special. Then $\rho(X) + \rho(Y) > \rho(X - Y) + \rho(Y - X) + \delta_F(V - X, X \cap Y) + \delta_F(V - Y, X \cap Y)$.*

(b) If Y is normal tight, $Y - X - S \neq \emptyset$, $s \notin X \cap Y$, then $\rho(Y) \leq \rho(Y - X) + \delta_F(V - Y, X \cap Y)$.

Proof. (a) By (8), it is enough to prove that $(\rho(X \cap Y) - \delta(X \cap Y)) + \bar{d}(X, Y) > \delta_F(V - X, X \cap Y) + \delta_F(V - Y, X \cap Y)$. By Claim 2.11, $\rho_F(X \cap Y) < \rho(X \cap Y) - \delta(X \cap Y)$ and obviously, $\delta_F(V - X - Y, X \cap Y) \leq \bar{d}(X, Y)$. These together imply the claim.

(b) Since $Y - X$ is not special, $\rho(Y - X) \geq \gamma(Y - X) + \rho_F(Y - X)$ and $\gamma(Y - X) = \gamma(Y)$ as $s \notin X \cap Y$. Using these,

$$\rho(Y) = \gamma(Y) + \rho_F(Y) = \gamma(Y) + \delta_F(V - Y, Y - X) + \delta_F(V - Y, X \cap Y) \leq \\ \leq \gamma(Y - X) + \rho_F(Y - X) + \delta_F(V - Y, X \cap Y) \leq \rho(Y - X) + \delta_F(V - Y, X \cap Y).$$

□

We are almost ready to prove Lemma 3.2. The following lemma is slightly weaker, but will easily imply it.

Lemma 5.6. *If F' is globally admissible and there exists a special tight set X with $|X| \geq 2$, then any $F \supseteq F'$ maximal globally admissible set will be sufficient.*

Proof. Let F be a maximal globally admissible set containing F'. Clearly, X is tight for F as well. Let X be chosen minimal subject to these conditions. We prove that F is sufficient.

First assume that X is a T-tight set with sprout z. By Lemma 5.2(iv) there is an edge $uz \in E - F$ with $u \in X$. By Claim 2.6, uz must enter a tight set Y which is either normal or T-tight with sprout z. Case (c) is excluded since u is special.

First assume Y is normal. If $V - Y \subseteq X$ then we have a contradiction by Lemma 5.4(iii) as $V - Y$ is an out-tight set satisfying the conditions. $Y \subset X$ is impossible as it would give $Y \subseteq S$. Thus X and Y are crossing.

$$\rho(X) = k + T(X) \leq \rho(X - Y) + \delta_F(V - X, X \cap Y) \tag{17}$$

as $z \in X \cap Y$ and $\rho(X - Y) \geq \gamma(X - Y) = k$. Using both Claim 5.5(b) and (a) we get a contradiction unless F is sufficient.

If Y is a T-tight set, by the minimality of X, X and Y are crossing. (17) holds again and also $\rho(Y) = k + T(Y) \leq \rho(Y - X) + \delta_F(V - Y, X \cap Y)$ as $z \in X \cap Y$ is also a sprout of Y. A contradiction again.

Assume now X is μ-tight with seed u. By Lemma 5.2(iii), we have a $uv \in E - F$ with $v \in X$ blocked by a tight set Y. In the first part of the proof we have already seen that no T-tight sets exist. Neither may Y be μ-tight since u is special. Thus Y should be normal. Again $V - Y \subseteq X$ would contradict Lemma 5.4(ii) and $Y \subset X$ is impossible, thus X and Y should be crossing. Using Claim 5.3 for X and $Z = X - Y$, $t(X - Y) = t(X)$. Thus

$$\begin{aligned}\rho(X) = k + \mu(X) - t(X) = k + \delta_F(V - S - X, X) - t(X - Y) = \\ k + \delta_F(V - S - X, X - Y) - t(X) + \delta_F(V - S - X, X \cap Y) \leq \\ \leq \rho(X - Y) + \delta_F(V - X, X \cap Y).\end{aligned}$$

Using again Claim 5.5(b) and (a) give a contradiction. □

Lemma 5.7. *Assume F is a maximal, insufficient globally admissible set of edges. If X and Y are crossing tight sets, then $X \cup Y$ and $X \cap Y$ are tight as well. If X or Y blocks an edge $uv \in E - F$, then either $X \cup Y$ or $X \cap Y$ blocks uv as well.*

