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A Proof to Cunningham's Conjecture on Restricted Subgraphs and Jump Systems

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Abstract

For an undirected graph and a fixed integer k, a 2-matching is said to be k-restricted if it has no cycle of length k or less. The problem of finding a maximum k-restricted 2-matching is polynomially solvable when $k \leq 3$, and NP-hard when $k \geq 5$. On the other hand, the degree sequences of the k-restricted 2-matchings form a jump system for $k \leq 3$, and do not always form a jump system for $k \geq 5$, which is consistent with the polynomial solvability of the maximization problem. In 2002, Cunningham conjectured that the degree sequences of 4-restricted 2-matchings form a jump system and the maximum 4-restricted 2-matching can be found in polynomial time.

In this paper, we show that the first conjecture is true, that is, the degree sequences of 4-restricted 2-matchings form a jump system. We also show that the weighted 4-restricted 2-matchings in a bipartite graph induce an M-concave function on the jump system if and only if the weight function is vertex-induced on every square. This result is also consistent with the polynomial solvability of the weighted 4-restricted 2-matching problem in bipartite graphs.

Keywords: restricted 2-matchings, jump systems, M-convex functions

1 Introduction

A jump system, introduced by Bouchet and Cunningham [5], is a set of integer lattice points with an exchange property (to be described in Section 2); see also [20, 26]. It is a generalization of a matroid [32, 38, 41], a delta-matroid [4, 6, 10], and a base polyhedron of an integral polymatroid (or a submodular system) [15]. Many efficiently solvable combinatorial optimization problems closely relate to these structures. For

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instance, the degree sequences of all matchings in an undirected graph form a delta-matroid, and the degree sequences of all even factors form a jump system if the given digraph has a certain property called odd-cycle-symmetric [25].

In the present paper, we investigate the relationship between jump systems and k-restricted 2-matchings, which was first considered by Cunningham [7]. We consider only simple undirected graphs in this paper. An edge set M is called a *t-matching* if at most t edges in M are incident to each vertex (these are usually called *simple* t-matchings in the literature). For an integer k, a 2-matching M is said to be *k-restricted* if M has no cycle of length k or less. The *k-restricted 2-matching problem* is to find a k-restricted 2-matching of maximum size for a given graph and given k. Note that the case $k \leq 2$ is exactly the classical simple 2-matching problem, which can be solved efficiently. Papadimitriou showed that the k-restricted 2-matching problem is NP-hard for $k \geq 5$ (see [9]), and Geelen [16] proved that it is NP-hard when $k = 6$ and the graph is bipartite. On the other hand, Hartvigsen [17] proved that the problem is polynomial-time solvable for $k = 3$. The case $k = 4$ is left open.

Cunningham [7] conjectured that the degree sequences of k-restricted 2-matchings form a jump system if and only if the k-restricted 2-matching problem is polynomial-time solvable (see Section 2 for the definition of degree sequences). He proved this conjecture for cases $k = 3$ and $k \geq 5$. That is, he proved that the degree sequences of the 3-restricted 2-matchings form a jump system and those of the 5-restricted 2-matchings do not. (His counterexample for the case $k \geq 5$ will be simplified in Section 3.2.)

For the case $k = 4$, Cunningham [7] conjectured that the 4-restricted 2-matching problem is polynomial-time solvable and that the degree sequences of the 4-restricted 2-matchings form a jump system. In this paper we prove the second conjecture. These conjectures of Cunningham were based on Russel's augmenting path theorem [35] and Király's min-max formula for the 4-restricted 2-matching problem in bipartite graphs [21]. Later, polynomial-time algorithms for the 4-restricted 2-matching problem in bipartite graphs are devised by Hartvigsen [18] and Pap [34].

Recently, generalizations of k-restricted 2-matchings to t-matchings are studied actively. Frank [14] first considered *$K_{t,t}$-free t-matchings*, which are t-matchings not containing a $K_{t,t}$ as a subgraph. Note that, when $t = 2$ and the given graph is bipartite, the $K_{t,t}$-free t-matchings are exactly the 4-restricted 2-matchings. Also, the notion of *K_{t+1}-free t-matchings*, which are t-matchings not containing a K_{t+1} as a subgraph, is a generalization of that of 3-restricted 2-matchings. The $K_{t,t}$-free t-matching problem and the K_{t+1}-free t-matching problem are solved in some classes of graphs. For the former problem in bipartite graphs, a min-max formula is given by Frank [14], and a combinatorial algorithm by Pap [34, 33]. For the both problems in graphs with degree at most $t + 1$, a min-max formula and a combinatorial algorithm are given by Bérczi and Végh [3].

In this paper, we prove Cunningham's conjecture, stating that the degree sequences of 4-restricted 2-matchings form a jump system, by showing a general theorem for t-matchings. Now we state our main result. We say that a graph $G = (V, E)$ is a *complete partite graph* if there exists a partition $\{V_1, \ldots, V_p\}$ of V such that $E = \{(u, v) \mid u \in V_i, v \in V_j, i \neq j\}$. In other words, a complete partite graph is the complement of the disjoint union of complete graphs. A graph is *t-regular* if the degree of every vertex is t. For a set $\mathcal{H}$ of graphs, a subgraph of G is an *$\mathcal{H}$-subgraph* if it is isomorphic to a member of $\mathcal{H}$, and a subgraph (or an edge set) is *$\mathcal{H}$-free* if it contains no $\mathcal{H}$-subgraph (resp. no edge set of an $\mathcal{H}$-subgraph). In particular, "k-restricted" and "$\{C_3, \ldots, C_k\}$-free" mean the same condition for a 2-matching, where C_t denote a cycle with t vertices. For a graph H, we denote "$\{H\}$-subgraph" and "$\{H\}$-free" simply by "H-subgraph" and "H-free," respectively. Our main result is stated as follows.

Theorem 1.1. *Let t be an integer and $\mathcal{H}$ be a set of t-regular graphs such that any proper subgraph of a member of $\mathcal{H}$ is not in $\mathcal{H}$. Then, the degree sequences of all $\mathcal{H}$-free t-matchings in G form a jump system for any graph G if and only if every member of $\mathcal{H}$ is a complete partite graph.*

As special cases of this theorem, we obtain the following as corollaries. In particular, Corollary 1.4 solves Cunningham's conjecture.

Corollary 1.2. *The degree sequences of all K_{t+1}-free t-matchings in a graph form a jump system.*

Corollary 1.3. *The degree sequences of all $K_{t,t}$-free t-matchings in a graph form a jump system.*

Corollary 1.4. *The degree sequences of all 4-restricted 2-matchings in a graph form a jump system.*

We also discuss the weighted version from the viewpoint of discrete convex analysis [29]. The concept of M-concave (M-convex) functions on constant-parity jump systems is a general framework of optimization problems on jump systems [30] (see Section 4.1 for a definition), and it is a generalization of valuated matroids [11, 13], valuated delta-matroids [12], and M-convex functions on base polyhedra [28].

We consider the weighted k-restricted 2-matching problem in bipartite graphs. When $k \geq 6$, it is NP-hard even for the unweighted case [16]. Moreover, Z. Király proved that the weighted 4-restricted 2-matching problem in bipartite graphs is also NP-hard (see [14]). This problem is, however, tractable if the weight function is vertex-induced on every $K_{2,2}$. A weight function is said to be *vertex-induced on H* for a subgraph H if there exists a function p_H on the vertex set of H such that $w(e) = p_H(u) + p_H(v)$ for every edge $e = (u, v)$ in H. If the given graph is bipartite and the given weight function is vertex-induced on every $K_{t,t}$, then the weighted $K_{t,t}$-free t-matching problem can be solved in polynomial time [27, 37].

In this paper, we show a relationship between the weighted $K_{t,t}$-free t-matchings in bipartite graphs and M-concave functions on constant-parity jump systems. For a weighted bipartite graph (G, w), let $J_{t,t}(G)$ be the set of degree sequences of all $K_{t,t}$-free t-matching in G, which is a jump system by Corollary 1.3. We define a function $f_{t,t}$ on $J_{t,t}(G)$ by

$$f_{t,t}(x) = \max \left\{ \sum_{e \in M} w(e) \;\middle|\; M \text{ is a } K_{t,t}\text{-free } t\text{-matching, } d_M = x \right\}.$$

Theorem 1.5. *For a weighted bipartite graph (G, w) and an integer $t \geq 2$, $f_{t,t}$ is an M-concave function on the constant-parity jump system $J_{t,t}(G)$ if and only if w is vertex-induced on every $K_{t,t}$ in G.*

This theorem suggests that assuming the weight function to be vertex-induced on every $K_{t,t}$ is reasonable in considering the weighted $K_{t,t}$-free t-matching problem in bipartite graphs. We also remark that a general algorithm maximizing an M-concave function on a constant-parity jump system [30, 31, 36] cannot be applied directly to the weighted $K_{t,t}$-free t-matching problem in this assumption. In such an algorithm, we compute the function value polynomially many times. Thus, in order to obtain a polynomial algorithm based on Theorem 1.5 and the general framework of maximizing M-concave functions on a constant-parity jump system, we need a polynomial-time algorithm computing $f_{t,t}$.

This paper is organized as follows. In Section 2, we give some definitions on graphs and jump systems. In Sections 3 and 4, we prove Theorems 1.1 and 1.5, respectively.

2 Definitions

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Assume that G is simple, that is, G has neither parallel edges nor self-loops. In what follows, we often omit to declare that the graph is simple or undirected. An edge connecting $u, v \in V$ is denoted by (u, v). The set of edges incident to $v \in V$ is denoted by $\delta(v)$. Recall that, for a positive integer t, an edge set $M \subseteq E$ is said to be a *t-matching* if $|M \cap \delta(v)| \leq t$ for every $v \in V$. In particular, a 2-matching is a vertex-disjoint collection of paths and cycles. The *degree sequence* $d_F \in \mathbf{Z}^V$ of an edge set $F \subseteq E$ is defined by

$$d_F(v) = |F \cap \delta(v)| \quad (v \in V).$$

We denote a bipartite graph with color classes V_1 and V_2 by $(V_1, V_2; E)$. For a positive integer t, K_t and C_t denote a complete graph with t vertices and a cycle with t vertices, respectively. For positive integers a and b, $K_{a,b}$ is a complete bipartite graph $(A, B; E)$ with $|A| = a$, $|B| = b$ and $E = \{(u, v) \mid u \in A,\ v \in B\}$. A graph $K_{1,t}$ is called a *star*.

