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A simple proof of a theorem of Benczúr and Frank

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Abstract

We give a simple proof of a theorem of Benczúr and Frank concerning covering symmetric crossing supermodular set functions with graph edges.

1 Introduction

A set function $p\colon 2^V \to \mathbb{Z}$ is called *positively crossing supermodular* if it satisfies the following inequality for every crossing pair $X, Y \subseteq V$ with $p(X), p(Y) > 0$:

$$p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y). \qquad (\cap\cup)$$

Observe that $(\cap\cup)$ trivially holds if $X \subseteq Y$ or $Y \subseteq X$. If furthermore p is symmetric (i.e. $p(X) = p(V - X)$ for any $X \subseteq V$) then it will also satisfy the following inequality for every crossing pair $X, Y \subseteq V$ with $p(X), p(Y) > 0$:

$$p(X) + p(Y) \le p(X - Y) + p(Y - X). \qquad (-)$$

Again, $(-)$ will always hold if $X \cap Y = \emptyset$ or $X \cup Y = V$. The argument given here, unlike that of Benczúr and Frank, will be simpler if we do not assume that our function is nonnegative.

A graph $G = (V, E)$ is said to **cover a set function** p if $d_G(X) \ge p(X)$ for any $X \subseteq V$, where $d_G(X)$ is the number of edges of G having exactly one endpoint in X. Assume that we are given a symmetric, positively crossing supermodular set function $p\colon 2^V \to \mathbb{Z}$ over the finite ground set V with $p(\emptyset) = 0$. In this paper we consider the question of finding a graph G covering the function p. The main objective would be to minimize the number of the edges of the graph to be found, but it is easier to speak about the more general **degree-specified** version of the problem, where we are also given a degree specification $m\colon V \to \mathbb{Z}_+$ and we want to find a graph G covering p that also satisfies this degree specification, that is $d_G(v) = m(v)$ for any $v \in V$ (note that we distinguish between $d_G(v)$ and $d_G(\{v\})$: the former counts the number of loops incident to v, too, so $d_G(v) = d_G(\{v\}) + 2|\{\text{loop edges incident to } v\}|$). Since $\sum_{v\in X} d_G(v) \ge d_G(X)$, a necessary condition of the existence of such a

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graph is that $m(X) = \sum_{v \in X} m(v) \geq p(X)$ for any $X \subseteq V$: let us say that such a degree-specification is **admissible**. Introduce the contrapolymatroid

$$C(p) = \{x \in \mathbb{R}^V : x(Z) \geq p(Z)\ \forall Z \subseteq V, x \geq 0\}.$$

Let $m \in C(p) \cap \mathbb{Z}^V$ (i.e. an admissible degree-specification). For a node $v \in V$ we say that v is **positive** if $m(v) > 0$, and **neutral** otherwise. The set of positive nodes will be denoted by V^+. Assume $u, v \in V^+$ are two positive nodes (possibly $u = v$, but then $m(u) \geq 2$ is assumed). The operation **splitting-off (at u and v)** is the following: let

$$m' = m - \chi_{\{u\}} - \chi_{\{v\}} \text{ and } p' = p - d_{(V,\{(uv)\})}. \tag{1}$$

If $m'(X) \geq p'(X)$ for any $X \subseteq V$ (i.e. $m' \in C(p')$) then we say that the splitting off is **admissible**. Clearly, splitting off at u and v is admissible if and only if there is no dangerous set X containing both u and v (a set X is **dangerous** if $m(X) - p(X) \leq 1$ and it is called **tight** if $m(X) - p(X) = 0$). We will also say that such a dangerous set X **blocks the splitting at u and v**, or simply that X **blocks u and v**.

The following lemma was proved in [3] under more general circumstances: for completeness we include a proof of this special case.

Lemma 1. *Let $p\colon 2^V \to \mathbb{Z}$ be a symmetric, positively crossing supermodular function and $m \in C(p) \cap \mathbb{Z}^V$. If $p(X) > 1$ for some $X \subseteq V$ then there is an admissible splitting-off.*