Proof. By Lemma 5.6, we know that both X and Y are normal tight. Assume first that $(X \cap Y) - S \neq \emptyset$. From (9) and the submodularity of the ρ function we have:

$$\begin{aligned}\rho_{E-F}(X) + \rho_{E-F}(Y) = \gamma(X) + \gamma(Y) = \gamma(X \cap Y) + \gamma(X \cup Y) \leq \\ \leq \rho_{E-F}(X \cap Y) + \rho_{E-F}(X \cup Y) \leq \rho_{E-F}(X) + \rho_{E-F}(Y),\end{aligned}$$

implying that both $X \cap Y$ and $X \cup Y$ are tight and $d_{E-F}(X, Y) = 0$. The second part of the claim follows as both of them are normal.

Suppose that $X \cap Y \subseteq S$. We show this is impossible. $X - Y$ and $Y - X$ are both non-special sets, thus Claim 5.5(b) applies for Y and also for X by exchanging the role of X and Y. Claim 5.5(a) leads to a contradiction again. □

An easy consequence of Lemma 5.7 is the following:

Claim 5.8. *If F is maximal insufficient globally admissible and $uv \in E - F$, either there is a unique minimal in-tight set B^{in}_{uv} blocking uv or a unique minimal out-tight B^{out}_{uv} blocking uv. If $u, v \in X$ for an in- or out-tight set X, then $B^{in}_{uv} \subseteq X$ or $B^{out}_{uv} \subseteq X$.*

Proof. By Lemma 5.7, for every edge $uv \in E - F$ there is a unique minimal B_1 and a unique maximal B_2 in-tight set entered by uv. If $s \notin B_1$ then B_1 is in-tight thus $B^{in}_{uv} = B_1$, if $s \in B_1$ then $B^{out}_{uv} = V - B_2$. (Note that both sets may exist). The second part also follows by Lemma 5.7. □

Now we are ready to prove Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. By Lemma 5.6, the only case left is if X is normal tight with $s \notin X$, $|X - S| = 1$. Let $X - S = \{u\}$. If there is no edge in $E - F$ from u to $X - u$, then by Lemma 5.2, $X - u$ is normal in-tight, a contradicting $X - u \subseteq S$. Thus there is an edge $uv \in E - F$ with $v \in X$. Let $Y = B^{in}_{uv}$ or $Y = B^{out}_{uv}$ as in Claim 5.8. In the first case $Y \subseteq S$ contradicting that it is a tight set and every tight set is normal. In the second case, X and $\mathcal{Y} = \{Y\}$ satisfy the conditions of Lemma 5.4(i), a contradiction again. □

Proof of Lemma 3.3. Let K denote the set of in-tight singletons and L the set of out-tight singletons.

Claim 5.9. $K \cap L = \emptyset$.

Proof. Let $u \in K \cap L$. Trivially, $u \neq s$. As a singleton tight set cannot be special, $\rho(u) = k$ and $\delta(u) \geq k$. However, the out-tightness of $\{u\}$ implies $\delta_{E-F}(u) = \ell$, thus $\delta_F(u) > 0$, a contradiction. □

Claim 5.10. *If an edge $f = uv \in E - F$ is blocked by an in-tight set, then $B^{in}_{uv} = \{v\}$. If it is blocked by an out-tight set, then $B^{in}_{out} = \{u\}$.*

Proof. Consider a minimal in-tight or out-tight set X for some edge $f = xy \in E - F$ which is not a singleton. By Lemma 5.2(i) or (ii) and the minimality of X, X is strongly connected in $E - F$. We show that either $X \subseteq K$ or $X \subseteq L$. Consider an edge $uv \in E - F$ with $u, v \in X$, guaranteed by the strong connectivity. By Claim 5.8, either uv enters a minimal in-tight or leaves a minimal out-tight Y with $Y \subseteq X$. By the minimal choice of X, Y is a singleton: $Y = \{u\} \in L$ or $Y = \{v\} \in K$. Thus either $X \cap K \neq \emptyset$ or $X \cap L \neq \emptyset$.

Assume first $X \cap K \neq \emptyset$ and let $Z = X \cap K$. If $X - Z \neq \emptyset$, then by the strongly connectedness there is an edge $uv \in E - F$ with $u \in Z$ and $v \in X - Z$ blocked by a minimal in- or out-tight set Y. Again, Y is a singleton and either $Y = \{u\} \in L$ or $Y = \{v\} \in K$. Both cases are impossible since $u \in X \cap K$, and $v \in X - K$. Thus we may conclude $X \subseteq K$.

