Balanced list edge-colourings of bipartite graphs

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Abstract

Galvin solved the Dinitz conjecture by proving that bipartite graphs are Δ-edge-choosable. We employ Galvin's method to show some further list edge-colouring properties of bipartite graphs. In particular, there exist balanced list edge-colourings for bipartite graphs. In the light of our result, it is a natural question whether a certain generalization of the well-known list colouring conjecture is true.

Keywords: chromatic index; list edge-colouring; Galvin's theorem, list colouring conjecture

1 Introduction

A *proper edge-colouring* of graph G is the assignment of colours to the edges of G such that no two edges incident with the same vertex have the same colour. Graph G is said to be *k-edge-choosable* if no matter how we assign lists $L(e)$ of k possible colours to each edge e of G, there always exists a proper edge-colouring of G such that each edge e of G is coloured from $L(e)$. The *list chromatic index* $\chi'_l(G)$ of G is the least integer k such that G is k-edge-choosable. As we may assign the same list to each edge, the chromatic index is always a lower bound on the list chromatic index: $\chi'(G) \leq \chi'_l(G)$. Dinitz conjectured (in terms of Latin squares) that complete bipartite graph $K_{n,n}$ has list chromatic index $\chi'_l(K_{n,n}) = n$ (see [1]). This is a special case of the famous list colouring conjecture stating that for any finite loopless graph G, we have $\chi'(G) = \chi'_l(G)$. Galvin's celebrated method shows that the list colouring conjecture is true for any finite bipartite graph and this immediately implies the Dinitz conjecture [2].

In this work, we employ Galvin's result and extend it to not necessarily proper edge-colourings. For this reason, we define a partial order on edge-colourings such

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that proper edge-colourings and 1-edge-colourings are at the best and worse elements, respectively. We deduce that for any k-edge-colouring of bipartite graph G, there exists a better list edge-colouring provided each edge has at least k possible colours. This result can be formulated such that a generalization of the list colouring conjecture to not necessarily proper edge-colourings holds for bipartite graphs. It is a natural question to ask the same question for nonbipartite graphs, as well.

In Section 2, we define a partial order on number theoretic partions and formulate our main result and prove the existence of balanced list k-edge-colourings of bipartite graphs that is the main motivation of this present work. It is well-known that for any bipartite graph G and for any positive integer k, there exists an edge-colouring of G with k colours such that no vertex v is incident with more than $\left\lceil \frac{d(v)}{k} \right\rceil$ edges of the same colour. We show that if each edge list contains at least k colours then there exists a list edge-colouring with the same property. We prove our main result in Section 3 and conclude in Section 4 with two open questions.

2 Main result

To define a partial order on edge-colourings, we start from a little afar. For a non-negative integer n, a *(number theoretic) partition of* n is a way to decompose n as a sum of positive integers where the order of the terms is indifferent. That is, if two such sums only differ in the order of the terms then those determine the same partition. For a number theoretic partition π let $\pi(i)$ denote the ith greatest term in π, where we count each addend with its multiplicity. That is, if π is partition $2+3+2+5+1+1$ of 14 then $\pi(3)=2$, $\pi(5)=1$ and (slightly abusing notation) $\pi(8)=0$. We say that partition π of n is *better* than partition π' of n' (denoted by $\pi \preceq \pi'$) if $\sum_{i=1}^{k} \pi(i) \leq \sum_{i=1}^{k} \pi'(i)$ holds for all positive integers k. It follows immediately from the definition that among partitions of n, $n = 1+1+\ldots+1$ is the best one and the one-term partition $n=n$ is the worst one.

Let us turn to edge-colourings now. By a *k-edge-colouring* of graph G we mean a function $c : E(G) \to \{1, 2, \ldots, k\}$, and $c(e)$ is called the *colour* of edge e of G. Each k-edge-colouring c and each vertex v of G induce a partition $\pi(c, v)$ of degree $d(v)$ of v into (at most k) terms that describe how many edges of each colour of c are incident with v. In particular, edge-colouring c is a proper one if and only if $\pi(c, v)$ is the best partition of $d(v)$ for each vertex v of G. An *edge-colouring* of graph G is a k-edge-colouring of G for some k.