For a subgraph H of G, the vertex set and edge set of H are denoted by $V(H)$ and $E(H)$, respectively. Recall that, for a set $\mathcal{H}$ of graphs, a subgraph of G is an *$\mathcal{H}$-subgraph* if it is isomorphic to a member of $\mathcal{H}$, and a subgraph (or an edge set) is *$\mathcal{H}$-free* if it contains no $\mathcal{H}$-subgraph (resp. no edge set of an $\mathcal{H}$-subgraph). While it causes no confusion, we sometimes identify a subgraph and its edge set. We also recall that for a positive integer k, we say that a 2-matching is *k-restricted* if it has no cycle of length k or less, that is, it is $\{C_3, \ldots, C_k\}$-free.

Let V be a finite set. For $u \in V$, we denote by χ_u the *characteristic vector* of u, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \in V \setminus \{u\}$. For $x, y \in \mathbf{Z}^V$, a vector $s \in \mathbf{Z}^V$ is called an *(x, y)-increment* if $x(u) < y(u)$ and $s = \chi_u$ for some $u \in V$, or $x(u) > y(u)$ and $s = -\chi_u$ for some $u \in V$.

Definition 2.1 (Jump system [5]). A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if it satisfies an exchange axiom, called the 2-*step axiom*:

Axiom 1. For any $x, y \in J$ and for any (x, y)-increment s_1 with $x + s_1 \notin J$, there exists an $(x + s_1, y)$-increment s_2 such that $x + s_1 + s_2 \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-parity system* if $\sum_{v \in V}(x(v) - y(v))$ is even for any $x, y \in J$. A constant-parity jump system is a generalization of the base family of a matroid, an even delta-matroid [39, 40], and a base polyhedron of an integral polymatroid. The degree sequences of all subgraphs in an undirected graph is a typical example of a constant-parity jump system. That is, for a graph $G = (V, E)$,

$$J_{\mathrm{SG}}(G) = \{d_F \mid F \subseteq E\}$$

is a constant-parity jump system [5, 26]. The set of all degree sequences of t-matchings is the intersection of $J_{\mathrm{SG}}(G)$ and a box $\{0, 1, \ldots, t\}^V$, and hence it is also a constant-parity jump system. However, Corollaries 1.2, 1.3, and 1.4 are not obvious since the additional conditions make the situation more complicated.

3 $\mathcal{H}$-free subgraphs and jump systems

In this section, we first investigate the relationship between $\mathcal{H}$-free t-matchings and jump systems. We prove the sufficiency and the necessity in Theorem 1.1 in Sections 3.1 and 3.2, respectively. After that, we consider the relationship between H-free subgraphs and jump systems in Section 3.3.

3.1 Sufficiency

First, we show the sufficiency ("if" part) in Theorem 1.1.

Proposition 3.1. *Let $\mathcal{H}$ be a set of t-regular complete partite graphs and $G = (V, E)$ be a graph. Then, the degree sequences of all $\mathcal{H}$-free t-matchings in G form a jump system.*

Proof. Let $J_{\mathcal{H}}(G)$ be the set of the degree sequences of all $\mathcal{H}$-free t-matchings in G. For $x, y \in J_{\mathcal{H}}(G)$, let M and N be $\mathcal{H}$-free t-matchings in G such that $d_M = x$ and $d_N = y$, and let s_1 be an (x, y)-increment. Note that $x + s_1 \notin J_{\mathcal{H}}(G)$. We present an algorithm for finding an $(x + s_1, y)$-increment s_2 satisfying Axiom 1. In what follows, we consider the case where $s_1 = \chi_u$ for some $u \in V$. The case where $s_1 = -\chi_u$ can be dealt with in a similar way.

Let $\mathcal{P}$ be the set of pairs of an $\mathcal{H}$-free t-matching and a vertex defined by

$$\mathcal{P} = \{(M', u') \mid M' \text{ is an } \mathcal{H}\text{-free } t\text{-matching in } G,\ u' \in V,\ d_{M'} + \chi_{u'} = x + s_1\}.$$

In order to show the proposition, we use the following lemma, whose proof is given below.

Lemma 3.2. *When we are given a pair $(M', u') \in \mathcal{P}$, we can find either an $(x + s_1, y)$-increment s_2 with $x + s_1 + s_2 \in J_{\mathcal{H}}(G)$ or a new pair $(M'', u'') \in \mathcal{P}$ such that $(|M'' \cup N|, d_{M'' \cap N}(u''))$ is lexicographically less than $(|M' \cup N|, d_{M' \cap N}(u'))$*

Since $(M, u) \in \mathcal{P}$ and $(|M' \cup N|, d_{M' \cap N}(u'))$ is finite for $(M', u') \in \mathcal{P}$, we obtain the proposition by using Lemma 3.2, repeatedly. □

Hence, what remains is the proof of Lemma 3.2.

Proof of Lemma 3.2. For $(M', u') \in \mathcal{P}$, $-\chi_{u'}$ is a desired $(x + s_1, y)$-increment if $d_{M'}(u') \geq y(u')$. Thus, we may assume that $d_{M'}(u') < y(u')$, that is, $\chi_{u'}$ is a $(d_{M'}, y)$-increment, which means that it suffices to consider the case where the given pair is (M, u).

To prove Lemma 3.2, we use the following claims.

Claim 3.3. *For $t \geq 3$, there exists at most one edge $e \in (N \setminus M) \cap \delta(u)$ such that $M \cup \{e\}$ contains an $\mathcal{H}$-subgraph.*

Proof. Assume that both $e_1 = (u, v_1)$ and $e_2 = (u, v_2)$ are in $(N \setminus M) \cap \delta(u)$ and both $M \cup \{e_1\}$ and $M \cup \{e_2\}$ contain $\mathcal{H}$-subgraphs. Let H_i be an $\mathcal{H}$-subgraph contained in $M \cup \{e_i\}$ for $i = 1, 2$. Since χ_u is a (d_M, y)-increment and $y(u) \leq t$, we have that $|M \cap \delta(u)| = t - 1$. Therefore, there exists an edge $(u, w_1) \in M \cap \delta(u)$, which is contained in both H_1 and H_2. Since M is a t-matching, $|M \cap \delta(w_1)| = t$ and all edges in $M \cap \delta(w_1)$ are contained in both H_1 and H_2.

Assume that there exists an edge (w_1, w_2) in M with $w_2 \neq u, v_1, v_2$. Then, $|M \cap \delta(w_2)| = t$ and all edges in $M \cap \delta(w_2)$ are contained in both H_1 and H_2. This means that, by the definition of complete partite graphs, both $V(H_1)$ and $V(H_2)$ are identical to the set of all end vertices of $M \cap (\delta(w_1) \cup \delta(w_2))$, denoted by $V(H)$. Hence, v_1 is adjacent to t vertices of $V(H)$ both in H_1 and H_2, which contradicts that $(u, v_1) \in E(H_1) \setminus E(H_2)$.

Thus, the remaining case is when $t = 3$ and $M \cap \delta(w_1) = \{(w_1, u), (w_1, v_1), (w_1, v_2)\}$. Note that 3-regular complete partite graph is either K_4 or $K_{3,3}$. Since (u, v_2) is not in H_1, H_1 is not a K_4 but a $K_{3,3}$. However, H_1 contains $\{(u, v_1), (w_1, u), (w_1, v_1)\}$, which is a contradiction. □

Claim 3.4. *For $t = 2$, either one of the following statements holds.*

- *There exists at most one edge $e \in (N \setminus M) \cap \delta(u)$ such that $M \cup \{e\}$ contains an $\mathcal{H}$-subgraph.*
- *$\mathcal{H} = \{K_3, K_{2,2}\}$ and there exist three vertices $v, w, z \in V$ such that $(u, v), (u, w) \in N \setminus M$ and $(u, z), (z, w), (w, v) \in M$.*

Proof. Assume that the former statements does not hold, that is, $M \cup \{e_1\}$ and $M \cup \{e_2\}$ contain $\mathcal{H}$-subgraphs for distinct edges $e_1 = (u, v_1)$ and $e_2 = (u, v_2)$ in $(N \setminus M) \cap \delta(u)$. Note that 2-regular complete partite graph is either K_3 or $K_{2,2}$. Let H_i be an $\mathcal{H}$-subgraph contained in $M \cup \{e_i\}$ for $i = 1, 2$. In the same way as Claim 3.3, we can take an edge $(u, w_1) \in M \cap \delta(u)$ such that $|M \cap \delta(w_1)| = 2$ and all edges in $M \cap \delta(w_1)$ are contained in both H_1 and H_2. Let (w_1, w_2) be an edge in M with $w_2 \neq u$. Since both $M \cup \{e_1\}$ and $M \cup \{e_2\}$ contain $\mathcal{H}$-subgraphs, we have that w_2 coincides with v_1 or v_2, one of H_1 and H_2 is K_3 and the other is $K_{2,2}$, and there exists an edge $(v_1, v_2) \in M$. This shows that we have the latter statement by setting $z = w_1$, $\{v, w\} = \{v_1, v_2\}$. $\square$

In order to prove Lemma 3.2, we consider the following two cases separately.
Case 1. Assume that there exists an edge $e = (u, v) \in (N \setminus M) \cap \delta(u)$ such that $M \cup \{e\}$ contains no $\mathcal{H}$-subgraph. Then, we define $M' = M \cup \{e\}$. If $d_M(v) < d_N(v)$, then $s_2 = \chi_v$ is an $(x + s_1, y)$-increment and $x + s_1 + s_2 = d_{M'} \in J_{\mathcal{H}}(G)$. Otherwise, since $d_{M'}(v) = d_M(v) + 1 > d_N(v)$, there exists an edge $(v, w) \in (M' \setminus N) \cap \delta(v)$. In this case, the pair (M'', w) defined by $M'' = M' \setminus \{(v, w)\}$ is in $\mathcal{P}$ and satisfies the condition in Lemma 3.2, because $|M'' \cup N| = |M \cup N| - 1$.
Case 2. Assume that there exist no such edges, that is, for every edge $e \in (N \setminus M) \cap \delta(u)$, $M \cup \{e\}$ contains an $\mathcal{H}$-subgraph. Since $d_N(u) > d_M(u)$ by the definition of an (x, y)-increment, at least one edge is in $(N \setminus M) \cap \delta(u)$. Hence, by Claims 3.3 and 3.4, we have the following two possibilities.

(2-1) There exists an $\mathcal{H}$-subgraph H containing u such that $\delta(u) \cap E(H) \subseteq N$ and $E(H) \setminus \{(u, v)\} \subseteq M$ for some $v \in V(H)$.