For next, consider $X \cap L \neq \emptyset$ and let $Z = X \cap L$. If $X - Z \neq \emptyset$, then an edge $uv \in E - F$ with $u \in X - Z$, $v \in Z$ gives the contradiction as above. Thus $X \subseteq L$ follows.

X was either in- or out-tight. If $X = B^{out}_{xy}$ is out-tight, then $X \subseteq L$ is excluded as it would give $B^{out}_{xy} = \{x\}$. Thus $X \subseteq K$. As $K \cap S = \emptyset$, for each $u \in X$, $\rho(u) = k$, $\delta(u) \geq k$. By the assumption that all edges in F have tail in $S + s$, $\delta_F(X) = 0$ thus $\delta(X) = \ell$. Now

$$k - \ell \leq \rho(Z) - \delta(Z) = \sum_{u \in Z} (\rho(u) - \delta(u)) \leq 0,$$

giving a contradiction.

If $X = B^{in}_{xy}$ is in-tight, then $X \subseteq K$ is excluded since it would give $B^{in}_{xy} = \{y\}$. Thus $X \subseteq L$. $X - S \neq \emptyset$ as all tight sets are normal by Lemma 5.6, thus the conditions of Lemma 5.4(i) apply with $\mathcal{Y}$ being the partition of X into singletons. □

$s \notin K$ implies $K \neq V$. Also $K \neq \emptyset$ as by Claim 5.10, all edges leaving s should enter members of K. As $\rho_{E-F}(V - K) \geq \ell$, there is an edge $uv \in E - F$ leaving K. This cannot be blocked by neither an in-tight nor an out-tight singleton. □

6 Further notes

6.1 Matroid property of locally admissible sets

First, we describe the structure of the locally admissible edge sets at a given special node z. We will prove

Theorem 6.1. *The set system $M_z = \{F : F$ is locally admissible at $z\}$ is a matroid.*

This together with Theorem 2.1 gives a straightforward way to find a sufficient locally admissible edge set. By the Theorem we know that special vertices exist and one of them has a sufficient locally admissible set. We check the special vertices one-by-one, and at each special vertex z we greedily choose a maximal locally admissible edge set. Note that this can be done easily as we just need to take care of the (k, ℓ)-edge connectedness in $V - z$ which can be checked by flow computations. Theorem 6.1 ensures that if z admits a sufficient global admissible edge set, we find it this way.

Proof of Theorem 6.1. The only nontrivial property we have to check is that if $|F| < |F'|$ and both $F, F' \in M_z$ then there is an edge $uz \in F' - F$ so that $F + uz$ is locally admissible as well. For a contradiction, assume this does not hold. A set X will now be called tight at z for F if $z \in X$, $X \neq \{z\}$ and it satisfies (1) with equality. (Actually this notion coincides with the tight sets containing z when we consider F as a globally admissible set of edges). Note that since $|F'| \leq k - \delta(z)$ by definition and $|F| < |F'|$, $|F'|$ is insufficient.

Claim 6.2. *If X and Y are crossing tight sets at z for F then so are $X \cap Y$ and $X \cup Y$.*

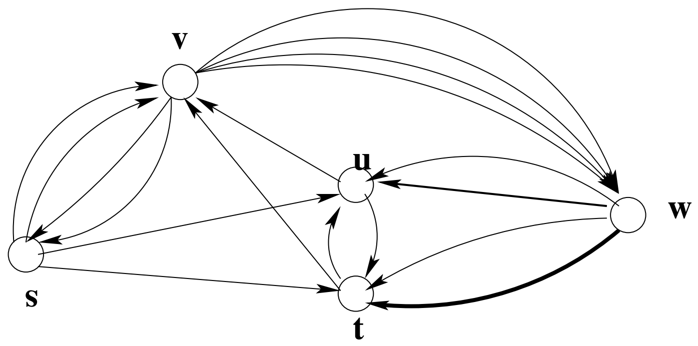
Proof. If $X \cap Y \neq \{z\}$, then (1) also holds for $X \cap Y$ and $X \cup Y$ thus the claim follows by the submodularity of the function ρ_{E-F}. We show that the case $X \cap Y = \{z\}$ is impossible. In this case we would have by (8) $2k = \rho_{E-F}(X) + \rho_{E-F}(Y) \geq \rho_{E-F}(X - Y) + \rho_{E-F}(Y - X) + \rho_{E-F}(z) - \delta_{E-F}(z) > \rho_{E-F}(X - Y) + \rho_{E-F}(Y - X) \geq 2k$, as F was insufficient. □

