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Generically globally rigid zeolites in the plane

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Abstract

A d-dimensional zeolite is a d-dimensional body-and-pin framework with a $(d+1)$-regular underlying graph G. That is, each body of the zeolite is incident with $d+1$ pins and each pin belongs to exactly two bodies. The corresponding d-dimensional combinatorial zeolite is a bar-and-joint framework whose graph is the line graph of G.

We show that a two-dimensional combinatorial zeolite is generically globally rigid if and only if its underlying 3-regular graph G is 3-edge-connected. The proof is based on a new rank formula for the two-dimensional rigidity matroid of line graphs.

1 Introduction

A *d-dimensional zeolite* is a d-dimensional body-and-pin framework in which each body is incident with $d+1$ pins and each pin belongs to exactly two bodies. In the underlying graph G of the zeolite vertices correspond to bodies and two vertices are adjacent if and only if the corresponding bodies share a pin. Thus the underlying graph of the zeolite is $(d+1)$-regular.

By replacing the bodies by complete bar frameworks one obtains a *d-dimensional combinatorial zeolite*. It is a bar-and-joint framework whose graph is the line graph of the underlying graph G of the zeolite. (The *line graph* $L(G)$ of a graph $G=(V,E)$ is the simple graph with vertex set $\{v_e : e \in E\}$, where two vertices v_e, v_f are adjacent if and only if e, f have a common end-vertex in G.) See Figure 1 for a two-dimensional example.

The investigation of these structures is motivated in part by the existence (and flexibility properties) of real zeolites, which are molecules formed by corner-sharing tetrahedra, see e.g. [3]. Planar plate frameworks (which contain planar zeolites as a special case), in which each body is a regular polygon, have also been studied in the combinatorial rigidity literature [2]. In this paper we shall consider the (global) rigidity properties of planar combinatorial zeolites in generic position.

Roughly speaking, a combinatorial zeolite is globally rigid if its bar lengths uniquely determine the whole framework, up to congruence. Brigitte Servatius and Herman

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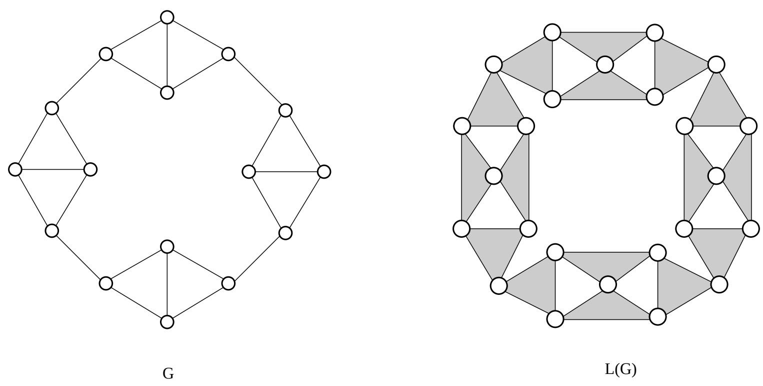

Figure 1: A 3-regular graph G and its line graph $L(G)$. The shaded triangles of the bar-and-joint framework on $L(G)$ correspond to the bodies in the two-dimensional zeolite whose underlying graph is G.

Servatius [9] asked whether there is a simple necessary and sufficient condition, in terms of its underlying graph, for the global rigidity of a planar zeolite whose vertices are in generic position. We shall give an affirmative answer in Section 3 by showing that a planar combinatorial zeolite is generically globally rigid if and only if its 3-regular underlying graph is 3-edge-connected. The proof is based on a new rank formula for the two-dimensional rigidity matroid of line graphs. This formula, along with the necessary definitions, is given in Section 2. The last section is devoted to some concluding remarks.

2 Rigidity of line graphs

We shall need the following basic notions of combinatorial rigidity. For a detailed survey of the area we refer the reader to [1, 10]. A d-dimensional (bar-and-joint) *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We also say that (G, p) is a *realization* of G in $\mathbb{R}^d$. We can think of the edges and vertices of G in the framework as rigid (fixed length) bars and universal joints, respectively. Two frameworks (G, p) and (G, q) are *equivalent* if corresponding edges have the same lengths, that is, if $||p(u)-p(v)|| = ||q(u)-q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$. We shall say that (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p).