If c and c' are two edge-colourings of G then edge-colouring c is *better* than c' if $\pi(c, v) \preceq \pi(c', v)$ holds for each vertex v of G, that is, if c induces a better partition on each degree than c' does. This definition yields in particular that the best edge-colourings are the proper ones. Now we can claim our main theorem.

Theorem 2.1. *Let $G = (V, E)$ be a finite bipartite graph and π_v be a partition of $d(v)$ into at most k terms. If $L(e)$ is a list of at least k possible colours for each edge e of G then we can pick a colour $c(e)$ of $L(e)$ for each edge e of G such that the partition c induces at v is better than π_v at each vertex v of G.*

An immediate corollary of Theorem 2.1 is the following.

Corollary 2.2. *If c' is a k-edge-colouring of bipartite graph G and $|L(e)| \geq k$ for each edge e of G then there exists a list edge colouring c of G such that $c \preceq c'$.*

From Kőnig's theorem we know that $\chi'(G) = \Delta(G)$ holds for bipartite graphs, so the fact that proper edge-colourings are the best ones shows that Galvin's theorem follows from Corollary 2.2.

Theorem 2.3 (Galvin [2])**.** *Each bipartite graph G is $\Delta(G)$-edge-choosable.*

Actually, Theorem 2.3 is a main ingredient in the proof of our main result. However, there is another consequence of Theorem 2.1 that has to do with balanced k-edge-colourings.

Corollary 2.4. *Assume that G is a bipartite graph and for each edge e of G, list $L(e)$ contains at least k colours. Then it is possible to pick a colour $c(e) \in L(e)$ for each edge e of G such that no vertex v is incident with more than $\left\lceil \frac{d(v)}{k} \right\rceil$ edges of the same colour.*

Proof. Appling Theorem 2.1 to G where π_v denotes the partition of $d(v)$ into k terms each of which is either $\left\lceil \frac{d(v)}{k} \right\rceil$ or $\left\lfloor \frac{d(v)}{k} \right\rfloor$ gives a list edge-colouring c such that $\pi(c, v)(1) \leq \pi_v(1) = \left\lceil \frac{d(v)}{k} \right\rceil$ for all vertices v of G. This is exactly what Corollary 2.4 requires. □

3 Proof of the main result

Before justifying our main theorem, we recall some definitions. If G is a graph and S is a set of vertices of G then by *merging the vertices of S* we mean the operation that we delete S from G, introduce a new vertex (say v_S) and in each edge e of G incident with some vertex of S we replace vertices of S by v_S. Note that we may create parallel edges and loops by merging. Clearly, if G' is obtained from G by merging the vertices of S then G and G' has the same number of edges and the degree of v_S in G' is the sum of the degrees of the vertices of S in G. If S contains k vertices then we say that we can get graph G from G' by *detaching v_S into k parts*. Note that merging vertices is a unique operation unlike detaching a vertex into k parts that can be done several ways.

We need some basics also on partitions. We say that partition π of n is the *conjugate* of partition σ of n if $\pi(i) = \max\{j : \sigma(j) \geq i\}$. It is well-known that turning the Ferrers diagram of a partition by 90 degrees (and taking mirror image) we get the Ferrers diagram of the conjugate partition hence if σ is the conjugate of π then π is the conjugate of σ, as well.

Proof of Theorem 2.1. Construct graph G' by detaching each vertex v of G into vertices $v_1, v_2, \ldots, v_{\pi_v(1)}$ in such a way that $d_{G'}(v_1)+d_{G'}(v_2)+\ldots+d_{G'}(v_k)$ is the conjugate partition of π_v. Clearly $k \geq d_{G'}(v_1) \geq d_{G'}(v_2) \geq d_{G'}(v_3) \geq \ldots$ holds by our choice, so $\Delta(G') \leq k$. For each edge e' of G' define $L(e') := L(e)$ where e' corresponds to edge e' of G. By Theorem 2.3 of Galvin, there exists a list edge-colouring of G', that is we can pick a colour $c'(e) \in L'(e)$ for each edge e' of G such that c' is a proper edge-colouring of G'. For each edge e of G define $c(e) := c'(e')$ where e' corresponds to e in G'. By definition, $c(e) \in L(e)$ holds.