Proof. Let $M_p = \max\{p(X) : X \subseteq V\}$, which is by assumption at least 2. Let Y be a minimal set satisfying $p(Y) = M_p$. By symmetry, $p(V - Y) = M_p$, too, so we can choose a minimal set $Z \subseteq V - Y$ satisfying $p(Z) = M_p$. Since $M_p \geq 1$ we can choose $y \in Y, z \in Z$ with $m(y), m(z) > 0$. We claim that the splitting at y and z is admissible. Assume not and consider a dangerous set X containing y and z. Since $m(X - Y) \leq m(X) - m(y) \leq m(X) - 1$ and $p(Y - X) < M_p$ by the minimality of Y, X and Y cannot satisfy $(-)$, since that would mean $m(X) - 1 + M_p \leq p(X) + p(Y) \leq p(X - Y) + p(Y - X) < m(X - Y) + M_p \leq m(X) - 1 + M_p$, a contradiction. So $Y \subseteq X$ must hold. Similarly, $Z \subseteq X$ holds, too. But then $m(X) \geq m(Y) + m(Z) \geq p(Y) + p(Z) = 2M_p$ contradicting $m(X) \leq p(X) + 1 \leq M_p + 1$. □

A consequence of this lemma is the following: assume $p\colon 2^V \to \mathbb{Z}$ is a function as in the statement of the lemma and $m \in C(p) \cap \mathbb{Z}^V$, and suppose that there is no admissible splitting-off. Then any pair $u, v \in V^+$ is in a dangerous set X: this means that $p(X) = 1$ and $m(X) = 2$, hence $m \leq 1$. We can further assume that $m(V) \geq 4$: then a set X blocking a pair $u, v \in V^+$ and another set Y blocking a pair $w, v \in V^+$ cross each other, meaning that $p(X \cap Y) = 1$, in other words every $v \in V^+$ is in a tight set. Let T_1, T_2 be two tight sets containing $v \in V^+$: then of course ($T_1 \cap T_2$ and) $T_1 \cup T_2$ is also a tight sets containing $v \in V^+$, thus there exists a unique maximal tight set T containing $v \in V^+$. The following lemma shows an important fact about these maximal tight sets.

Lemma 2. *Let $p\colon 2^V \to \mathbb{Z}$ be a symmetric, positively crossing supermodular function and $m \in C(p) \cap \mathbb{Z}^V$. Assume that there does not exist an admissible splitting-off and*

$m(V) \geq 4$. Let $V^+ = \{v_1, v_2, \dots, v_k\}$ and V_i be the maximal tight set containing v_i for any $i \in \{1, 2, \dots, k\}$. Then

(i) the set blocking v_i and v_j is $V_i \cup V_j$ for any $i, j \in \{1, 2, \dots, k\}$,

(ii) $p(\cup_{i \in I} V_i) = 1$ for any nonempty $I \subsetneq \{1, 2, \dots, k\}$,

(iii) the sets $V_1, V_2, \dots, V_k$ form a partition of V,

(iv) furthermore a set $X \subseteq V$ having $p(X) = 1$ cannot cross any of the sets $\{V_i : i = 1, 2, \dots, k\}$.

Proof. The sets that we consider will always have positive p value, so we can use $(\cap\cup)$ and $(-)$ if two of them cross. Let i, j be two different indices between 1 and k. It is straightforward that V_i and V_j have to be disjoint (otherwise $p(V_i \cap V_j) = 1$ would follow from $(\cap\cup)$). Similarly, a set X blocking v_i and v_j must contain V_i (and V_j), otherwise $p(V_i - X) = 1$ would follow from $(-)$. On the other hand, if $l \in \{1, 2, \dots, k\}$ is different from i and j and Y is a set blocking v_i and v_l then $(\cap\cup)$ implies that $X \cap Y = V_i$ (since it is tight) and $(-)$ implies that $X - Y = V_j$ (since it is tight again).

Now a simple induction on $|I|$ shows that $p(\cup_{i \in I} V_i) = 1$ for any nonempty $I \subsetneq \{1, 2, \dots, k\}$. The case $|I| \leq 2$ is clear, so assume $I = I' + j$ where $i \in I' \subsetneq I \subsetneq \{1, 2, \dots, k\}$. Let $X = \cup_{i \in I'} V_i$ and $Y = V_i \cup V_j$. The conditions imply that X and Y cross and $p(X)$ and $p(Y)$ are both positive by the inductive hypothesis. Applying $(\cap\cup)$ for X and Y and using $p(X \cap Y) = 1$ gives (ii).

The only thing to be proved to get (iii) is that $\cup_{i=1}^k V_i = V$: but if this was not the case then the above induction would also imply that $p(\cup_{i=1}^k V_i) = 1$, which would give a contradiction, since $m(V - \cup_{i=1}^k V_i) = 0$ and $p(V - \cup_{i=1}^k V_i) = p(\cup_{i=1}^k V_i) = 1$.