(2-2) $t = 2$, $\mathcal{H} = \{K_3, K_{2,2}\}$, and there exist three vertices $v, w, z \in V$ such that $(u, v), (u, w) \in N \setminus M$ and $(u, z), (z, w), (w, v) \in M$.

In the case (2-2), the pair $(M', z) \in \mathcal{P}$ defined by $M' = (M \cup \{(u, v)\}) \setminus \{(v, w)\}$ satisfies the condition in Lemma 3.2, because $|M' \cup N| = |M \cup N| - 1$. Note that $(v, w) \notin N$, because N contains no $\mathcal{H}$-subgraph.

In the case (2-1), there exists an edge $(w_1, w_2) \in E(H)$ that is not contained in N, because N does not contain H. Since at least one of (v, w_1) and (v, w_2) is in $E(H)$ by the definition of complete partite graphs, we can assume that $(v, w_1) \in E(H)$ without loss of generality. Now we show the following claim.

Claim 3.5. *Suppose that there exists an $\mathcal{H}$-subgraph H such that $u \in V(H)$, $\delta(u) \cap E(H) \subseteq N$ and $E(H) \setminus \{(u, v)\} \subseteq M$ for some $v \in V(H)$. Then, H is the unique $\mathcal{H}$-subgraph in $M \cup \{(u, v)\}$.*

Proof. Let $M' = M \cup \{(u, v)\}$. Assume that M' contains an $\mathcal{H}$-subgraph $H' \neq H$. Since $|M \cap \delta(u)| \leq t - 1$, we have $|M' \cap \delta(u)| = t$ and all edges in $M' \cap \delta(u)$ are contained in both H and H'. Let (u, w) be an edge in $M' \cap \delta(u)$ with $w \neq v$. Then, w is contained in both H and H'. Since $|M' \cap \delta(w)| = t$, all edges in $M' \cap \delta(w)$ are contained in both H and H'. By the definition of complete partite graphs, both $V(H)$ and $V(H')$ are the set of all end vertices of $M' \cap (\delta(u) \cup \delta(w))$, which contradicts that $H' \neq H$. $\square$

Define $M'' = (M \cup \{(u, v)\}) \setminus \{(v, w_1)\}$. Since H is the unique $\mathcal{H}$-subgraph in $M \cup \{(u, v)\}$ by Claim 3.5, M'' contains no $\mathcal{H}$-subgraph, which implies that $(M'', w_1) \in \mathcal{P}$. Then, (M'', w_1) is a desired pair in Lemma 3.2, because $|M'' \cup N| \leq |M \cup N|$,

$d_{M\cap N}(u) = t-1$, and $d_{M''\cap N}(w_1) \le |(M''\cap\delta(w_1))\setminus\{(w_1,w_2)\}| = t-2$. This completes the proof of Lemma 3.2. □

3.2 Necessity

We prove the necessity ("only if" part) in Theorem 1.1.

Proposition 3.6. *Let $\mathcal{H}$ be a set of t-regular graphs such that any proper subgraph of a member of $\mathcal{H}$ is not in $\mathcal{H}$. If the degree sequences of all $\mathcal{H}$-free t-matchings in G form a jump system for any graph $G = (V,E)$, then every member of $\mathcal{H}$ is a complete partite graph.*

Instead of proving Proposition 3.6, we show the following stronger theorem.

Theorem 3.7. *Let t be a positive integer and $\mathcal{H}$ be a set of graphs such that any proper subgraph of a member of $\mathcal{H}$ is not in $\mathcal{H}$, and every member of $\mathcal{H}$ has no isolated vertices and has maximum degree at most t. If the degree sequences of all $\mathcal{H}$-free t-matchings in G form a jump system for any graph G, then every member of $\mathcal{H}$ is a complete partite graph.*

Proof. Assume to the contrary that $H \in \mathcal{H}$ is not a complete partite graph. We prove that the degree sequences of all $\mathcal{H}$-free t-matchings in $H = (V,E)$ itself do not form a jump system. The complete partite graphs are exactly those graphs in which non-adjacency is a transitive relation, and hence H has three vertices $v_1, v_2, v_3 \in V$ such that $(v_1,v_2),(v_1,v_3) \notin E$ and $(v_2,v_3) \in E$. Since H has no isolated vertices, there exists a vertex $u \in V$ adjacent to v_1. Let $M = E\setminus\{(v_1,u)\}$, and

$$N = \{e \in E \mid e \text{ is incident to some vertex in the neighborhood of } v_1\},$$

where the *neighborhood* of v_1 is the set of vertices adjacent to v_1. Since N does not contain (v_2,v_3), N is an $\mathcal{H}$-free t-matchings, and so is M. Let $x = d_M$, $y = d_N$, and $s_1 = \chi_u$. Then, x and y are degree sequences and s_1 is an (x,y)-increment. However, there exists no $(x+s_1,y)$-increment s_2 such that $x+s_1+s_2$ is a degree sequence of an $\mathcal{H}$-free t-matching. This is because,

- for a vertex v in the neighborhood of v_1, $(x+s_1)(v) = y(v)$,
- for a vertex v not in the neighborhood of v_1, $x+s_1-\chi_v = d_H-\chi_{v_1}-\chi_v$ is not a degree sequence of a subgraph of H since $(v_1,v)\notin E$, and
- for a vertex v not in the neighborhood of v_1, $x+s_1+\chi_v$ is a degree sequence of a subgraph of H only if $v = v_1$, but in this case it is a degree sequence of $E(H)$, which is not H-free.

□

It follows that C_k itself is an example for which the degree sequences of k-restricted 2-matchings do not form a jump system for $k \ge 5$. This counterexample is somewhat simpler than that of Cunningham [7]. It also follows that the degree sequences of C_k-free 2-matchings do not always form a jump system for $k \ge 5$.

3.3 Degree sequences of H-free subgraphs

In this subsection, we investigate when the degree sequences of all H-free subgraphs form a jump system for a graph H.

Theorem 3.8. *Let H be a graph. The degree sequences of all H-free subgraphs form a jump system in any graph if and only if H is a star.*

Proof. By Theorem 3.7, if H is not a complete partite graph then the degree sequences of H-free subgraphs of H itself do not form a jump system.

Suppose that H is a complete p-partite graph for $p \geq 3$, and denote the color classes by $V_1, \ldots, V_p$, where $|V_1| \leq \cdots \leq |V_p|$. Construct a new p-partite graph G by adding a new element r to V_p, that is, $V(G) = V(H) \cup \{r\}$ and $E(G) = E(H) \cup \{(v, r) \mid v \in V \setminus V_p\}$. Let $u \in V_1$, $v \in V_2$, $z_1, z_2 \in V_p \cup \{r\}$, $M = E(G) \setminus \{(u, v)\}$, and $N = E(G) \setminus \{(u, z_1), (u, z_2)\}$. Observe that M and N are H-free subgraphs and $s_1 = \chi_v$ is a (d_M, d_N)-increment. We have the following possibilities for $(d_M + s_1, d_N)$-increments: $-\chi_u$ and $-\chi_{z_i}$ for $i = 1, 2$. For the first case, there is no subgraph whose degree sequence is $d_M + \chi_v - \chi_u = d_{E(G)} - 2\chi_u$. For the second case, the only subgraph whose degree sequence is $d_M + \chi_v - \chi_{z_i}$ is $E(G) \setminus \{(u, z_i)\}$, which contains an H-subgraph $G - z_i$. Thus, there is no $(d_M + s_1, d_N)$-increment s_2 such that $d_M + s_1 + s_2$ is a degree sequence of an H-free subgraph.

Suppose that H is a complete bipartite graph $K_{a,b}$ with $a, b > 1$. Let $G = (V, E)$ be the graph defined by

$$V = U_1 \cup U_2 \cup V_1 \cup V_2 \cup \{u\} \cup \{v\},$$
$$E = \{(u', v') \mid u' \in U_1 \cup \{u\}, v' \in V_1 \cup \{v\}\} \cup \{(u', v') \mid u' \in U_2 \cup \{u\}, v' \in V_2 \cup \{v\}\},$$

where U_1, U_2, V_1, V_2, $\{u\}$ and $\{v\}$ are disjoint vertex set, $|U_1| = |U_2| = a - 1$, and $|V_1| = |V_2| = b - 1$. Let $M = E \setminus \{(u, v)\}$ and $N = \delta(u) \cup \delta(v)$. Then, M and N are clearly H-free subgraphs and $s_1 = \chi_u$ is a (d_M, d_N)-increment. We have the following possibilities for $(d_M + s_1, d_N)$-increments: χ_v and $-\chi_z$ for $z \in U_1 \cup U_2$. For the first case, $d_M + \chi_u + \chi_v$ is the degree sequence of G, which contains a $K_{a,b}$-subgraph. For the second case, $d_M + \chi_u - \chi_z$ is the degree sequence of $G - (v, z)$, in which the vertex set $U_1 \cup V_1 \cup \{u, v\}$ or $U_2 \cup V_2 \cup \{u, v\}$ induces a $K_{a,b}$-subgraph.

If H is a star with $k+1$ vertices, then the H-free subgraphs are exactly the subgraphs with degree at most $k - 1$. The degree sequences of these form a jump system in an arbitrary graph G as we described in Section 2. $\square$

4 Weighted t-matchings and M-concave functions

4.1 A main result on the weighted problem

In this section, we give a proof of Theorem 1.5, which shows the relationship between the weighted $K_{t,t}$-free t-matching problem in bipartite graphs and M-concave functions.

An *M-concave* (*M-convex*) *function on a constant-parity jump system* is a quantitative extension of a jump system, which is a generalization of valuated matroids [11, 13], valuated delta-matroids [12], and M-concave (M-convex) functions on base polyhedra [28, 29].

Definition 4.1 (M-concave function on a constant-parity jump system [30])**.** For $J \subseteq \mathbf{Z}^V$, we call $f : J \to \mathbf{R}$ an *M-concave function on a constant-parity jump system* if it satisfies the following exchange axiom:

Axiom 2. For any $x, y \in J$ and for any (x, y)-increment s_1, there exists an $(x+s_1, y)$-increment s_2 such that $x + s_1 + s_2 \in J$, $y - s_1 - s_2 \in J$, and $f(x) + f(y) \leq f(x + s_1 + s_2) + f(y - s_1 - s_2)$.

