Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2008-15. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Recognizing conic TDI systems is hard

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October 2008

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Abstract

In this note we prove that the problem of deciding whether or not a set of integer vectors forms a Hilbert basis is co-NP-complete. Equivalently, deciding whether a conic linear system is totally dual integral (TDI) or not, is co-NP-complete. These are true even if the vectors in the set or respectively the coefficient vectors of the inequalities are $0-1$ vectors having at most three ones.

1 Introduction

Total dual integrality of systems of linear inequalities was introduced by Edmonds and Giles [3] and plays an important role in polyhedral combinatorics. A linear system $Ax \leq b$ with rational A and b is called *totally dual integral* (or *TDI*) if for each integer vector c, the dual system $\{\min y^\top b : \ y \geq 0, \ y^\top A = c^\top\}$ has an integral optimal solution provided the optimum is finite. Edmonds and Giles celebtated theorem states that if $Ax \leq b$ is TDI and b is integer, then $\{x : \ Ax \leq b\}$ is an integer polyhedron. By consequence the TDI property is a common framework to prove a bunch of min-max relations in combinatorial optimization.

Giles and Pulleyblank [4] introduced a related notion, namely that of Hilbert bases. A set of integer vectors is called a *Hilbert basis* if every integer vector in their cone can be written as a nonnegative integral combination of them. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ ($\mathbf{v}_i \in \mathbb{Z}^n$) is a Hilbert basis if $\text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) = \text{cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) \cap \mathbb{Z}^n$, where $\text{cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ denotes the cone generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and
$\text{int.cone}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ (their integral cone) is the set of vectors $\{\mathbf{z} \in \mathbb{Z}^n : \mathbf{z} = \sum_{i=1}^m \lambda_i \mathbf{v}_i, (\lambda_i \in \mathbb{Z}_+)\}$.

The connection between Hilbert bases and TDI-ness was established by Giles and Pulleyblank [4], who showed that for an integer matrix A, $Ax \leq b$ is TDI if and only if for every minimal face F of $\{x : \ Ax \leq b\}$ the active rows in F form a Hilbert basis where a row $a_i^\top$ of A is called *active in* F if $a_i^\top x = b_i$ holds for every x in F. Giles and Pulleyblank used this characterization to prove that for every rational polyhedron P there exists a TDI system which describes P, with an integer constraint matrix.

The complexity of deciding whether a system is TDI or not, was open for a long time. First Cook, Lovász and Schrijver [1] showed that if the dimension is fixed then

⋆This work was done while the author was visiting Laboratoire G-SCOP, Grenoble. Research supported by OTKA Grant No. K60802.

this problem is polynomially solvable. Recently Ding, Feng and Zang [2] proved that in general the problem is co-NP-complete, even if A is the incidence matrix of a graph.

In this note we strengthen this by answering a question about the complexity of deciding TDI-ness of a conic system $Ax \geq 0$. We prove that this problem is also co-NP-complete, which, by the above result of Giles and Pulleyblank, is equivalent to the following.

Theorem 1.1. *The problem of deciding whether or not a set of integer vectors forms a Hilbert basis is co-NP-complete even if the set consists of* $0-1$ *vectors having at most three ones.*

Let $\mathcal{H} = (V, E)$ be a hypergraph. We call a hyperedge of $\mathcal{H}$ of size 2 a *2-edge* and one of size 3 a *3-edge*. We say that $\mathcal{H}$ is a *2-3-hypergraph* if each hyperedge is of size 2 or 3. Let us denote by cone($\mathcal{H}$) and int.cone($\mathcal{H}$) the cone and integer cone of the characteristic vectors of the hyperedges of $\mathcal{H}$. Sometimes we will not distinguish between a hyperedge and its characteristic vector. The binary vectors in Theorem 1.1 will consist of the characteristic vectors of a 2-3-hypergraph.

We will denote the set $\{1, 2, \dots i\}$ by $[i]$ for $i \in \mathbb{N}$.

Related questions were studied in [5]. For a survey of the connection of Hilbert bases to combinatorial optimization see [7].

2 Proof of Theorem 1.1

Proof. For the sake of completeness we sketch the proof of the problem being in co-NP. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a set of integer vectors which is not a Hilbert basis. and let F be the minimal face of cone(S). It can be seen that int.cone($S \cap F$) is equal to the lattice generated by $S \cap F$. Thus if there exists an integer vector in F which can not be written as a nonnegative integer combination of vectors in S, then the lattice generated by $S \cap F$ is a proper subset of $F \cap \mathbb{Z}^n$, for which there is a certificate, see [6]. If there does not exist such a vector then it can be seen that there is an integer vector $\mathbf{z}$ in the zonotope of the vectors in S (that is in the set $\{\mathbf{v} : \ \mathbf{v} = \sum_{i=1}^{m} \lambda_i \mathbf{v}_i, \ 0 \leq \lambda_i < 1\}$) for which $\mathbf{z} - \mathbf{v}_i \notin$ cone(S) for all $\mathbf{v}_i \in S \setminus F$. In this case, $\mathbf{z}$ is a certificate.