To prove the last statement, assume that X crosses V_1. By possibly complementing X we can assume that $m(V_1 \cap X) = 0$. But $(\cap\cup)$ implies that $p(V_1 \cap X) = 1$, a contradiction. □

Let us introduce another necessary condition for the existence of a graph covering our function p and satisfying the degree-specification m. A partition $\mathcal{X} = \{X_1, X_2, \dots, X_t\}$ of V is called p-**full** if $p(\cup_{i \in I} X_i) > 0$ for any nonempty $I \subsetneq \{1, 2, \dots, t\}$. The maximum cardinality of a p-full partition is the **dimension of** p and is denoted by $\dim(p)$. It is easy to see that any graph covering p must have at least $\dim(p) - 1$ edges. The following simple claim due to Benczúr and Frank can be checked easily.

Lemma 3. *If $p \colon 2^V \to \mathbb{Z}$ is a symmetric, positively crossing supermodular function and $\{X_1, X_2, \dots, X_t\}$ is a partition of V satisfying $p(X_1) = 1$ and $p(X_1 \cup X_i) > 0$ for any $i = 1, 2, \dots, t$, then this partition is p-full.*

However we will need the following, slightly more complicated lemma.

Lemma 4. *Let $p \colon 2^V \to \mathbb{Z}$ be a symmetric, positively crossing supermodular function and $\{V_1, V_2, \dots, V_k\}$ be a partition of V satisfying $p(V_i) = 1$ for any $i = 1, 2, \dots, k$ (where $k \geq 4$). Let furthermore $U_i^1, U_i^2, \dots, U_i^{t_i}$ be a partition of V_i (where $t_i \geq 1$*

is an integer) for any $i = 1, 2, \dots, k$ such that $p(V_i \cup U_j^l) > 0$ for any possible i, j, l. Assume furthermore that $p(U_1^1) = 1$. Then the partition $\mathcal{U} = \{U_i^j : i = 1, 2, \dots, k$ and $j = 1, 2, \dots, t_i\}$ is p-full.

Proof. Let $i \in \{1, 2, \dots, k\}$ be arbitrary and $\mathcal{U}' \subseteq \mathcal{U}$ such that $V_i \cup (\bigcup \mathcal{U}') \neq V$. First we prove by induction on $|\mathcal{U}'|$ that $p(V_i \cup (\bigcup \mathcal{U}')) > 0$: the base case $|\mathcal{U}'| = 0$ is obvious, so let $U \in \mathcal{U}'$ be arbitrary and let $\mathcal{U}'' = \mathcal{U} - U$, $X = V_i \cup (\bigcup \mathcal{U}'')$ and $Y = V_i \cup U$. By the inductive hypothesis, $p(X) > 0$, and by the assumption in the lemma, $p(Y) > 0$. We can apply $(\cap\cup)$ for X and Y, and using that $p(X \cap Y) = 1$ gives that $p(X \cup Y) = p(V_i \cup (\bigcup \mathcal{U}')) > 0$, as claimed. By the symmetry of p this implies that $p(U_1^1 \cup U_j^l) > 0$ for any possible j, l. But then we can apply Lemma 3 in order to finish this proof. □

2 Proof of the theorem of Benczúr and Frank

In this note we want to prove the following theorem due to Benczúr and Frank [2].

Theorem 5. *Let $p_0 \colon 2^V \to \mathbb{Z}$ be a symmetric, positively crossing supermodular set function and $m_0 \in C(p_0) \cap \mathbb{Z}^V$ with $m_0(V)$ even. There exists a graph G covering p_0 with $d_G(v) = m_0(v)$ for any $v \in V$ if and only if $m_0(V)/2 \geq \dim(p_0) - 1$.*