It follows from Axiom 2 that J is a constant-parity jump system (see [30]). We call a function $f : J \to \mathbf{R}$ an *M-convex function* if $-f$ is an M-concave function on a constant-parity jump system. M-concave functions on constant-parity jump systems appear in many combinatorial optimization problems such as the weighted matching problem [30], the minsquare factor problem [1], and the weighted even factor problem in odd-cycle-symmetric digraphs [7, 8, 25]. Some properties of M-concave functions are investigated in [23, 24], and efficient algorithms for maximizing an M-concave function on a constant-parity jump system are given in [31, 36].

Recall that, for a weighted bipartite graph (G, w), $J_{t,t}(G)$ is the set of degree sequences of all $K_{t,t}$-free t-matching in G and a function $f_{t,t}$ on $J_{t,t}(G)$ is defined by

$$f_{t,t}(x) = \max \left\{ \sum_{e \in M} w(e) \;\middle|\; M \text{ is a } K_{t,t}\text{-free } t\text{-matching, } d_M = x \right\}.$$

We restate Theorem 1.5 here.

Theorem 1.5. For a weighted bipartite graph (G, w) and an integer $t \geq 2$, $f_{t,t}$ is an M-concave function on the constant-parity jump system $J_{t,t}(G)$ if and only if w is vertex-induced on every $K_{t,t}$ in G.

Our proof of Theorem 1.5 consists of three parts: the necessity for $t \geq 2$ (Proposition 4.2), the sufficiency for $t \geq 3$ (Proposition 4.3), and the sufficiency for $t = 2$ (Proposition 4.14).

4.2 Necessity

This subsection is devoted to proving the necessity in Theorem 1.5.

Proposition 4.2. *For a weighted bipartite graph (G, w) and for an integer $t \geq 2$, if $f_{t,t}$ is an M-concave function on the constant-parity jump system $J_{t,t}(G)$, then w is vertex-induced on every $K_{t,t}$ in G.*

Proof. Let H be a $K_{t,t}$-subgraph in G such that $V(H) = \{u_1, \ldots, u_t, v_1, \ldots, v_t\}$ and $E(H) = \{(u_i, v_j) \mid i, j = 1, \ldots, t\}$. Denote $d_{E(H)\setminus\{(u_1,v_1)\}} \in J_{t,t}(G)$ by x, and $d_{E(H)\setminus\{(u_i,v_j)\}} \in J_{t,t}(G)$ by y for some $i, j \in \{2, \ldots, t\}$. Then, $M = E(H) \setminus \{(u_1, v_1)\}$ and $N = E(H) \setminus \{(u_i, v_j)\}$ are the unique edge sets such that $d_M = x$ and $d_N = y$, and hence $f_{t,t}(x) = w(M)$ and $f_{t,t}(y) = w(N)$.

For an (x, y)-increment $s_1 = \chi_{u_1}$, one can see that $s_2 = -\chi_{u_i}$ is the only $(x + s_1, y)$-increment such that $x + s_1 + s_2 \in J_{t,t}(G)$ and $y - s_1 - s_2 \in J_{t,t}(G)$. Since $M' = E(H) \setminus \{(u_i, v_1)\}$ and $N' = E(H) \setminus \{(u_1, v_j)\}$ are the unique edge sets such that achieve the degree sequences $x + s_1 + s_2$ and $y - s_1 - s_2$, respectively, we have that $f_{t,t}(x+s_1+s_2) = w(M')$ and $f_{t,t}(y-s_1-s_2) = w(N')$. If $f_{t,t}$ is an M-concave function on $J_{t,t}(G)$, by Axiom 2, we have $w(M) + w(N) \leq w(M') + w(N')$, which means that

$$w(u_1, v_1) + w(u_i, v_j) \geq w(u_i, v_1) + w(u_1, v_j). \tag{1}$$

A similar argument shows that

$$w(u_i, v_1) + w(u_1, v_j) \geq w(u_1, v_1) + w(u_i, v_j). \tag{2}$$

By (1) and (2), we have

$$w(u_1, v_1) + w(u_i, v_j) = w(u_i, v_1) + w(u_1, v_j). \tag{3}$$

Note that this equality is obvious when $i = 1$ or $j = 1$.

Define a function $p : V(H) \to \mathbf{R}$ by

$$p(u_i) = w(u_i, v_1), \quad p(v_j) = w(u_1, v_j) - w(u_1, v_1)$$

for $i, j = 1, \ldots, t$. Then, $w(u_i, v_j) = p(u_i) + p(v_j)$ holds for any $i, j \in \{1, \ldots, t\}$ by (3), which shows that w is induced by p on H. □

4.3 Sufficiency for the case of $t \geq 3$

In this subsection, we show the sufficiency for the case of $t \geq 3$ in Theorem 1.5.

Proposition 4.3. *Let $t \geq 3$ be an integer and $G = (V_1, V_2; E)$ be a weighted bipartite graph with a weight function w. If w is vertex-induced on every $K_{t,t}$ in G, then $f_{t,t}$ is an M-concave function on the constant-parity jump system $J_{t,t}(G)$.*

We prove Proposition 4.3 by presenting an algorithm for finding an $(x + s_1, y)$-increment s_2 satisfying Axiom 2 for given $x, y \in J_{t,t}(G)$ and (x, y)-increment s_1. In what follows, we consider the case where $s_1 = -\chi_v$ with $v \in V_1$. The other cases can be dealt with in a similar way.

4.3.1 Properties of triples

Our algorithm to find an $(x + s_1, y)$-increment keeps a triple (M, N, u) of $M, N \subseteq E$ and $u \in V_1 \cup V_2$ satisfying a certain condition. The purpose of this subsection is to define this condition and to show some properties of the triples. Note that the definitions in this subsection make sense only for the case where $s_1 = -\chi_v$ with $v \in V_1$.

Definition 4.4. For two edge sets $M, N \subseteq E$ and for a vertex $u \in V_1 \cup V_2$, the *semi-degree* of (M, N, u) is a pair (x', y') of vectors in $\mathbf{Z}^{V_1 \cup V_2}$ such that

- $x' = d_M - \chi_u$ and $y' = d_N + \chi_u$ if $u \in V_1$,
- $x' = d_M + \chi_u$ and $y' = d_N - \chi_u$ if $u \in V_2$.

For an integer t and vectors $x', y' \in \{0, 1, \dots, t\}^{V_1 \cup V_2}$, we define

$$\mathcal{T}_t(x', y') = \{(M, N, u) \mid M, N \subseteq E,\ u \in V_1 \cup V_2,\ M \text{ and } N \text{ are } K_{t,t}\text{-free}, \\ \text{the semi-degree of } (M, N, u) \text{ is } (x', y')\}.$$

Definition 4.5. For $(M_1, N_1, u_1), (M_2, N_2, u_2) \in \mathcal{T}_t(x', y')$, we say that (M_1, N_1, u_1) is *adjacent* to (M_2, N_2, u_2) if they satisfy one of the following conditions:

- $u_1 \in V_1$, $(u_1, u_2) \in M_1 \setminus N_1$, $M_2 = M_1 \setminus \{(u_1, u_2)\}$, and $N_2 = N_1 \cup \{(u_1, u_2)\}$.
- $u_1 \in V_2$, $(u_1, u_2) \in N_1 \setminus M_1$, $M_2 = M_1 \cup \{(u_1, u_2)\}$, and $N_2 = N_1 \setminus \{(u_1, u_2)\}$.

It is obvious that if (M_1, N_1, u_1) is adjacent to (M_2, N_2, u_2), then (M_2, N_2, u_2) is adjacent to (M_1, N_1, u_1).

We say that $(M, N, u) \in \mathcal{T}_t(x', y')$ is *active*, if $u \in V_1$ and $d_M(u) > d_N(u)$, or $u \in V_2$ and $d_M(u) < d_N(u)$. A triple $(M, N, u) \in \mathcal{T}_t(x', y')$ is *stable* if $u \in V_1$ and $d_M(u) \leq d_N(u) + 1$, or $u \in V_2$ and $d_N(u) \leq d_M(u) + 1$. Note that if $(M, N, u) \in \mathcal{T}_t(x', y')$ is stable, then M and N are t-matchings. We also note that if $(M, N, u) \in \mathcal{T}_t(x, y)$ is not stable, then it is active. The definition of stable triples means that χ_u or $-\chi_u$, say s_2, is an (x', y')-increment such that $d_M = x' + s_2$ and $d_N = y' - s_2$ (see Claim 4.13). Hence, our algorithm stops when we find a stable triple.

We now show some properties of the triples, which will be used in our algorithm.

Lemma 4.6. *Let (M, N, u_1) be a triple in $\mathcal{T}_t(x', y')$, $u_1 \in V_1$, and $e = (u_1, u_2) \in M \setminus N$. If $N \cup \{e\}$ is $K_{t,t}$-free, then $(M \setminus \{e\}, N \cup \{e\}, u_2)$ is in $\mathcal{T}_t(x', y')$ and adjacent to (M, N, u_1).*

Proof. The semi-degree of $(M \setminus \{e\}, N \cup \{e\}, u_2)$ is

$$(d_{M \setminus \{e\}} + \chi_{u_2}, d_{N \cup \{e\}} - \chi_{u_2}) = (d_M - \chi_{u_1}, d_N + \chi_{u_1}) = (x', y'),$$

which means $(M \setminus \{e\}, N \cup \{e\}, u_2) \in \mathcal{T}_t(x', y')$. It is obvious that $(M \setminus \{e\}, N \cup \{e\}, u_2)$ and (M, N, u_1) are adjacent by the definition. □

Lemma 4.7. *Let (M, N, u) be a triple in $\mathcal{T}_t(x', y')$, $u \in V_1$, and $d_M(u) - d_N(u) \geq 2$. Then, one of the following conditions holds:*

- *(M, N, u) is adjacent to at least two triples in $\mathcal{T}_t(x', y')$.*
- *(M, N, u) is adjacent to exactly one triple in $\mathcal{T}_t(x', y')$ and there exists a $K_{t,t}$-subgraph H containing u in G such that $\delta(u) \cap E(H) \subseteq M$ and $E(H) \setminus \{(u, v)\} \subseteq N$ for some $v \in V(H)$.*

Proof. Suppose that (M, N, u) is adjacent to at most one triple in $\mathcal{T}_t(x', y')$. Since $|(M \cap \delta(u)) \setminus N| \geq d_M(u) - d_N(u) \geq 2$, by Lemma 4.6, we have that $|(M \cap \delta(u)) \setminus N| = 2$ and $N \cup \{e\}$ has a $K_{t,t}$ for one edge $e \in (M \cap \delta(u)) \setminus N$. Note that, since $d_N(u) \leq t-1$, there exists at most one edge $e \in \delta(u)$ such that $N \cup \{e\}$ contains a $K_{t,t}$.

