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Stable roommates with free edges

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Abstract

In the well-known stable roommates problem we have given a graph with preferences on the stars and we are looking for a matching that is not blocked by a nonmatching edge. There are well-known algorithms to find such a matching or to conclude that no such matching exists. Here we consider a relaxed problem motivated by kidney exchanges, where not all edges of the graph can block a matching. We show that this problem is NP-complete and apply the result to give an alternative proof of an NP-completeness result of Ronn.

Keywords: stable marriages; stable roommates problem; NP-completeness; polynomial reduction

1 Introduction

In the stable marriage problem of Gale and Shapley [4], there are n men and n women and each person ranks the members of the opposite gender by an arbitrary, strict preference order. A *marriage scheme* in this model is a set of marriages between different men and women. Such a scheme is *unstable* if there exists a *blocking pair*, that is, a man m and a woman w in such a way that m is either unmarried or m prefers w to his wife, and at the same time, w is either unmarried or prefers m to her partner. A marriage scheme is *stable* if it is not unstable, and a natural problem is finding a stable marriage scheme if it exists at all.

Nowadays, it is already folklore that for any preference rankings of the n men and n women, a stable marriage scheme exists. This theorem was proved first by Gale and Shapley in [4]. They constructed a special stable marriage scheme with the help of a finite procedure, the so-called deferred acceptance algorithm. It also turned out that for the existence of a stable scheme it is not necessary that the number of men is the same as the number of women or that for each person, all members of the opposite group are acceptable.

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The notion of a stable marriage scheme relies on the two-sidedness of the "marriage market". But in fact there is no mathematical reason to restrict ourselves to a model where certain agents cannot "marry" one another. The stable roommates problem is the generalization along these lines. Let $G = (V, E)$ be a graph and for each vertex v of G let $<_v$ be a linear order on the set $E(v)$ of edges incident with v. If $\mathcal{O} := \{<_v : v \in V\}$ denotes the set of these then $(G, \mathcal{O})$ is called a *preference system*. If in a preference system $(G, \mathcal{O})$, $e <_v f$ holds for some edges e, f and vertex v then we say that v *prefers* e to f and e *dominates* f. A *matching* of a graph G is a set of disjoint edges of G. If $(G, \mathcal{O})$ is a preference system and M is a matching of G then M is *stable* if every edge of $E \setminus M$ is dominated by some other edge of M. If M is a matching and edge $e \notin M$ is not dominated by M then we say that M is *blocked* by e. The *stable roommates problem* is finding a stable matching in a given preference system $(G, \mathcal{O})$. It is easy to see that in a preference system $(G, \mathcal{O})$, a stable matching might not exist. The first efficient algorithm to solve the stable roommates problem is due to Irving [6].

In the stable roommates problem, ordinary edges between two agents have two different features: on one hand, an edge can dominate other edges by being present in a matching, and on the other hand it may block a matching. A possible way to generalize the stable roommates problem is to allow edges to have only one of these properties. That is, we call an edge *forbidden* if it may block a matching, but it cannot be present in the stable matching that we look for. So the presence of a forbidden edge reduces the chance to find a stable matching. A *free* edge is the opposite: we may use it in a stable matching, but it never blocks. Clearly, the presence of a free edge improves our chance to find a stable matching.

It turned out that the stable roommates problem with forbidden edges is tractable and there exists an efficient algorithm to find a stable matching in a preference system if some edges are forbidden. The first such result is due to Dias et al. for the bipartite case [2]. Fleiner et al. extended this result to nonbipartite preference systems, where one may allow indifferences in the preferences [3].

Our main result concerns the presence of free edges. We shall show that the stable roommates problem with free edges is NP-complete. We apply our result to give an alternative proof for another NP-completeness result by Ronn [7].

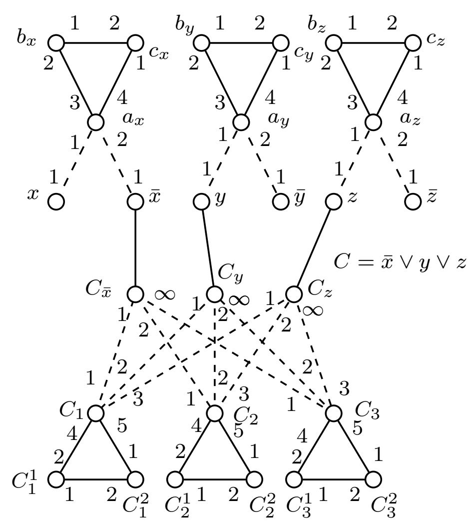
The motivation of the present work is kidney exchanges, a method to boost the number of kidney transplantations that recently became popular in some countries. In one possible model of it, vertices of the underlying graph are the incompatible patient-donor pairs and there is an edge between pairs A and B if the cross-transplantation is possible. There is a natural preference on the edges: pair A prefers edge AB to AC if the kidney of the donor of pair B is more suitable to the patient of pair A than the kidney of the donor of pair C. A feasible kidney exchange scheme is a matching of this graph. If we assume that everybody has full information and patients act according to their preferences then we look for a stable matching in this exchange graph. However, patients and donors do not have full information so in a more realistic model we assume that the vertices of the exchange graph are partitioned along transplant centers that have full information. So a stable kidney exchange scheme is an exchange scheme where no edge within a transplant center is blocking, or, in other words a stable matching where all edges between transplant centers are free. Péter Biró [1] asked the complexity of this problem. Rob Irving [5] proposed the "inverse" of this, where free edges are the ones that spanned by the partition sets.