Proof. The necessity of the conditions is clear: see details in [2]. The proof of the other direction uses the splitting-off technique. We will give a simple algorithm that starts with an arbitrary $m_0 \in C(p_0) \cap \mathbb{Z}^V$ (with $m_0(V)$ even) and either finds the graph in question or shows that the condition $m_0(V)/2 \geq \dim(p_0) - 1$ did not hold. The algorithm goes as follows: perform an arbitrary sequence of admissible splitting-off steps. Assume that no further admissible splitting-off is possible and let the graph of the edges split so far be denoted by G, $p = p_0 - d_G$ and $m(v) = m_0(v) - d_G(v)$ for any $v \in V$. If $m(V) = 0$ then we are done, so assume that $m(V) \geq 4$ (one can simply check that $m(V) = 2$ cannot be the case). Lemmas 1 and 2 show us that $m \leq 1$: let the positive nodes be $v_1, v_2, \dots, v_k$ and $V_1, V_2, \dots, V_k$ be the partition of V into maximal tight sets with $v_i \in V_i$ for any $i \in \{1, 2, \dots, k\}$ (here $k = m(V)$ is of course even, but we will not really use this). One simple observation shows that G does not have edges between two classes V_i and V_j of this partition: if it had, then choosing a third index $l \in \{1, 2, \dots, k\}$ and using that $X = V_i \cup V_l$ and $Y = V_j \cup V_l$ has to satisfy $(\cap\cup)$ with equality would give a contradiction (here and later on we will use that the edges of G strenghten the inequalities $(\cap\cup)$ and $(-)$ in the following way: if X and Y are crossing sets with $p_0(X), p_0(Y) > 0$ then $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) - 2d_G(X, Y)$, and similarly for $(-)$). It is possible that the splitting-off sequence we have performed contained some foolish steps and we could have split off more edges by taking some extra care. Our approach is the following: we try to undo the splitting-off of **a single edge** of G which allows us to split-off two edges instead, thus decreasing $m(V)$. Interestingly, this step already leads us to our target. Let us give the details.

Pick an edge $uv = e \in G$. The **unsplitting** operation of e is simply the reverse of the splitting-off operation: $m^e = m + \chi_{\{u\}} + \chi_{\{v\}}$, $G^e = G - e$ and $p^e = p + d_{(V, \{(uv)\})} =$

$p_0 - d_{G^e}$. Of course, this is always admissible, that is $m^e \in C(p^e)$. If v_r and v_s are two (distinct) positive nodes then an **admissible improvement (at** v_r, u, v, v_s**)** is the following operation: $m' = m - \chi_{\{v_r\}} - \chi_{\{v_s\}}$, $G' = G - e + uv_r + vv_s$ and $p' = p_0 - d_{G'}$, where $m' \in C(p')$. Observe that the admissible improvement can be considered as the sequence of unsplitting e followed by two admissible splitting-offs (in any order) at u, v_r and v, v_s.

Our aim is to find an edge $uv = e$ of G (spanned by V_i, say) and two positive nodes v_r and v_s such that an admissible improvement can be performed. Let us investigate the obstacles of an admissible improvement. First note that $r = i$ is not a good choice, since $p(V_i \cup V_s) = 1$; $s = i$ is not good, either, so v_r and v_s are both distinct from v_i. Now since $p \leq 1$ implies that $p^e \leq 2$, we only have to worry about sets X having $p(X) \geq 0$. Since we have seen in Lemma 2 that sets with p-value 1 are quite rare, one can check that they will not obstruct the simple improvement, if v_r and v_s are both distinct from v_i (note that $m(V) \geq 4$). A set $X \subseteq V$ having $p(X) = 0$ **obstructs the admissible improvement at** v_r, u, v, v_s if and only if it is entered by the edge uv, $u, v_r \in X$ or $v, v_s \in X$ and $m(X) = 1$. Let us describe such sets: assume that $0 = p(X) = m(X) - 1$, $v_r, u \in X$ but $v \notin X$: note that such a set has $p_0(X) > 0$. First of all, using $(\cap\cup)$ for X and V_i gives that $p(X \cup V_i) = 1$, i.e. $X - V_i = V_r$. The next simple observation is the following: if such a set X exists then there is no set Y having $0 = p(Y) = m(Y) - 1$, $v, v_r \in Y$ but $u \notin Y$. Assume indirectly that both X and Y exist and apply $(\cap\cup)$ for them: since $d_G(X, Y) \geq 1$ this implies that $p(X \cup Y) \geq 1$, which is impossible.

Claim 1. *If there is a set X such that $0 = p(X) = m(X) - 1$, $v_r, u \in X$ but $v \notin X$ (where $uv \in E(G)$ is induced by V_i for some i and $v_r \in V^+ - v_i$) then* ***there is no admissible improvement at all at the edge*** *uv, since $p(V_j \cup (X \cap V_i)) = 0$ for any $j \neq i$.*

Proof. We will use that $m(V) \geq 4$: let i, j, r be the indices of the statement and let s be a fourth index out of $1, 2, \ldots, k$. Apply $(\cap\cup)$ to X and $Y = V_r \cup V_j$ to get that $p(X \cup Y) \geq 0$, but since it cannot be 1 it must be 0. Now apply $(-)$ to $X \cup Y$ and $V_r \cup V_s$ to obtain that $p(V_j \cup (X \cap V_i)) \geq 0$, but again it cannot be one, so the claim is proved. □