Therefore, there exists a $K_{t,t}$-subgraph H containing u such that $E(H) \setminus \{(u, v)\} \subseteq N$ for some $v \in V(H)$. To the end, since $d_M(u) - d_N(u) = |(M \cap \delta(u)) \setminus N| = 2$, we have that $\delta(u) \cap N \subseteq M$, and hence $\delta(u) \cap E(H) = (\delta(u) \cap N) \cup \{e\} \subseteq M$. □

The following lemma can be proved similarly.

Lemma 4.8. *Suppose that* (M, N, u) *is a triple in* $\mathcal{T}_t(x', y')$*,* $u \in V_1$*, and* $d_M(u) - d_N(u) \geq 1$*. Then, one of the following conditions holds:*

- (M, N, u) *is adjacent to at least one triple in* $\mathcal{T}_t(x', y')$*.*
- (M, N, u) *is adjacent to no triple in* $\mathcal{T}_t(x', y')$ *and there exists a* $K_{t,t}$*-subgraph* H *containing* u *in* G *such that* $\delta(u) \cap E(H) \subseteq M$ *and* $E(H) \setminus \{(u, v)\} \subseteq N$ *for some* $v \in V(H)$*.*

4.3.2 Updating a triple

In this subsection, we consider a procedure of updating a given triple, which is a subroutine of our main algorithm. Roughly speaking, when a triple (M, N, u) is given as the input, this procedure increases $w(M) + w(N)$ and decreases $|M \cup N|$, maintaining its semi-degree. The procedure is described as follows.

Procedure A

Input. An integer $t \geq 3$, a bipartite graph $G = (V_1, V_2; E)$ with a weight function w that is vertex-induced on every $K_{t,t}$, vectors $x', y' \in \{0, 1, \ldots, t\}^{V_1 \cup V_2}$, and an active triple $(M, N, u) \in \mathcal{T} := \mathcal{T}_t(x', y')$.

Output. A triple $(M^*, N^*, u^*) \in \mathcal{T}$ satisfying one of the following:

1. $w(M^*) + w(N^*) > w(M) + w(N)$.
2. $w(M^*) + w(N^*) = w(M) + w(N)$ and $|M^* \cup N^*| < |M \cup N|$.
3. $w(M^*) + w(N^*) = w(M) + w(N)$, $|M^* \cup N^*| = |M \cup N|$, and (M^*, N^*, u^*) is stable.

Step 0. Set $\tau := 0$, $M^{(0)} := M$, $N^{(0)} := N$, and $u^{(0)} := u$. Then, go to Step 1.

Step 1. If $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ has an adjacent triple $(M', N', u') \in \mathcal{T}$ which is different from $(M^{(\tau-1)}, N^{(\tau-1)}, u^{(\tau-1)})$ (we ignore this condition if $\tau = 0$), then set $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)}) := (M', N', u')$ and $\tau := \tau + 1$, and go to Step 2. Otherwise, go to Step 4.

Step 2. If $u^{(\tau)} = u^{(\tau')}$ for some $\tau' < \tau$, then execute one of the following:

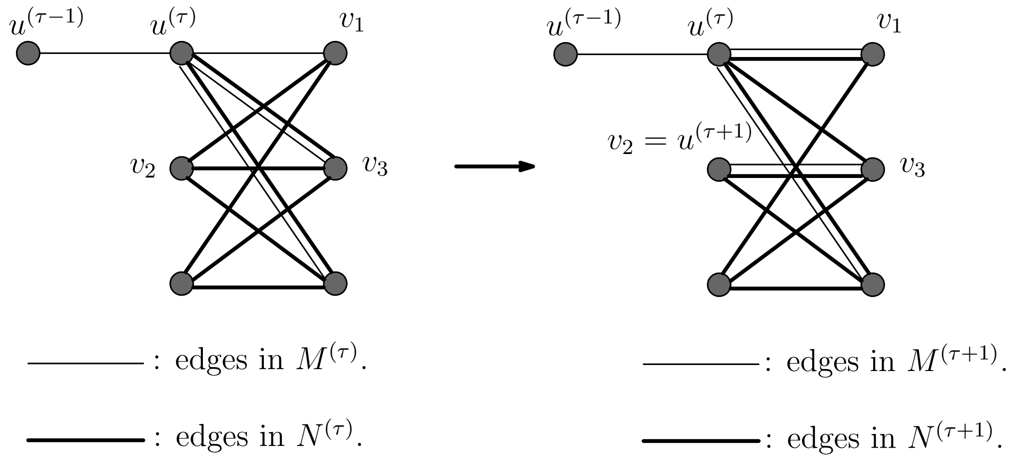

Figure 1: Definitions of $M^{(\tau+1)}$, $N^{(\tau+1)}$, and $u^{(\tau+1)}$.

- If $w(M^{(\tau')}) > w(M^{(\tau)})$, then output $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure.
- If $w(M^{(\tau')}) < w(M^{(\tau)})$, then output $(M^{(\tau)}, N^{(\tau')}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure.
- If $w(M^{(\tau')}) = w(M^{(\tau)})$, then either $|M^{(\tau')} \cup N^{(\tau)}| < |M^{(\tau')} \cup N^{(\tau')}|$ or $|M^{(\tau)} \cup N^{(\tau')}| < |M^{(\tau')} \cup N^{(\tau')}|$ holds (see Claim 4.10). In the former case, output $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure. In the latter case, output $(M^{(\tau)}, N^{(\tau')}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure.

Otherwise, go to Step 3.

Step 3. If $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ is a stable triple, then output $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure. Otherwise, go to Step 1.

Step 4. If $u^{(\tau)} \in V_1$, then execute Step 4-1. Otherwise, execute Step 4-2.

Step 4-1. If $\tau \geq 1$, then $d_{M^{(\tau)}}(u^{(\tau)}) - d_{N^{(\tau)}}(u^{(\tau)}) \geq 2$, because $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ is not stable by Step 3. If $\tau = 0$, then $d_{M^{(\tau)}}(u^{(\tau)}) - d_{N^{(\tau)}}(u^{(\tau)}) \geq 1$ by the activeness of the input. Therefore, by Lemmas 4.7 and 4.8, there exists a $K_{t,t}$-subgraph H containing $u^{(\tau)}$ in G such that $\delta(u^{(\tau)}) \cap E(H) \subseteq M^{(\tau)}$ and $E(H) \setminus \{(u^{(\tau)}, v_1)\} \subseteq N^{(\tau)}$ for some $v_1 \in V(H)$.

Then, there exists an edge $(v_2, v_3) \in E(H) \setminus M^{(\tau)}$ such that $v_2 \in V_1$, $v_3 \in V_2$ (possibly $v_3 = v_1$), and $(M^{(\tau)} \setminus \{(u^{(\tau)}, v_3)\}) \cup \{(v_2, v_3)\}$ contains no $K_{t,t}$ (see Claim 4.11). As shown in Figure 1, we define

$$\begin{aligned} M^{(\tau+1)} &:= (M^{(\tau)} \setminus \{(u^{(\tau)}, v_3)\}) \cup \{(v_2, v_3)\}, \\ N^{(\tau+1)} &:= (N^{(\tau)} \setminus \{(v_1, v_2)\}) \cup \{(u^{(\tau)}, v_1)\}, \\ u^{(\tau+1)} &:= v_2. \end{aligned}$$

Then, $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)}) \in \mathcal{T}$ (see Claim 4.12). Set $\tau := \tau + 1$, and go to Step 2.

Step 4-2. Execute a similar procedure to Step 4-1 by switching $M^{(\tau)}$ and $N^{(\tau)}$.

If $u^{(\tau_1)} = u^{(\tau_2)}$ for distinct τ_1 and τ_2, then Procedure A stops in Step 2, which assures that each step is executed at most $|V_1| + |V_2|$ times. We now show the correctness of the procedure. First, we can easily show the following claim.

Claim 4.9. *In Steps 1 and 4,* $w(M^{(\tau+1)}) + w(N^{(\tau+1)}) = w(M^{(\tau)}) + w(N^{(\tau)})$ *and* $M^{(\tau+1)} \cup N^{(\tau+1)} \subseteq M^{(\tau)} \cup N^{(\tau)}$.

Proof. Since w is vertex-induced on every $K_{t,t}$,

$$w(u^{(\tau)}, v_3) + w(v_1, v_2) = w(v_2, v_3) + w(u^{(\tau)}, v_1)$$

in Step 4, which shows that $w(M^{(\tau)}) + w(N^{(\tau)}) = w(M^{(\tau+1)}) + w(N^{(\tau+1)})$ holds in Step 4. The other parts are obvious. □

By this claim, if Procedure A outputs a stable triple $(M^*, N^*, u^*) \in \mathcal{T}$ in Step 3, then $w(M^*) + w(N^*) = w(M) + w(N)$ and $|M^* \cup N^*| \leq |M \cup N|$, which shows that (M^*, N^*, u^*) is a desired output. Similarly, we can see that the output in Step 2 is also a desired triple.

The correctness of Steps 2 and 4 of Procedure A is guaranteed by the following claims.

Claim 4.10. *If* $u^{(\tau_1)} = u^{(\tau_2)}$ *for some* $\tau_2 < \tau_1$ *and* $u^{(\tau_2)}, u^{(\tau_2+1)}, \ldots, u^{(\tau_1-1)}$ *are distinct, then either* $|M^{(\tau_2)} \cup N^{(\tau_1)}| < |M^{(\tau_1)} \cup N^{(\tau_1)}|$ *or* $|M^{(\tau_1)} \cup N^{(\tau_2)}| < |M^{(\tau_1)} \cup N^{(\tau_1)}|$ *holds.*

Proof. Suppose we update a triple in Step 1 when $\tau = \tau_3$, and let $e_3 = (u^{(\tau_3)}, u^{(\tau_3+1)})$. Since we update a triple in Step 1 at least twice while $\tau_2 \leq \tau \leq \tau_1 - 1$, we may assume that $u^{(\tau_3)} \neq u^{(\tau_1)}$ and $u^{(\tau_3+1)} \neq u^{(\tau_1)}$ by choosing appropriate τ_3.