Thus for each edge $uz \in F' - F$ there is a minimal tight set X_{uz} for F entered by uz. For different $uz, wz \in F' - F$, X_{uz} and X_{wz} cannot be crossing as $X_{uz} \cap X_{wz}$ would also be tight contradicting their minimality. Thus $X_{uz} \cup X_{wz} = V$. Let $\mathcal{T} = \{V - X_{uz} : uz \in F' - F\}$. $\mathcal{T}$ forms a subpartition of $V - z$ so that for each $uz \in E - F$, u is in some member of $\mathcal{T}$. For each $Y \in \mathcal{T}$, $\delta(Y) = \gamma(V - Y) + \delta_F(Y)$. Let $m(Y) = |(F' - F) \cap \delta(Y)|$. As F' is locally admissible, $\delta_{F'}(Y) \leq \delta(Y) - \gamma(V - Y) = \delta_F(Y)$, thus $m(Y) \leq \delta_{F-F'}(Y)$. Summing up for all $Y \in \mathcal{T}$ we get $|F' - F| = \sum_{Y \in \mathcal{T}} m(Y) \leq \sum_{Y \in \mathcal{T}} \delta_{F-F'}(Y) \leq |F - F'|$, contradicting $|F| < |F'|$. □

6.2 Example of an insufficient maximal globally admissible set

An example for an insufficient maximal globally admissible set is shown on the figure for $k = 4$, $\ell = 2$. G is minimally $(4, 2)$-edge-connected. It contains two special vertices u and t with in-degree 4 and out-degree 2. Both of them have a sufficient locally admissible edge set: for both u and t the two edges coming from w are sufficient locally admissible. However, if we consider F consisting of one wu and on wt edge (the thick edges), F is maximal as the following sets block every edge entering u and t: $\{u\}$, $\{t\}$ $\{w\}$ are out-tight and $\{u, t, v, w\}$ is in-tight. However, F is insufficient.

The proof of the case $\ell = k - 1$ by Frank and Király [5] used an argument similar to the proof of Lemma 3.3. It is for the following reason why this argument cannot be applied in the general case to prove that every maximal globally admissible set is sufficient (and in fact, this is not true). Claim 5.9 fails to hold unless F satisfies the condition in Lemma 3.3: in this example the singleton set $\{w\}$ is both in- and out-tight.



Acknowledgement

We are grateful to András Frank for drawing our attention to this problem. We wish to thank Tamás Király his valuable comments and suggestions.

References

[1] J. Edmonds. Edge disjoint branchings. In B. Rustin, editor, *Combinatorial Algorithms*, pages 91–96. Academic Press, New York, 1973.

[2] A. Frank. On the orientation of graphs. *J. Comb. Theory, Ser. B.*, 28(3):251–261, 1980.

[3] A. Frank. Connectivity and network flows. In L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of combinatorics (vol. 1)*, pages 111–177. MIT Press, Cambridge, MA, USA, 1995.

[4] A. Frank. Connectivity augmentation problems in network design. In J. Birge and K. Murty, editors, *Mathematical Programming: State of the Art*, pages 34–63. The University of Michigan, 1999.

[5] A. Frank and Z. Király. Graph orientations with edge-connection and parity constraints. *Combinatorica*, 22(1):47–70, 2002.

[6] A. Frank and L. Szegő. Constructive characterizations for packing and covering with trees. *Discrete Appl. Math.*, 131(2):347–371, 2003.

[7] T. Király. The splitting off operation and its applications. Master's thesis, Eötvös University, Budapest, 2002. available online at `http://www.cs.elte.hu/`.

[8] E. R. Kovács and L. A. Végh. Constructive characterization theorems in combinatorial optimization. In *Proceedings of the Kyoto RIMS Workshop on Combinatorial Optimization and Discrete Algorithms*, 2008.

[9] W. Mader. Konstruktion aller n-fach kantenzusammenhängenden digraphen. *Europ. J. Combinatorics*, 3:63–67, 1982.

[10] W. T. Tutte. On the problem of decomposing a graph into n connected factors. *J. London Math. Soc.*, pages 1–36, 1961.