The procedure goes as follows: while there is an admissible improvement, perform this admissible improvement and decrease $m(V)$ (observe that an admissible improvement will not create an admissible splitting). Do this until you cannot find any more admissible improvements: for simplicitiy let us denote the remaining degree specification again by m, the obtained graph by G and $p = p_0 - d_G$. Again, if $m(V) \leq 3$ then we are done, so assume that $m(V) \geq 4$. We will now show how to obtain a p_0-full partition of size greater than $m_0(V)/2 + 1$, which finishes the proof of the theorem. Furthermore this partition will be of size $m(V) + |E(G)|$, which shows that the dimension of p_0 is not greater than this, since adding a spanning tree on V^+ to G covers p_0 and has size $m(V) + |E(G)| - 1$. To this end let us describe the structure of the obstacles of further admissible improvements. Let again $uv = e$ be an edge of G (spanned by V_i): since for any $j \neq i$ the same endpoint of this edge is contained in an obstacle for v_j and e, this is best denoted by defining an orientation $\vec{G}$ of G in the following way: if an obstacle for v_j and e contains u then e is oriented from v to u. Let v_r and v_s be two positive nodes and consider the sets X (Y) obstructing e and v_r (e and v_s, resp.). Applying $(-)$ for X and Y (and p) gives that $p(X-Y)=p(Y-X)=1$ and $\overline{d}_G(X,Y)=1$ (it is positive by the edge e). This further implies that $X-Y=V_r$ and $Y-X=V_s$, so $X_e=X\cap V_i=Y\cap V_i$ is uniquely defined (it does not depend on the choice of r) and the obstacle X for v_r and $\vec{vu}\in\vec{G}$ is equal to $X_e\cup V_r$ (so it is also unique in the sense that if $u,v_r\in X$, $v\notin X$, $p(X)=m(X)-1=0$ for some $X\subseteq V$ then $X=X_e\cup V_r$). We will also use the notation $X_{\vec{vu}}$ for X_e to emphasize that $\vec{vu}$ enters X_e.

Another very important consequence of $\overline{d}_G(X,Y)=1$: G cannot contain a cycle. So $\vec{G}$ consists of directed trees: next we show that these are in fact arborescences (out-trees). This will follow from the following claim.

Claim 2. *Let $x_1\vec{y}_1, x_2\vec{y}_2$ be two (distinct) arcs of $\vec{G}$ spanned by V_i (where any two of this four nodes may coincide). Then $X_{x_1y_1}$ and $X_{x_2y_2}$ are either disjoint or one of them contains the other.*

Proof. Assume the contrary and consider two positive nodes v_r and v_s (distinct from v_i). Apply $(-)$ for $X=V_r\cup X_{x_1y_1}$ and $Y=V_s\cup X_{x_2y_2}$ and in each of the cases suggested by the position of the two arcs $x_1\vec{y}_1$ and $x_2\vec{y}_2$ you get a contradiction. □

This claim shows that the sets $\{X_e : e\in G\}$ form a laminar family (as suggested by the arborescences: it is known that an arborescence naturally defines a laminar family). For any $e\in G$ we define the set $Y_e=X_e-\cup_{f\subseteq X_e}X_f$: observe that $Y_e\neq\emptyset$ since the head of $\vec{e}$ is in Y_e. Moreover, for any $i=1,2,\dots,k$ we define $U_i=V_i-\cup_{f\subseteq V_i}X_f$, which is again not empty, since $v_i\in U_i$. So the family $\{Y_e : e\in G\}\cup\{U_i : i=1,2,\dots,k\}$ form a partition: the following claim almost implies that it is a p_0-full partition.