Now we observe that, for an edge e, if we update $\{e\} \cap M^{(\tau)}$ or $\{e\} \cap N^{(\tau)}$ in Step 4, i.e., $\{e\} \cap M^{(\tau)} \neq \{e\} \cap M^{(\tau+1)}$ or $\{e\} \cap N^{(\tau)} \neq \{e\} \cap N^{(\tau+1)}$, then $u^{(\tau)}$ or $u^{(\tau+1)}$ is an end vertex of e. Since $u^{(\tau_2)}, u^{(\tau_2+1)}, \ldots, u^{(\tau_1-1)}$ are distinct, by the above observation, $\{e_3\} \cap M^{(\tau)}$ is updated only when $\tau = \tau_3$. Thus, either $e_3 \in (M^{(\tau_1)} \setminus N^{(\tau_1)}) \cap (N^{(\tau_2)} \setminus M^{(\tau_2)})$ or $e_3 \in (N^{(\tau_1)} \setminus M^{(\tau_1)}) \cap (M^{(\tau_2)} \setminus N^{(\tau_2)})$.

In the former case, $e_3 \notin M^{(\tau_2)} \cup N^{(\tau_1)}$ and $M^{(\tau_2)} \cup N^{(\tau_1)} \subseteq M^{(\tau_1)} \cup N^{(\tau_1)}$, which implies $|M^{(\tau_2)} \cup N^{(\tau_1)}| < |M^{(\tau_1)} \cup N^{(\tau_1)}|$. Similarly, we have $|M^{(\tau_1)} \cup N^{(\tau_2)}| < |M^{(\tau_1)} \cup N^{(\tau_1)}|$ in the latter case. This completes the proof. □

Claim 4.11. *In Step 4-1, there exists an edge* $(v_2, v_3) \in E(H) \setminus M^{(\tau)}$ *such that* $v_2 \in V_1$, $v_3 \in V_2$ *(possibly* $v_3 = v_1$*), and* $(M^{(\tau)} \setminus \{(u^{(\tau)}, v_3)\}) \cup \{(v_2, v_3)\}$ *contains no* $K_{t,t}$.

Proof. Since $M^{(\tau)}$ does not contain H, one of the following holds:

Case 1. There exists a vertex $v_2 \in V_1 \setminus \{u^{(\tau)}\}$ such that $|\delta(v_2) \cap (E(H) \setminus M^{(\tau)})| \geq 2$.

Case 2. There exists a vertex $v_2 \in V_1 \setminus \{u^{(\tau)}\}$ such that $|\delta(v_2) \cap (E(H) \setminus M^{(\tau)})| = 1$.

We consider these cases separately.

Case 1. Let $e_1 = (v_2, z_1), e_2 = (v_2, z_2)$ be edges in $\delta(v_2) \cap (E(H) \setminus M^{(\tau)})$. We show that at least one of e_1 and e_2 satisfies the condition. To the contrary, we assume that $M^{(\tau)} \cup \{e_i\}$ contains a $K_{t,t}$-subgraph H_i for $i = 1, 2$. Since $|\delta(v_2) \cap (M^{(\tau)} \cup \{e_1, e_2\})| \leq t + 2$ and $t \geq 3$, there exists a vertex $z \in V_2 \setminus \{z_1, z_2\}$ contained in both H_1 and H_2. Then, all the t edges in $\delta(z) \cap M^{(\tau)}$ are contained in both H_1 and H_2, and hence $V(H_1) \cap V_1 = V(H_2) \cap V_1$. This means that there exists a vertex $z' \in V_1 \setminus \{u^{(\tau)}, v_2\}$ contained in both H_1 and H_2. Since all the t edges in $\delta(z') \cap M^{(\tau)}$ are contained in both H_1 and H_2, we have $V(H_1) \cap V_2 = V(H_2) \cap V_2$, which is a contradiction.

Case 2. Let $e_1 = (v_2, z_1)$ be the edge in $\delta(v_2) \cap (E(H) \setminus M^{(\tau)})$. In order to show that e_1 satisfies the condition, we assume that $(M^{(\tau)} \setminus \{(u^{(\tau)}, z_1)\}) \cup \{e_1\}$ contains a $K_{t,t}$-subgraph H_1 to derive a contradiction. Since $|(\delta(v_2) \cap M^{(\tau)}) \setminus E(H)| = |\delta(v_2) \cap M^{(\tau)}| - (t-1) \leq 1$ and $t \geq 3$, there exists a vertex $z \in (V(H) \cap V_2) \setminus \{z_1\}$ contained in H_1. Then, all the t edges in $\delta(z) \cap M^{(\tau)}$ are contained in H_1, which implies that $(u^{(\tau)}, z)$ is contained in H_1. On the other hand, since $M^{(\tau)}$ contains no $K_{t,t}$, H_1 contains $e_1 = (v_2, z_1)$. This contradicts that H_1 does not contain $(u^{(\tau)}, z_1)$. □

Claim 4.12. *In Step 4-1,* $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)}) \in \mathcal{T}$.

Proof. By the definition, $M^{(\tau+1)}$ contains no $K_{t,t}$. Since $N^{(\tau)}$ is $K_{t,t}$-free, H is an unique $K_{t,t}$ contained in $N^{(\tau)} \cup \{(u^{(\tau)}, v_1)\}$, and hence $N^{(\tau+1)} = (N^{(\tau)} \setminus \{(v_1, v_2)\}) \cup \{(u^{(\tau)}, v_1)\}$ contains no $K_{t,t}$. It is obvious that $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)})$ satisfies the degree constraints, which completes the proof. □

Note that we can show the correctness of Step 4-2 in the same way. The above claims show the correctness of Procedure A.

4.3.3 A main algorithm

In this subsection, we give an algorithm for finding an $(x + s_1, y - s_1)$-increment s_2 using Procedure A. The algorithm is described as follows.

Algorithm FIND-INCREMENT

Input. An integer $t \geq 3$, a bipartite graph $G = (V_1, V_2; E)$ with a weight function w that is vertex-induced on every $K_{t,t}$, $K_{t,t}$-free t-matchings M and N in G with $d_M = x$ and $d_N = y$, and an (x, y)-increment $s_1 = -\chi_u$ with $u \in V_1$.

Output. An $(x + s_1, y - s_1)$-increment s_2 and $K_{t,t}$-free t-matchings M' and N' in G such that $d_{M'} = x + s_1 + s_2$, $d_{N'} = y - s_1 - s_2$, and $w(M') + w(N') \geq w(M) + w(N)$.

Step 1. Execute Procedure A for $(M, N, u) \in \mathcal{T}_t(x + s, y - s)$ to obtain a triple $(M^*, N^*, u^*) \in \mathcal{T}_t(x + s, y - s)$ satisfying one of the following:

1. $w(M^*) + w(N^*) > w(M) + w(N)$.

2. $w(M^*) + w(N^*) = w(M) + w(N)$ and $|M^* \cup N^*| < |M \cup N|$.

3. $w(M^*) + w(N^*) = w(M) + w(N)$, $|M^* \cup N^*| = |M \cup N|$, and (M^*, N^*, u^*) is stable.

Then, go to Step 2.

Step 2. If (M^*, N^*, u^*) is stable, then output $s_2 := d_{M^*} - x - s_1$, $M' := M^*$, and $N' := N^*$, and stop the algorithm. Otherwise, update M, N, and u as $M := M^*$, $N := N^*$, and $u := u^*$, and go to Step 1.

In the algorithm, $(w(M) + w(N))\ell - |M \cup N|$ increases monotonically, where ℓ is a sufficiently large number, and this shows that FIND-INCREMENT terminates in finite steps. The correctness of FIND-INCREMENT is guaranteed by the following claim.

Claim 4.13. *If we obtain s_2 in Step 2 of FIND-INCREMENT, then $s_2 = d_{M^*} - x - s_1$ is an $(x + s_1, y - s_1)$-increment.*

Proof. Suppose that $u^* \in V_1$. Then, $d_{M^*} - x - s_1 = -d_{N^*} + y - s_1 = \chi_{u^*}$ by the definition of $\mathcal{T}_t(x + s_1, y - s_1)$. Since $d_{M^*}(u^*) - d_{N^*}(u^*) \leq 1$ by the definition of a stable triple, $(y - s_1)(u^*) - (x + s_1)(u^*) = (d_{N^*}(u^*) + 1) - (d_{M^*}(u^*) - 1) \geq 1$, which means that $s_2 = \chi_{u^*}$ is an $(x + s_1, y - s_1)$-increment. We can deal with the case when $u^* \in V_2$ in the same way. □

Now we show how we obtain an $(x + s_1, y)$-increment s_2, and prove Proposition 4.3.

Proof for Proposition 4.3. For $x, y \in J_{t,t}(G)$ and an (x, y)-increment s_1, we choose $K_{t,t}$-free t-matchings M and N in G such that $d_M = x$, $d_N = y$, $w(M) = f_{t,t}(x)$, and $w(N) = f_{t,t}(y)$. Furthermore, we choose M and N minimizing $|M \cup N|$.

By executing FIND-INCREMENT, we find $K_{t,t}$-free t-matchings M' and N' and an $(x + s_1, y - s_1)$-increment s_2 that satisfy $d_{M'} = x + s_1 + s_2$, $d_{N'} = y - s_1 - s_2$, and $w(M') + w(N') \geq w(M) + w(N)$. We now prove that the output s_2 of FIND-INCREMENT is an $(x + s_1, y)$-increment, that is, $d_{M'} \neq x$, $d_{N'} \neq y$, when we assume the minimality of $|M \cup N|$.

If Procedure A outputs $(M^*, N^*, u^*) \in \mathcal{T}_t(x + s, y - s)$ with $w(M^*) + w(N^*) > w(M) + w(N)$ or $|M^* \cup N^*| < |M \cup N|$ at least once in Step 1 of FIND-INCREMENT, then the output (M', N') of Algorithm FIND-INCREMENT satisfies that either

- $w(M') + w(N') > w(M) + w(N)$ or
- $w(M') + w(N') = w(M) + w(N)$ and $|M' \cup N'| < |M \cup N|$,

which implies $d_{M'} \neq d_M$ and $d_{N'} \neq d_N$ by the minimality of $|M \cup N|$.

Otherwise, when we execute Procedure A for the first time, it outputs a stable triple $(M^*, N^*, u^*) \in \mathcal{T}_t(x + s_1, y - s_1)$ with $w(M^*) + w(N^*) = w(M) + w(N)$ and $|M^* \cup N^*| = |M \cup N|$. Therefore, (M^*, N^*, u^*) is not outputted in Step 2 of Procedure A, and hence $u^* \neq u$, which means $s_2 \neq -s_1$.