Rigidity is a weaker property of frameworks than global rigidity. Intuitively, a framework is rigid if it has no continuous deformations. Equivalently, and more formally, a framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p).

A framework (G, p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. It is known that rigidity as well as global rigidity are generic properties of d-dimensional frameworks for all d, that is, the (global) rigidity of a generic realization of a graph G depends only on the graph G and not the particular realization. We say that the graph G is *rigid*, respectively *globally rigid*, in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid, respectively globally rigid. Many of the (global) rigidity properties of a generic framework (G, p) are determined by an associated matroid, the *d-dimensional rigidity matroid* $\mathcal{R}_d(G)$, defined on the edge set of G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$.

In what follows we shall focus on the case $d = 2$. In this case rigidity and the rank function of the rigidity matroid are well characterized. It is known that a graph $G = (V, E)$ is rigid in $\mathbb{R}^2$ if and only if $r_2(G) = 2|V| - 3$. It is also known that the edge set of G is independent in $\mathcal{R}_2(G)$ if and only if each subset $X \subseteq V$ with $|X| \geq 2$ induces at most $2|X| - 3$ edges [7]. Lovász and Yemini [8] characterized rigid graphs in $\mathbb{R}^2$ by providing a formula for $r_2(G)$, in terms of 'thin covers' of G. We shall use the following refinement of their result, which uses rigid components, see [1, Section 4.4]. We define a *rigid component* of a graph $G = (V, E)$ to be a maximal rigid subgraph of G. By the *glueing lemma* (see [10, Lemma 3.1.4]), which says that the union of two rigid graphs with at least two vertices in common is rigid, it follows that any two rigid components of G intersect in at most one vertex. Thus their vertex sets form a special 'thin cover' of G.

Theorem 2.1. *[1, 8] Let $H = (V, E)$ be a graph with rigid components $H_1, H_2, ..., H_t$. Then*

$$r_2(H) = \sum_{i=1}^{t} (2|C_i| - 3),$$

where $C_i = V(H_i)$, $1 \leq i \leq t$.

Let $G = (V, E)$ be a graph. For a family $\mathcal{F}$ of pairwise disjoint subsets of V let $E_G(\mathcal{F})$ denote the set, and $e_G(\mathcal{F})$ the number, of edges of G connecting distinct members of $\mathcal{F}$. For a partition $\mathcal{P}$ of V let

$$\mathrm{def}_G(\mathcal{P}) = 3(|\mathcal{P}| - 1) - 2e_G(\mathcal{P})$$

denote the *deficiency* of $\mathcal{P}$ in G and let

$$\mathrm{def}(G) = \max\{\mathrm{def}_G(\mathcal{P}) : \mathcal{P} \text{ is a partition of } V\}.$$

We say that a partition $\mathcal{P}$ of V is *tight* if $\mathrm{def}_G(\mathcal{P}) = \mathrm{def}(G)$ holds. Note that $\mathrm{def}(G) \geq 0$, since $\mathrm{def}_G(\{V\}) = 0$. For example, the graph G on Figure 1 has $\mathrm{def}(G) = 1$. The vertex sets of the four disjoint copies of 'K_4 minus an edge' in G form a tight partition of G.

The following rank formula (which is implicit in [6]) shows that the 'degree of freedom' of $L(G)$ is equal to the deficiency of G.

Theorem 2.2. *Let $G = (V, E)$ be a graph with minimum degree at least two. Then*

$$r_2(L(G)) = 2|E| - 3 - \mathit{def}(G). \tag{1}$$

Proof: First we prove that the right hand side is an upper bound on $r_2(L(G))$. Since $|V(L(G))| = |E|$, we have $r_2(L(G)) \leq 2|E|-3$. Thus we may assume that $\text{def}(G) \geq 1$. Let $\mathcal{Q} = \{Q_1, Q_2, ..., Q_t\}$ be a tight partition of V. Since $\text{def}(G) \geq 1$, we must have $t \geq 2$.