To prove completeness we reduce the 3-SATISFIABILITY (3SAT) problem to the complement of this problem. Let $X = \{x_1, \dots, x_p\}$ be the set of variables and $\mathcal{C} = \{c_1, \dots, c_q\}$ be the set of clauses of an arbitrary 3SAT-instance.

Let the clause c_i be $c_i^1 \vee c_i^2 \vee c_i^3$, where $c_i^j \in X \cup \bar{X}$ ($j \in [3]$, $\bar{X}$ denotes the set of negated literals $\{\bar{x}_1, \dots, \bar{x}_p\}$).

We aim at constructing a hypergraph $\mathcal{H} = (V, E)$ (with $|V|$ and $|E|$ linear in p and q and maximal edge size three) such that $\mathcal{C}$ is satisfiable if and only if the characteristic vectors of the hyperedges of $\mathcal{H}$ do not form a Hilbert basis.

Let the groundset V of the hypergraph $\mathcal{H}$ be

$$V = \{u_i, v_j, \bar{v}_j, w_k^l \ (i \in \{0, 1, \dots p+q\}, j \in [p], k \in [q], l \in [3])\},$$

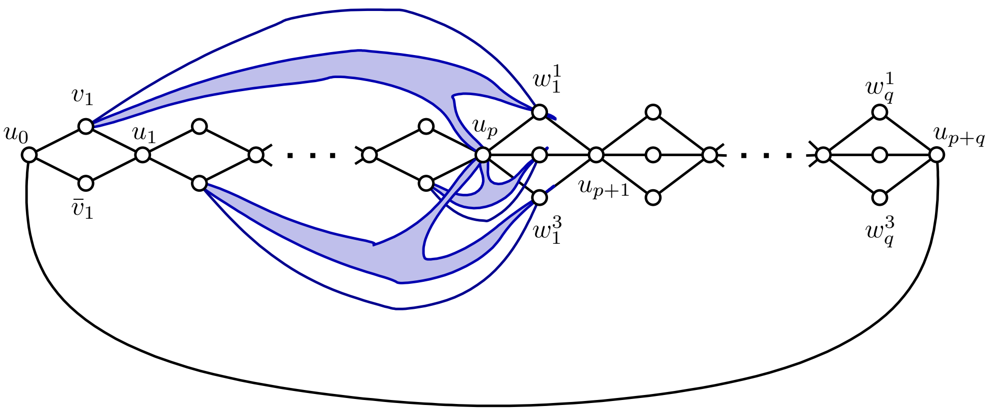

Figure 1: Part of hypergraph $\mathcal{H}$ where $c_1 = \bar{x}_1 \vee x_p \vee x_2$

where we say that the nodes v_j and $\bar{v}_j$ correspond to the literals x_j and $\bar{x}_j$, and nodes w_k^1, w_k^2, w_k^3 correspond to the three literals of clause c_k.

Let the hyperedge-set of $\mathcal{H}$ be $E = E_1 \cup E_2 \cup E_3$ (in figure 1, the the black 2-edges are in E_0, the blue 2-edges in E_1 and the 3-edges in E_3), where

$$E_1 = \{u_0 u_{p+q}\} \cup \{u_{i-1} v_i, u_{i-1} \bar{v}_i, u_i v_i, u_i \bar{v}_i \ (i \in [p])\} \\ \cup \{u_{p+k-1} w_k^l, u_{p+k} w_k^l \ (k \in [q], l \in [3])\},$$

$$E_2 = \{v_j w_k^l : \text{if } c_k^l = \bar{x}_j \ (j \in [p], k \in [q], l \in [3])\} \\ \cup \{\bar{v}_j w_k^l : \text{if } c_k^l = x_j \ (j \in [p], k \in [q], l \in [3])\},$$

$$E_3 = \{u_p v_j w_k^l : \text{if } c_k^l = \bar{x}_j \ (j \in [p], k \in [q], l \in [3])\} \\ \cup \{u_p \bar{v}_j w_k^l : \text{if } c_k^l = x_j \ (j \in [p], k \in [q], l \in [3])\}.$$

Notice that $\mathcal{H} - u_p$ is a bipartite graph.