By the above arguments, we obtain new $K_{t,t}$-free t-matchings M' and N' and an $(x + s_1, y)$-increment s_2 that satisfy $d_{M'} = x + s_1 + s_2$, $d_{N'} = y - s_1 - s_2$, and $w(M) + w(N) \leq w(M') + w(N')$. Then, we have

$$\begin{aligned} f_{t,t}(x) + f_{t,t}(y) &= w(M) + w(N) \\ &\leq w(M') + w(N') \\ &\leq f_{t,t}(x + s_1 + s_2) + f_{t,t}(y - s_1 - s_2). \end{aligned}$$

Hence $f_{t,t}$ is an M-concave function on $J_{t,t}(G)$. □

4.4 Sufficiency for the case of $t = 2$

Next, we show the sufficiency for the case of $t = 2$ in Theorem 1.5.

Proposition 4.14. *For a bipartite graph $G = (V_1, V_2; E)$ with a weight function w, if w is vertex-induced on every C_4 in G, then $f_{2,2}$ is an M-concave function on the constant-parity jump system $J_{2,2}(G)$.*

In the same way as Proposition 4.3, we prove Proposition 4.14 by presenting an algorithm for finding an $(x + s_1, y)$-increment s_2 satisfying Axiom 2 for given $x, y \in J_{2,2}(G)$ and (x, y)-increment s_1. In what follows, we consider the case where $s_1 = -\chi_u$ with $u \in V_1$.

4.4.1 $\mathcal{S}$-squares

We call a C_4 in a bipartite graph as a *square*. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_p\}$ be a set of vertex-disjoint squares in G. We say that a square S is an *$\mathcal{S}$-square* if $S \in \mathcal{S}$ or S is vertex-disjoint from every member of $\mathcal{S}$. We say that an edge set is *$\mathcal{S}$-square-free* if it contains no $\mathcal{S}$-square.

In the same way as $\mathcal{T}_t(x', y')$ in Section 4.3.1, we define $\mathcal{T}_{\text{sq}}(\mathcal{S}, x', y')$ as follows. For a bipartite graph $G = (V_1, V_2; E)$, vectors $x', y' \in \{0, 1, 2\}^{V_1 \cup V_2}$, and a set $\mathcal{S}$ of vertex-disjoint squares, we define

$$\begin{aligned} \mathcal{T}_{\text{sq}}(\mathcal{S}, x', y') = \{(M, N, u) \mid\ & M, N \subseteq E,\ u \in V_1 \cup V_2,\ M \text{ and } N \text{ are } \mathcal{S}\text{-square-free}, \\ & \text{the semi-degree of } (M, N, u) \text{ is } (x', y')\}. \end{aligned}$$

We define *stable triples*, *active triples*, *adjacency of triples* in $\mathcal{T}_{\text{sq}}(\mathcal{S}, x', y')$ in the same way as those in $\mathcal{T}_t(x', y')$. Then, Lemmas 4.6, 4.7, and 4.8 can be modified to lemmas for $\mathcal{T}_{\text{sq}}(\mathcal{S}, x', y')$ by replacing $K_{t,t}$'s with $\mathcal{S}$-squares.

4.4.2 Updating a triple

In this subsection, we consider a procedure of updating a given triple in G, which modifies the procedure in Section 4.3.2. Note that we need a modification because Claim 4.11 does not hold when $t = 2$. In the procedure, we use edge sets M, N and a maximal set $\mathcal{S}$ of vertex-disjoint squares such that $E(S) \subseteq M \cup N$ and $|E(S) \cap M| = |E(S) \cap N| = 3$ for each $S \in \mathcal{S}$. In other words, M, N and $\mathcal{S}$ satisfy the following assumption.

Assumption 4.15. For an $\mathcal{S}$-square S, it holds that $S \in \mathcal{S}$ if and only if $E(S) \subseteq M \cup N$ and $|E(S) \cap M| = |E(S) \cap N| = 3$.

The modified procedure is described as follows.

Procedure B

Input. A bipartite graph $G = (V_1, V_2; E)$ with a weight function w that is vertex-induced on every square, vectors $x', y' \in \{0, 1, 2\}^{V_1 \cup V_2}$, a set $\mathcal{S}$ of vertex-disjoint squares, and an active triple $(M, N, u) \in \mathcal{T} := \mathcal{T}_{\mathrm{sq}}(\mathcal{S}, x', y')$ satisfying Assumption 4.15.

Output. A triple $(M^*, N^*, u^*) \in \mathcal{T}$ such that $E(S) \subseteq M \cup N$ and $|E(S) \cap M| = |E(S) \cap N| = 3$ for any $S \in \mathcal{S}$, and one of the following holds:

1. $w(M^*) + w(N^*) > w(M) + w(N)$.
2. $w(M^*) + w(N^*) = w(M) + w(N)$ and $|M^* \cup N^*| < |M \cup N|$.
3. $w(M^*) + w(N^*) = w(M) + w(N)$, $|M^* \cup N^*| = |M \cup N|$, and (M^*, N^*, u^*) is stable.

Step 0. Set $\tau := 0$, $M^{(0)} := M$, $N^{(0)} := N$, and $u^{(0)} := u$. Then, go to Step 1.

Step 1. If $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ has an adjacent triple $(M', N', u') \in \mathcal{T}$ which is different from $(M^{(\tau-1)}, N^{(\tau-1)}, u^{(\tau-1)})$ (we ignore this condition if $\tau = 0$), then set $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)}) := (M', N', u')$ and $\tau := \tau + 1$, and go to Step 2. Otherwise, go to Step 4.

Step 2. If $u^{(\tau)} = u^{(\tau')}$ for some $\tau' < \tau$, then execute one of the following:

- If $w(M^{(\tau')}) \geq w(M^{(\tau)})$, then output $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure.
- If $w(M^{(\tau')}) < w(M^{(\tau)})$, then output $(M^{(\tau)}, N^{(\tau')}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure.

Otherwise, go to Step 3.

Step 3. If $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ is a stable triple, then output $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{T}$ and stop the procedure. Otherwise, go to Step 1.

Step 4. If $u^{(\tau)} \in V_1$, then go to Step 5. Otherwise, execute a similar procedure to Step 5 by switching $M^{(\tau)}$ and $N^{(\tau)}$.

Step 5. If $\tau \geq 1$, then $d_{M^{(\tau)}}(u^{(\tau)}) - d_{N^{(\tau)}}(u^{(\tau)}) \geq 2$, because $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ is not stable by Step 3. On the other hand, if $\tau = 0$, then $d_{M^{(\tau)}}(u^{(\tau)}) - d_{N^{(\tau)}}(u^{(\tau)}) \geq 1$ by the activeness of the input. By the modifications of Lemmas 4.7 and 4.8, there exists an $\mathcal{S}$-square $S = (u^{(\tau)}, v_1, v_2, v_3)$ in G such that $\{(u^{(\tau)}, v_1), (u^{(\tau)}, v_3)\} \subseteq M^{(\tau)}$ and $\{(u^{(\tau)}, v_1), (v_1, v_2), (v_2, v_3)\} \subseteq N^{(\tau)}$.

If $S \in \mathcal{S}$, then go to Step 5-1. Otherwise, go to Step 5-2.

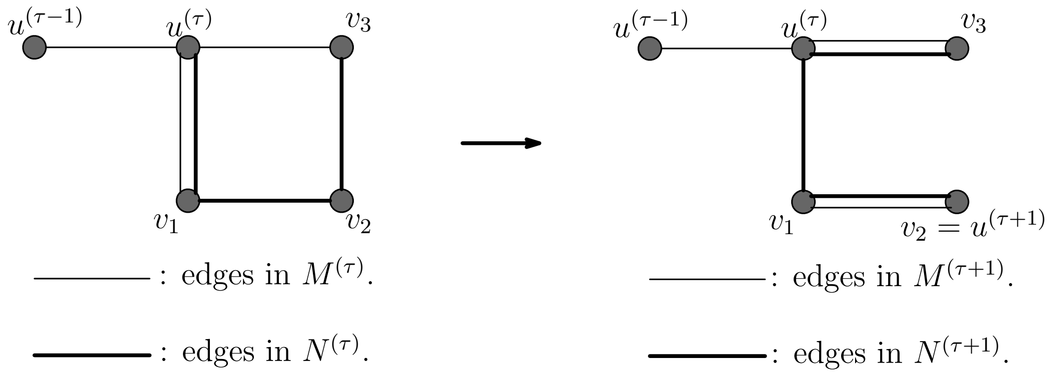

Figure 2: Definitions of $M^{(\tau+1)}$, $N^{(\tau+1)}$, and $u^{(\tau+1)}$.

Step 5-1. By Assumption 4.15, we have that $E(S) \cap N^{(\tau)} = E(S) \setminus \{(u^{(\tau)}, v_3)\}$ and $E(S) \cap M^{(\tau)} = E(S) \setminus \{(v_2, z)\}$, where z is either v_1 or v_3.

Define $M^{(\tau+1)}$, $N^{(\tau+1)}$, and $u^{(\tau+1)}$ by

$$\begin{aligned} M^{(\tau+1)} &:= (M^{(\tau)} \setminus \{(u^{(\tau)}, z)\}) \cup \{(v_2, z)\}, \\ N^{(\tau+1)} &:= (N^{(\tau)} \setminus \{(v_2, v_3)\}) \cup \{(u^{(\tau)}, v_3)\}, \\ u^{(\tau+1)} &:= v_2, \end{aligned}$$

and then go to Step 2.

Step 5-2. By Assumption 4.15, we have that $E(S) \cap M^{(\tau)} = \{(u^{(\tau)}, v_1), (u^{(\tau)}, v_3)\}$ and $E(S) \cap N^{(\tau)} = \{(u^{(\tau)}, v_1), (v_1, v_2), (v_2, v_3)\}$.

As shown in Figure 2, define $M^{(\tau+1)}$, $N^{(\tau+1)}$, and $u^{(\tau+1)}$ by

$$\begin{aligned} M^{(\tau+1)} &:= (M^{(\tau)} \setminus \{(u^{(\tau)}, v_1)\}) \cup \{(v_2, v_1)\}, \\ N^{(\tau+1)} &:= (N^{(\tau)} \setminus \{(v_2, v_3)\}) \cup \{(u^{(\tau)}, v_3)\}, \\ u^{(\tau+1)} &:= v_2. \end{aligned}$$

If $M^{(\tau+1)}$ is $\mathcal{S}$-square-free, then output a triple $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)}) \in \mathcal{T}$, which satisfies that $|M^{(\tau+1)} \cup N^{(\tau+1)}| = |M \cup N| - 1$ (see Claim 4.18), and stop the procedure. Otherwise, there exists an $\mathcal{S}$-square $S' = (v_2, v_1, v_4, v_5)$ in $M^{(\tau+1)}$, where $\{u^{(\tau)}, v_3\} \cap \{v_4, v_5\} = \emptyset$ (see Figure 3). Then define

$$\begin{aligned} M^{(\tau+2)} &:= M^{(\tau+1)} \setminus \{(v_2, v_5)\}, \\ N^{(\tau+2)} &:= N^{(\tau+1)} \cup \{(v_2, v_5)\}, \\ u^{(\tau+2)} &:= v_5. \end{aligned}$$

Output a triple $(M^{(\tau+2)}, N^{(\tau+2)}, u^{(\tau+2)}) \in \mathcal{T}$, which satisfies that $|M^{(\tau+2)} \cup N^{(\tau+2)}| = |M \cup N| - 1$ (see Claim 4.19), and stop the procedure.