2 Main result

By the definition, if we declare an ordinary edge free then all stable matchings remain stable and some new stable matchings may emerge, hence we have more chance to find one. Forbidding an ordinary edge may kill some stable matchings but never creates a new one, so it reduces our chance to find one. For the decision problem the opposite holds: it is known that the problem with forbidden edges is tractable and we shall show that the presence of free edges makes it hard.

Theorem 2.1. *For a preference system $(G, \mathcal{O})$ and subset $F \subseteq E(G)$ of free edges it is NP-complete to decide the existence of a stable matching. The problem is NP-complete already in each of the following cases. (1) F consists of disjoint edges. (2) F is the set of those edges that connect different V_i's and (3) F is the set of those edges that is spanned by some V_i, where $V_1 \cup V_2 \cup \ldots \cup V_k$ is a partition of the vertices of G.*

Sketch of the proof. We show a polynomial reduction of the 3-SAT problem to the the stable roommates problem with free edges. For this reason, we have to construct in polynomial time for each 3-CNF boolean formula Φ a preference system $(G, \mathcal{O})$ and set of free edges such that Φ is satisfiable if and only if there is a matching in $(G, \mathcal{O})$ that can be blocked only by free edges.



For each variable x of Φ, let us define vertices a_x, b_x, c_x, x and $\bar{x}$, edges a_xb_x, b_xc_x, c_xa_x free edges a_xx and $a_x\bar{x}$. These latter edges are first choices of x and $\bar{x}$ respectively, the preference order of a_x is $a_xx, a_x\bar{x}, a_xb_x, a_xc_x$, b_x prefers b_xc_x to b_xa_x and c_x prefers c_xa_x to c_xb_x. For each clause C we have vertices C_i and C_i^j for $i = 1, 2, 3$, $j = 1, 2$ and C_l, for each literal l in C. Construct free edges C_lC_i that are the ith choices of C_l for $i = 1, 2$ and C_lC_3 is the very last choice of C_l. For each C_i, the edges C_iC_l are the first three choices in an arbitrary order.

For $i = 1, 2, 3$, add all edges between C_i, C_i^1 and C_i^2 such that these edges are ordinary and preferences along these triangles are cyclic. To finish the construction of G, for each literal l in some clause C, connect vertex l with C_l. All nonspecified preferences are arbitrary. The figure shows the part of G corresponding to clause $C = \bar{x} \vee y \vee z$ and variables x, y and z. It is easy to see that if there is a truth assigment to Φ then there is a matching in G that is not blocked by ordinary edges. For this, we choose edges xa_x for each true variable and edges $\bar{x}a_x$ for each false one, all edges $b_x c_x$ and $C_i^1 C_i^2$. It is straightforward to complete this matching it to one we need.

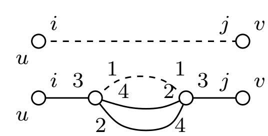
If there is a matching M that is not blocked by an ordinary edge then for each variable x, exactly one of $a_x x$ and $a_x \bar{x}$ is present in M, as otherwise M would contain a stable matching of triangle $a_x b_x c_x$ that does not exist. Similarly, each vertex C_i is covered by a free edge of M as otherwise M would contain a stable matching of triangle $C_i C_i^1 C_i^2$, which is impossible. This means that M contains no edge lC_l.

If $a_x x \in M$ then declare x true, otherwise (when $a_x \bar{x} \in M$) let x be false. We prove that this is a truth assigment of Φ, that is, each clause has a true literal. If C is a clause then there is an edge $C_3 C_l$ of M. As lC_l cannot block this means that l is covered by a free edge, hence l is a true literal in C. This follows that Φ has a truth assignment.