We call a cycle a *choice-cycle* if its edges are in E_1 and has length $2(p+q)+1$. Such a cycle uses exactly one of $\{v_j, \bar{v}_j\}$ for each $j \in [p]$ and exactly one of $\{w_k^1, w_k^2, w_k^3\}$ for each $k \in [q]$. A cycle is *induced* if its node set does not induce other hyperedges from E.

Claim 2.1. *$\mathcal{C}$ is satisfiable if and only if there exists an induced choice-cycle in $\mathcal{H}$.*

Proof. Suppose that $\tau : X \mapsto \{\text{TRUE}, \text{FALSE}\}$ is a satisfying truth assignment for $\mathcal{C}$. Then the nodes u_i $(i \in \{0, 1, \dots p+q\})$, and the nodes in $\{v_j, \bar{v}_j : j \in [p]\}$ corresponding to the true literals, and for each $k \in [q]$ one node from $\{w_k^1, w_k^2, w_k^3\}$ which corresponds to a true literal induce a choice-cycle.

On the other hand, if Q is an induced choice-cycle then the assignment

$$\tau(x_j) := \begin{cases} \text{TRUE if } v_j \in V(Q) \\ \text{FALSE if } \bar{v}_j \in V(Q) \end{cases}$$

satisfies $\mathcal{C}$. □

Using Claim 2.1 one can check that the satisfiability of $\mathcal{C}$ implies that $\{\chi_e : e \in E\}$ is not a Hilbert basis: for an induced choice-cycle Q the incidence vector of its vertex-set, $\chi_{V(Q)}$ is in cone($\mathcal{H}$) but is not in int.cone($\mathcal{H}$) because every nonnegative integer linear combination which gives $\chi_{V(Q)}$ can only use the hyperedges of Q (C being an induced cycle), and the characteristic vectors of these hyperedges are linearly independent so there is a unique linear combination of hyperedges of Q that gives $\chi_V(Q)$ and that is the all-1/2 vector.

It remains to prove that if $\mathcal{C}$ is not satisfiable then the incidence vectors of E form a Hilbert basis. Let $0 \neq \mathbf{z} \in \mathbb{Z}^V \cap \text{cone}(\mathcal{H})$. Since $\mathbf{z} \in \text{cone}(\mathcal{H})$, using Carathéodory's theorem, $\mathbf{z} = \sum_{e \in E} \lambda_e \chi_e$ ($\lambda_e \geq 0$ $\forall e \in E$), where $\{\chi_e : \lambda_e > 0\}$ are linearly independent. We have to show that there exist $\lambda'_e \in \mathbb{Z}_+$ ($e \in E$) for which $\mathbf{z} = \sum_{e \in E} \lambda'_e \chi_e$. It suffices to show that $\sum_{e \in E} \{\lambda_e\} \chi_e$ can be obtained as a nonnegative integer combination of hyperedges ($\{\ .\ \}$ denotes the fractional part), so we can assume that $\lambda_e < 1$ ($\forall e \in E$).

Let us call a hyperedge $e \in E$ positive if $\lambda_e > 0$ (these are exactly the hyperedges with non-integer coefficient) and let us denote the set of positive hyperedges by E^+. For a hyperedge e, let $t(e)$ denote e itself if it is a 2-edge and $e \setminus \{u_p\}$ if it is a 3-edge, and let $G = (V, E')$ be the multigraph with $E' = \{t(e) : e \in E^+\}$.

Claim 2.2. *G is a cycle (and isolated nodes).*

Proof. A node $v \in V \setminus \{u_p\}$ can not be a leaf of G because then $z(v)$ would be non-integer.

If Q is a cycle in G then adding the vectors $\{\chi_e : e \in E^+, t(e) \in Q\}$ with coefficients $+1$ and -1 alternately regarding $t(e)$ going round Q, starting at u_p if it lies on Q, we get $k\chi_{\{u_p\}}$ where $k \neq 0$ because of the linear independence of the positive edges.

From this and the linear independence of the positive edges it follows that there cannot be two different cycles in G.

From the above observations and that $\mathcal{H} \setminus u_p$ is a bipartite graph it follows that either G is a cycle or an even cycle and a path from u_p to a node v on the cycle with no other common nodes. But this latter cannot happen either because then the coefficients on the cycle could only be alternately λ and $1 - \lambda$ for some $0 < \lambda < 1$, so $\mathbf{z}_v$ would be non-integer. □

Let us denote this cycle by Q. $|V(Q)|$ is greater than 2 because if $|V(Q)| = 2$ then in E^+ vertex u_p would have degree one and hence z_{u_p} would be non-integer. So by Claim 2.2 the hypergraph of the positive edges looks like in Figure 2. The cycle Q can be odd or even, and u_p can be on the cycle or not, but if it is not on Q then Q is even since $\mathcal{H} - u_p$ is a bipartite graph.