The only thing left is to show that $\pi(c, v) \preceq \pi_v$ for each vertex v of G. To this end, it is enough to prove that for any positive integer i and any set C of i colours, no more than $\pi_v(1) + \pi_v(2) + \ldots + \pi_v(i)$ edges incident with v have been coloured to a colour of C. So fix set C of i colours and let $E(C, v) := \{e \in E(v) : c(e) \in C\}$ be the set of edges incident with v with a colour of C. (Here $E(v)$ stands for the set of edges incident with v.) Let $E'(C, v)$ be the set of edges of G' that correspond to edges $E(C, v)$. Clearly, each vertex v_j of G' is incident with at most $\min(d_{G'}(v_j), i)$ edges of $E'(C, v)$. This means that

$$|E(C, v)| = |E'(C, v)| \leq \sum_{j=1}^{\pi_v(1)} \min(d_{G'}(v_j), i) = \pi_v(1) + \pi_v(2) + \ldots + \pi_v(i) \ ,$$

where the last equality follows from the fact that partitions $\pi_v(1) + \pi_v(2) + \ldots + \pi_v(i)$ and $\min(d_{G'}(v_1), i) + \min(d_{G'}(v_2), i) + \ldots + \min(d_{G'}(v_{\pi_v(1)}), i)$ are conjugates of one another.

We got that for each vertex v of G there cannot be more edges incident with v coloured with at most i colours than the sum of the i greatest term of prescribed vertex-partition π_v. This means that list edge-colouring c induces a better partition on $d(v)$ than π_v, and this is exactly what we wanted to prove. □

4 Conclusion

The list colouring conjecture can be interpreted such that if k colours are enough to properly colour the edges of a graph then from arbitrarily given edge lists of size k it is possible to pick a colour for each edge to form a proper edge-colouring. The list colouring conjecture is known to be true for bipartite graphs due to [2] by Galvin. Corollary 2.2 shows that for bipartite graphs, an even stronger statement is true: if we fix some k-edge-colouring of G, it never hurts if we assign *different* lists of k colours to the edges in the sense that we can always find a better edge-colouring from the lists than our initial colouring. It is a natural question whether the following generalized form of the list colouring conjecture is true.

> **Generalized list colouring conjecture.** Is it true that any graph G and for any k-edge-colouring c of G, no matter how sets $L(e)$ of size k are assigned to each edge e of G, there always exist elements $c'(e)$ of $L(e)$ such that c' is a better edge-colouring of G than c is?

The motivation of this work is the study of the existence of balanced list edge-colourings. We proved in Corollary 2.4 that for any assignment of k-lists to the edges, there exists a "balanced" list edge-colouring in which no edge is incident with "too many" edges of the same colour. Instead of Theorem 2.1, Corollary 2.4 can also be deduced from Corollary 2.2 and the well-known fact that for any bipartite graph there exists a "balanced" k-edge-colouring c' such that the number of edges of the same c'-colour incident with v is either $\left\lceil \frac{d(v)}{k} \right\rceil$ or $\left\lfloor \frac{d(v)}{k} \right\rfloor$ for each vertex v of G. But it is also well-known that we can require some additional properties of balanced k-edge-colouring c'. Namely, if we fix a nested family on the vertices of each colour class then we can require that $\left\lfloor \frac{|E(X)|}{k} \right\rfloor \leq |c^{-1}(i) \cap E(X)| \leq \left\lceil \frac{|E(X)|}{k} \right\rceil$ holds for each member X of the nested families, where $E(X)$ denotes the set of edges of G incident with X. This stronger balanced property means that for each member X more or less exactly the kth fraction of the edges incident with X receive the same colour. So it is a natural question whether Corollary 2.4 can be generalized the following way.

> Let G be bipartite, $k > 0$, let $\mathcal{A}$ and $\mathcal{B}$ be nested set-systems on colour classes A and B of G and let $|L(e)| \geq k$ for each edge e of G. Does there always exist list edge-colouring c from L in such a way that no more than $\left\lceil \frac{|E(X)|}{k} \right\rceil$ edges o $E(X)$ has the same colour for any member X of $\mathcal{A} \cup \mathcal{B} \cup V$?

We leave this open question to the reader.

References

[1] Paul Erdős, Arthur L. Rubin, and Herbert Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157, Winnipeg, Man., 1980. Utilitas Math.

[2] Fred Galvin. The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B*, 63(1):153–158, 1995.