For $v \in V$ let $B(v)$ denote the set of vertices in $L(G)$ corresponding to the edges incident with v in G. Since G has minimum degree at least two, we have $|B(v)| \geq 2$ for all $v \in V$. Let $X_i = \cup_{v \in Q_i} B(v)$, for $1 \leq i \leq t$. Since each set $B(v)$ contains at least two vertices, we have $|X_i| \geq 2$ for $1 \leq i \leq t$. Furthermore, $|\{X_i : v_e \in X_i\}| \leq 2$ for each vertex v_e of $L(G)$ with equality if and only if $e \in E_G(\mathcal{Q})$. Thus $\sum_{i=1}^{t} |X_i| = |E| + e_G(\mathcal{Q})$. Since every edge of $L(G)$ is induced by some X_i and each set $X \subseteq V(L(G))$ with $|X| \geq 2$ induces at most $2|X|-3$ independent edges in $\mathcal{R}_2(L(G))$, we can deduce that

$$\begin{aligned} r_2(L(G)) &\leq \sum_{i=1}^{t}(2|X_i| - 3) = 2|E| + 2e_G(\mathcal{Q}) - 3t \\ &= 2|E| - 3 - \text{def}(G). \end{aligned}$$

To prove that equality holds consider the rigid components $H_1, H_2, ..., H_t$ of $L(G)$ and let $C_i = V(H_i)$ for $1 \leq i \leq t$. Since each set $B(v)$, $v \in V$, induces a complete (and hence rigid) subgraph in $L(G)$, we must have $B(v) \subseteq C_i$ for some $1 \leq i \leq t$. Furthermore, since $|B(v)| \geq 2$ for all $v \in V$, the maximality of the C_i's and the glueing lemma imply that each $B(v)$ is contained in exactly one set C_i. Let $Q_i = \{v \in V : B(v) \subseteq C_i\}$, $1 \leq i \leq t$. Observe that $Q_i \neq \emptyset$ for all $1 \leq i \leq t$, since each rigid component H_i has at least one edge, say $v_e v_f$. Hence there is a vertex $x \in V$ which is a common end-vertex of edges e, f in G. Thus $|B(x) \cap C_i| \geq 2$ and hence, by the glueing lemma, $B(x) \subseteq C_i$ and $x \in Q_i$ must hold. It follows that $\mathcal{Q} = \{Q_1, Q_2, ..., Q_t\}$ is a partition of V.

Claim 2.3. *$v_e \in C_i \cap C_j$ for some $v_e \in V(L(G))$ and $1 \leq i < j \leq t$ if and only if $e \in E_G(Q_i, Q_j)$.*

Proof: First suppose $v_e \in C_i \cap C_j$. Consider an edge $v_e v_f \in E(H_i)$. As above, we may deduce that there is a vertex $x \in V$, incident with e, f, with $x \in Q_i$. Similarly, by considering an edge $v_e v_h \in E(H_j)$ we obtain that there is a vertex $y \in V$, incident with e, h, with $y \in Q_j$. This implies that $e = xy$ and $e \in E_G(Q_i, Q_j)$.

Conversely, suppose that $e = xy \in E_G(Q_i, Q_j)$. Then $B(x) \subseteq C_i$, $B(y) \subseteq C_j$. Since $v_e \in (B(x) \cap B(y))$, we have $v_e \in C_i \cap C_j$, as required. •

By using Theorem 2.1 and Claim 2.3 we obtain

$$\begin{aligned} r_2(L(G)) &= \sum_{i=1}^{t}(2|C_i| - 3) = 2|E| + 2e_G(\mathcal{Q}) - 3t \\ &= 2|E| - 3 - \text{def}(\mathcal{Q}) \geq 2|E| - 3 - \text{def}(G), \end{aligned}$$

which completes the proof. •

The lower bound on the minimum degree of G in Theorem 2.2 cannot be weakened. This follows by observing that if G is a star then $L(G)$ is rigid but G is highly deficient.

3 Globally rigid zeolites

Globally rigid graphs in $\mathbb{R}^2$ have been characterized by Jackson and Jordán [5]. We say that a graph G is *redundantly rigid* in $\mathbb{R}^2$ if $G - e$ is rigid in $\mathbb{R}^2$ for all $e \in E(G)$.

Theorem 3.1. *[5] Let H be a graph. Then H is globally rigid in $\mathbb{R}^2$ if and only if H is a complete graph on at most three vertices or H is 3-vertex-connected and redundantly rigid in $\mathbb{R}^2$.*

If H is a line graph of a 3-regular graph then a simpler characterization follows from the next theorem.