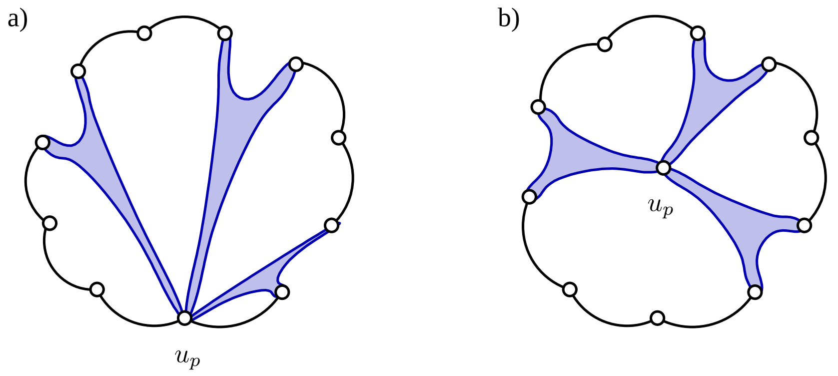

Figure 2: Structure of hypergraph (V, E^+) if a) $u_p \in V(Q)$ and b) $u_p \notin V(Q)$

Let us denote the edges of Q by $h_1, h_2, \ldots h_{|E(Q)|}$, beginning from u_p if it lies on Q. We color a hyperedge $e \in E^+$ red or green if $t(e)$ has an odd resp. even index, and we color a 2-edge vw red or green if the 3-edge u_pvw is already red resp. green. So we colored every positive hyperedge and $t(e)$ for every positive 3-edge e.

It follows from Claim 2.2 that there is a $0 < \lambda < 1$ for which $\lambda_e = \lambda$ if $e \in E_+$ is red and $\lambda_e = 1 - \lambda$ if $e \in E_+$ is green. Thus $\mathbf{z} = \chi_{V(Q)} + c\chi_{\{u_p\}}$ where $c \in \mathbb{Z}_+$.

Suppose there are r red and g green 3-edges.

If $|Q|$ is even then (no matter whether u_p is on Q or not) $c = r\lambda + g(1-\lambda) \leq \max(r, g)$. Let us assume that $r \leq g$ (the other case is similar). Then z can be obtained as the sum of characteristic vectors of only green hyperedges: we can take c arbitrary green 3-edges and the $|Q|/2 - c$ green 2-edges disjoint from them (except in u_p).

Thus we can suppose that $|Q|$ is odd. In this case u_p is on Q and the two 2-edges in E^+ incident to it have coefficient λ so $c = 2\lambda - 1 + r\lambda + g(1-\lambda) = (r+1)\lambda + (g-1)\lambda \leq \max(r+1, g-1)$.

All vectors of the form $\chi_{V(Q)} + c'\chi_{\{u_p\}}$ (where $c' \in \{1, 2, \ldots, r+1\}$) can be obtained as the sum of $(|Q|+1)/2$ red hyperedges which are disjoint except in u_p. On the other hand, all vectors of the form $\chi_{V(Q)} + c''\chi_{\{u_p\}}$ (where $c'' \in \{0, 1, \ldots, g-1\}$) can be obtained as the sum of $(|Q|-1)/2$ green hyperedges which are disjoint except in u_p. Thus we may assume that $\mathbf{z}$ is not among these from which follows that $\mathbf{z} = \chi_{V(Q)}$ and $g = 0$.

If Q is a choice cycle then because of Claim 2.1 $V(Q)$ induces a 3-edge Δ. It follows from the construction of $\mathcal{H}$ that Δ divides Q into three odd length paths so $\mathbf{z}$ can be obtained by adding the characteristic vectors of Δ and every second edge on these paths.

If Q is not a choice cycle then there is an edge vw on Q for which $u_pvw \in E$. We claim that there is one for which the two edge-disjoint paths on Q from u_p to v and w are odd. If the two paths are even then each path either contains the edge u_0u_{p+q} or contains another edge $v'w'$ with $u_pv'w' \in E$. So in one of the two directions the first edge from u_p with this property will have odd paths from u_p to its endnodes. Adding the characteristic vectors of this 3-edge and every second edge on the two odd length paths yields **z** and the proof is complete.

□

2.1 Acknowledgments

I thank András Sebő for bringing the problem to my attention and for stimulating discussions on the subject.

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