Claim 3. *For any distinct $i,j\in 1,2,\dots,k$ and $\vec{xy}\in\vec{G}$ induced by V_i one has $p_0(V_j\cup Y_{xy})=1$.*

Proof. The claim follows from the following induction: let $i,j\in 1,2,\dots,k$ and $\vec{xy}\in\vec{G}$ as in the claim and let $x_1y_1,x_2y_2,\dots x_ly_l$ be some edges induced by X_{xy} such that there is no $p,q\in\{1,2,\dots,l\}$ with $x_py_p\subseteq X_{x_qy_q}$. Let $Z=X_{xy}-\cup_{h=1}^{l}X_{x_hy_h}$: note that $d_G(Z)=l+1$, since all the arcs $x_1\vec{y}_1,x_2\vec{y}_2,\dots x_l\vec{y}_l$ leave Z and $\vec{xy}$ enters Z and no other edge enters Z (note that in the interesting case the edges $x_1y_1,x_2y_2,\dots x_ly_l$ are either successors of xy or edges going out of some of the roots of the arborescences induced by X_{xy}). Then we claim that $p(V_j\cup Z)=-l$, i.e. $p_0(V_j\cup Z)=1$. We prove this by induction on l: the $l=0$ case is obvious, so let $l\geq 1$. Let $Z'=X_{xy}-\cup_{h=1}^{l-1}X_{x_hy_h}$. Let v_r be a positive node distinct from v_i and v_j and let $e=x_ly_l$. Apply $(-)$ for $V_j\cup Z'$ and $V_r\cup X_e$ (and p) to get that $p(V_j\cup Z)\geq p(V_j\cup Z')+p(V_r\cup X_e)-p(V_r)=-(l-1)+0-1=-l$ (observe that the sets in question have positive p_0 value, indeed). On the other hand, using $(-)$ again for $V_j\cup Z$ and $V_r\cup Z$ gives the opposite inequality (using that the choice of j was arbitrary, so we also have that $p(V_r\cup Z)\geq -l$): $p(V_j\cup Z)+p(V_r\cup Z)\leq p(V_j)+p(V_r)-2d_G(Z,V_i-Z)=-2l$. □

The same argument proves that $p_0(V_j \cup U_i) = 1$ for any distinct $i, j \in 1, 2, \ldots, k$. One can simply check that $p_0(X_e) = 1$ for any $e \in G$ (apply $(\cap\cup)$ for $V_i \cup X_e$ and $V_j \cup X_e$ and p, where $i, j \in \{1, 2, \ldots, k\}$ and $e \in G - V_i - V_j$), which implies that $p_0(Y_e) = 1$ for some $e \in G$ (if $X_f = Y_f$). Applying Lemma 4 shows that the family $\{Y_e : e \in G\} \cup \{U_i : i = 1, 2, \ldots, k\}$ is a p_0-full a partition of size $m(V) + |E(G)|$. This finishes the proof of the theorem.

□

The proof above clearly proves the following deficient form of Theorem 5 of Benczúr and Frank.

Theorem 6. *Let $p_0 \colon 2^V \to \mathbb{Z}$ be a symmetric, positively crossing supermodular set function and $m_0 \in C(p_0) \cap \mathbb{Z}^V$ with $m_0(V)$ even. If $m_0(V)/2 < \dim(p_0) - 1$ then the longest admissible splitting sequence consists of $m_0(V) - \dim(p_0)$ splitting-offs. If $m_0(V)/2 \geq \dim(p_0) - 1$ then there exists a complete admissible splitting-off.*

Proof. Consider an arbitrary running of the algorithm sketched above. If it gets stuck with remaining degree specification m and graph G, then we have seen that $m(V) + |E(G)| = \dim(p_0)$. Since $m(V) + 2|E(G)| = m_0(V)$, this shows that (after arbitrary choices in the algorithm) $|E(G)| = m_0(V) - \dim(p_0)$. Since a longest admissible splitting sequence is clearly a valid running of the algorithm (there cannot exist an admissible improvement after a longest splitting sequence), this finishes the proof. □

Using standard methods (detailed for example in [1]) one can prove the following version of Theorem 5.

Theorem 7 (Benczúr and Frank [2])**.** *Let $p \colon 2^V \to \mathbb{Z}$ be a symmetric, positively crossing supermodular set function. The minimum number of graph edges covering p is equal to the maximum of the following two quantities:*

$$\max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} p(X) \rceil : \ \mathcal{X} \text{ is a subpartition of } V\}, \tag{2}$$

$$\dim(p) - 1. \tag{3}$$

We mention that the algorithm given in this paper can be implemented to run in polynomial time **only** if the function $p : 2^V \to \mathbb{Z}$ (given with a function evaluation oracle) is not only positively crossing supermodular, but it is **crossing supermodular**. The example $p(X) = 1$ if $X = X_0$ or $X = V - X_0$ for some fixed X_0 (and 0 otherwise) shows that for the class of positively crossing supermodular functions we need exponentially many oracle calls just to decide whether a given graph (e.g. the empty graph) covers the function or not.

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