Theorem 3.2. *Let $G = (V, E)$ be a 3-regular graph. Then $L(G)$ is 3-vertex-connected and redundantly rigid in $\mathbb{R}^2$ if and only if G is 3-edge-connected.*

Proof: First suppose that $G - F$ has two connected components D_1, D_2 for some $F \subseteq E$ with $|F| \leq 2$. Since G is 3-regular, there must be an edge in D_i for $i = 1, 2$. This implies that the vertex set in $L(G)$ corresponding to F is a separating vertex set in $L(G)$. Thus $L(G)$ is not 3-vertex-connected. This proves the 'only if' direction.

To see the 'if' part, suppose that G is 3-edge-connected. This implies that $L(G)$ is 3-vertex-connected, since each separating vertex set in $L(G)$ gives rise to a separating edge set of G of the same size.

Next we show that $L(G)$ is redundantly rigid. We need the following claim.

Claim 3.3. *Let H be a graph with minimum degree at least two and suppose that H can be made 3-edge-connected by adding at most one edge. Then $L(H)$ is rigid.*

Proof: By Theorem 2.2 it suffices to show that $\text{def}(H) = 0$. Consider a partition $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of $V(H)$ with $t \geq 2$. Since H can be made 3-edge-connected by adding at most one edge, all but at most two members X_i of $\mathcal{P}$ satisfy $e_H(X_i, V(H) - X_i) \geq 3$, and all members satisfy $e_H(X_i, V(H) - X_i) \geq 2$. Hence

$$2e_H(\mathcal{P}) \geq 3t - 2 > 3(|\mathcal{P}| - 1).$$

Thus $\text{def}(H) = 0$. •

Now consider and edge $p = v_e v_f$ of $L(G)$. This edge corresponds to a pair of edges $e = xy, f = xz$ in G with a common end-vertex. Since G is 3-edge-connected, we can apply Claim 3.3 to $H = G - e$ to deduce that $L(H)$ is rigid.

It is easy to check that $L(G) - p$ can be obtained from $L(H)$ by adding a new vertex and connecting it to three distinct vertices of $L(H)$. This operation is known to preserve rigidity (in fact, connecting the new vertex to two vertices of $L(H)$ would already preserve rigidity, see e.g. [10, Lemma 2.1.3]). Thus $L(G) - p$ is rigid. This proves that $L(G)$ is redundantly rigid, as required. •

By Theorems 3.1 and 3.2 we obtain:

Corollary 3.4. *A two-dimensional combinatorial zeolite is globally rigid if and only if its underlying graph is 3-edge-connected.*

4 Concluding remarks

The charaterization of global rigidity provided by Corollary 3.4 has algorithmic implications, too. Given a 3-regular graph G on n vertices, the best known running time bound for testing whether $L(G)$ satisfies both conditions of Theorem 3.1 is $O(n^2)$. However, testing whether G is 3-edge-connected can be done in linear time. Since G is 3-regular, this gives rise to an improved $O(n)$ time bound for testing global rigidity of two-dimensional combinatorial zeolites.

The characterization of rigid (or globally rigid) graphs in $\mathbb{R}^d$, for $d \geq 3$, is not known. Even the special case of three-dimensional combinatorial zeolites appears to be difficult. Nevertheless, one might conjecture that for all positive integers d a d-dimensional combinatorial zeolite is globally rigid if and only if its underlying $(d+1)$-regular graph is $(d+1)$-edge-connected (or possibly $(d+1)$-vertex-connected). This natural extension of Corollary 3.4, which is correct for $d \leq 2$, fails in $\mathbb{R}^3$: a counterexample, due to Bill Jackson [4], is shown in Figure 2.

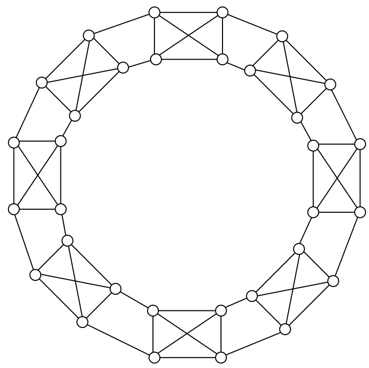

Figure 2: A 4-vertex-connected 4-regular graph G for which $L(G)$ is not rigid (and hence not globally rigid) in $\mathbb{R}^3$. The line graph of G behaves like a body-and-hinge framework whose underlying graph is a cycle of length 8, and hence it is easily shown to be flexible. (Thus a cycle of length 7 would also be a counterexample.)

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