Let $V(\Phi)$ and $C(\Phi)$ denote the set of variables and clauses of Φ. Define
$V_a := \{a_x, x, \bar{x} : x \in V(\Phi)\}$, $V_b := \{b_x : x \in V(\Phi)\}$, $V_c := \{c_x : x \in V(\Phi)\}$,
$V_0 := \{C_l, C_i : C \in C(\Phi), i = 1,2,3 \text{ and } l \text{ is a literal of } C\}$,
$V_1 := \{C_i^1 : C \in C(\Phi), i = 1,2,3\}$ and $V_2 := \{C_i^2 : C \in C(\Phi), i = 1,2,3\}$, moreover let
$V(1) := \{a_x, b_x, c_x : x \in V(\Phi)\}$, $V(2) := \{x, \bar{x}, C_x, C_{\bar{x}} : x \in V(\Phi)\}$ and
$V(3) := \{C_1, C_1^1, C_1^2 : C \in C(\Phi)\}$.

Observe that $V(G) = (V_a \cup V_1) \cup (V_b \cup V_0) \cup (V_c \cup V_2) = (V(1) \cup V(3)) \cup V(2)$ are two partitions of the vertices of G into 3 and 2 parts such that set F of free edges have the property described in (3) and (2), respectively. This proves (2) and (3). Note that if we can partiton the vertices into 2 parts such that no free edge connect the parts then there always exists a stable matching and the deferred acceptance algorithm of Gale and Shapley finds it.

To show the Theorem in case (1), we construct for each stable roommates problem with free edges an equivalent problem one where free edges are disjoint.



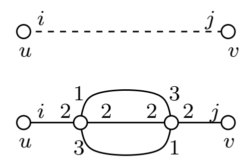
To achieve this, it is enough to substitute each free edge with a little graph shown on the figure. That is, we delete each free edge $e = uv$, we add new vertices u_e and v_e, edges uu_e, vv_e, and two parallel copies of $u_e v_e$ and an extra free edge $u_e v_e$ that is first choice for both u_e and v_e. Preferences along u is unchanged, where edge e is substitued by uu_e. The first choice of u_e is the free edge, the second is a parallel copy of $u_e v_e$ that will be the last choice of v_e. The third choice is $u_e u$ and the last one is the second choice of v_e. After the changes we get a new preference system where all free edges are disjoint. Moreover, there is a matching that is not blocked by ordinary edges in $(G, \mathcal{O})$ if and only if there is one after the construction. □

In both the stable marriage and the stable roommates problems, strict (linear) preferences of the participating agents play a crucial role. However, in many practical situations, one has to deal with indifferences in the preference orders. A natural model for this is that preference orders are partial (rather than linear) orders. One can extend the notion of a stable matching to this model in at least three different ways. One possibility is that a matching is *weakly stable* if no pair of agents a, b exists such that they mutually strictly prefer one another to their eventual partner. Ronn proved that deciding the existence of a weakly stable matching is NP-complete [7]. Based on Theorem 2.1, there is an alternative proof.

Theorem 2.2 (see Ronn [7])**.** *It is NP-complete to decide the existence of a weakly stable matching in the stable roommates problem where each vertex has a weak linear preference order on the incident edges with at most two edges in tie.*

Sketch of the proof. Construct graph G' the following way. Substitute each free edge of G with a path of three edges with three parallel copies of the middle edge. Keep the preferences of the original vertices. At the new vertices the unique first choices is one of the parallel copies, the other ends of the first choices are last choices and the remaining two edges are tied and second choices. (See the figure.)

It is easy to check that there there is a matching of G not blocked by any ordinary edge if and only if there is a weakly stable matching in $(G', \mathcal{O}')$, where preferences of $\mathcal{O}'$ are the same as of $\mathcal{O}$ at vertices of G and given by the figure at the new vertices.



Hence for each instance of the stable roommates problem with free edges, one can construct in polynomial time an equivalent instance of the problem described in Theorem 2.2, that is we have a polynomial reduction of the former problem to the latter. As the former problem is NP-complete by Theorem 2.1, the latter one has this property as well. □

References

[1] P. Biró, Personal communication.

[2] V.M.F. Dias, G.D. da Fonseca, C.M.H. de Figueiredo, and J.L. Szwarcfiter, The stable marriage problem with restricted pairs. *Theoret. Comput. Sci.*, (2003) **306(1-3)**,391–405

[3] T. Fleiner, R.W. Irving, and D.F. Manlove, Efficient algorithms for generalized stable marriage and roommates problems. *Theoret. Comput. Sci.* (2007) **381(1-3)**, 162–176

[4] D. Gale and L.S. Shapley, College admissions and stability of marriage. *Amer. Math. Monthly* (1962) **69(1)**, 9–15

[5] R.W. Irving, Personal communication.

[6] R.W. Irving, An efficient algorithm for the "stable roommates" problem. *J. Algorithms* (1985) **6(4)**, 577–595,

[7] E. Ronn, NP-complete stable matching problems. *J. Algorithms* (1990) **11(2)**, 285–304,