If $u^{(\tau_1)} = u^{(\tau_2)}$ for distinct τ_1 and τ_2, then Procedure B stops in Step 2, which assures that each step is executed at most $|V_1| + |V_2|$ times. We now show the correctness of the procedure.

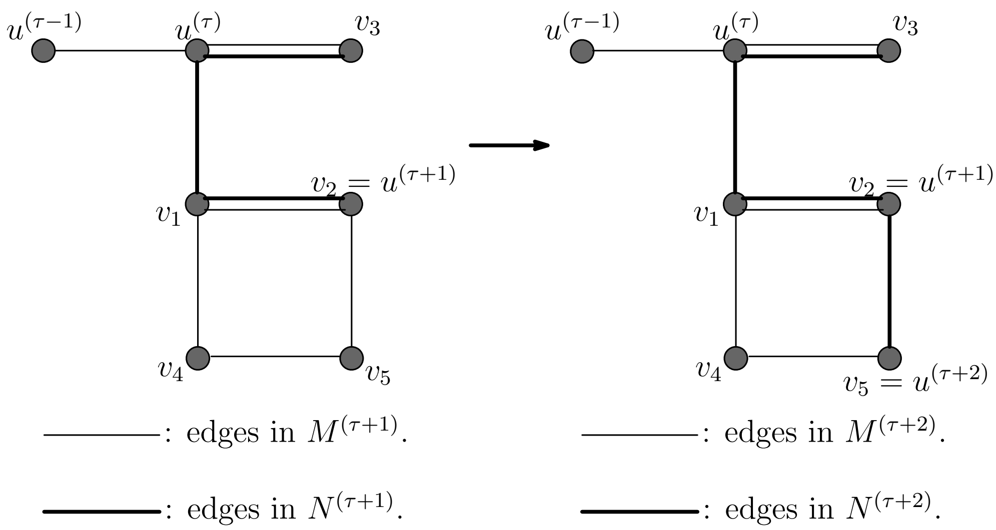

Figure 3: Definitions of $M^{(\tau+2)}$, $N^{(\tau+2)}$, and $u^{(\tau+2)}$.

First, since w is vertex-induced on every square, we can see the following in the same way as Claim 4.9.

Claim 4.16. *In Steps 1 and 5-1,* $w(M^{(\tau+1)}) + w(N^{(\tau+1)}) = w(M^{(\tau)}) + w(N^{(\tau)})$ *and* $M^{(\tau+1)} \cup N^{(\tau+1)} = M^{(\tau)} \cup N^{(\tau)}$.

By this claim, if Procedure B outputs a stable triple $(M^*, N^*, u^*) \in \mathcal{T}$ in Step 3, then $w(M^*) + w(N^*) = w(M) + w(N)$ and $|M^* \cup N^*| = |M \cup N|$, which shows that (M^*, N^*, u^*) is a desired output. To show that the output in Step 2 is also a desired triple, we prove the following claim.

Claim 4.17. *In Step 2,* $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)})$ *and* $(M^{(\tau)}, N^{(\tau')}, u^{(\tau)})$ *are both in* $\mathcal{T}$ *and* $|M^{(\tau')} \cup N^{(\tau)}| < |M \cup N|$.

Proof. As both $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ and $(M^{(\tau')}, N^{(\tau')}, u^{(\tau')})$ are in $\mathcal{T}$, both $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)})$ and $(M^{(\tau)}, N^{(\tau')}, u^{(\tau)})$ are in $\mathcal{T}$.

Since we update a triple in Step 1 at least once when $u^{(\tau)} \in V_1$, $(M^{(\tau)} \setminus N^{(\tau)}) \cap (N^{(\tau')} \setminus M^{(\tau')}) \neq \emptyset$. Thus, we have

$$\begin{aligned} |M^{(\tau')} \cup N^{(\tau)}| &= |M^{(\tau)} \cup N^{(\tau)}| - |(M^{(\tau)} \setminus M^{(\tau')}) \setminus N^{(\tau)}| + |(M^{(\tau')} \setminus M^{(\tau)}) \setminus N^{(\tau)}| \\ &= |M^{(\tau)} \cup N^{(\tau)}| - |(M^{(\tau)} \setminus M^{(\tau')}) \setminus N^{(\tau)}| \\ &< |M^{(\tau)} \cup N^{(\tau)}| \\ &= |M \cup N|. \end{aligned}$$

□

Next, in order to show the correctness of Step 5-2, we prove the following claims.

Claim 4.18. *In Step 5-2,* $N^{(\tau+1)}$ *is* $\mathcal{S}$*-square-free and* $|M^{(\tau+1)} \cup N^{(\tau+1)}| = |M \cup N| - 1$.

Proof. First, $(u^{(\tau)}, v_1)$ is the unique edge in $N^{(\tau)} \cap \delta(u^{(\tau)})$ and $N^{(\tau)} \cap \delta(v_1) = \{(u^{(\tau)}, v_1), (v_1, v_2)\}$. Thus, $N^{(\tau+1)}$ does not have an $\mathcal{S}$-square containing $(u^{(\tau)}, v_3)$, and hence $N^{(\tau+1)}$ is $\mathcal{S}$-square-free because $N^{(\tau)}$ does not have an $\mathcal{S}$-square. Furthermore, by Claim 4.16, $|M^{(\tau+1)} \cup N^{(\tau+1)}| = |M^{(\tau)} \cup N^{(\tau)}| - 1 = |M \cup N| - 1$. $\square$

Claim 4.19. *In Step 5-2, $(M^{(\tau+2)}, N^{(\tau+2)}, u^{(\tau+2)}) \in \mathcal{T}$ and $|M^{(\tau+2)} \cup N^{(\tau+2)}| = |M \cup N| - 1$.*

Proof. Since $C = (v_2, v_1, v_4, v_5)$ is the unique $\mathcal{S}$-square in $M^{(\tau+1)}$, $M^{(\tau+2)}$ is $\mathcal{S}$-square-free. On the other hand, since $N^{(\tau+2)} \cap \delta(v_2) = \{(v_1, v_2), (v_2, v_5)\}$, $N^{(\tau+2)} \cap \delta(v_1) = \{(u^{(\tau)}, v_1), (v_1, v_2)\}$, and $(u^{(\tau)}, v_5) \notin N^{(\tau+2)}$, $N^{(\tau+2)}$ does not have an $\mathcal{S}$-square containing (v_2, v_5). Thus, by Claim 4.18, $N^{(\tau+2)}$ is $\mathcal{S}$-square-free, and hence $(M^{(\tau+2)}, N^{(\tau+2)}, u^{(\tau+2)}) \in \mathcal{T}$. Claim 4.18 also implies that $|M^{(\tau+2)} \cup N^{(\tau+2)}| = |M^{(\tau+1)} \cup N^{(\tau+1)}| = |M \cup N| - 1$. $\square$

The above claims show the correctness of Procedure B.

4.4.3 A main algorithm

In this subsection, we give an algorithm for finding an $(x + s_1, y - s_1)$-increment s_2 using Procedure B. We modify Algorithm FIND-INCREMENT by using a set $\mathcal{S}$ of squares.

Algorithm FIND-INCREMENT II

Input. A bipartite graph $G = (V_1, V_2; E)$ with a weight function w that is vertex-induced on every square, square-free 2-matchings M and N in G with $d_M = x$ and $d_N = y$, and an (x, y)-increment $s_1 = -\chi_u$ with $u \in V_1$.

Output. An $(x + s_1, y - s_1)$-increment s_2 and square-free 2-matchings M' and N' in G such that $d_{M'} = x+s_1+s_2$, $d_{N'} = y-s_1-s_2$, and $w(M')+w(N') \geq w(M)+w(N)$.

Step 1. Let $S_1, S_2, \dots, S_p$ be vertex-disjoint squares in G such that $E(S_i) \subseteq M \cup N$ and $|E(S_i) \cap M| = |E(S_i) \cap N| = 3$ for $i = 1, 2, \dots, p$. We take such $S_1, S_2, \dots, S_p$ maximally and define $\mathcal{S} = \{S_1, S_2, \dots, S_p\}$.

Step 2. Execute Procedure B for $(M, N, u) \in \mathcal{T}_{\rm sq}(\mathcal{S}, x + s, y - s)$ to obtain a triple $(M^*, N^*, u^*) \in \mathcal{T}_{\rm sq}(\mathcal{S}, x + s, y - s)$ satisfying one of the following:

1. $w(M^*) + w(N^*) > w(M) + w(N)$.
2. $w(M^*) + w(N^*) = w(M) + w(N)$ and $|M^* \cup N^*| < |M \cup N|$.
3. $w(M^*) + w(N^*) = w(M) + w(N)$, $|M^* \cup N^*| = |M \cup N|$, and (M^*, N^*, u^*) is stable.

Then, go to Step 3.

Step 3. If (M^*, N^*, u^*) is stable, then output $s_2 := d_{M^*} - x - s_1$, $M' := M^*$, and $N' := N^*$, and stop the algorithm. Otherwise, update M, N, and u as $M := M^*$, $N := N^*$, and $u := u^*$. While there exists an $\mathcal{S}$-square S such that $S \notin \mathcal{S}$, $E(S) \subseteq M \cup N$, and $|E(S) \cap M| = |E(C) \cap N| = 3$, add S to $\mathcal{S}$. Then, go to Step 2.

The correctness of FIND-INCREMENT II can be shown in the same way as FIND-INCREMENT. By using FIND-INCREMENT II, we can prove Proposition 4.14 in the same way as Proposition 4.3, and so we omit the proof. We only mention here that if a 2-matching L is $\mathcal{S}$-square-free and $|E(S) \cap L| = 3$ for any $S \in \mathcal{S}$, then L is a square-free 2-matching.

5 Discussion

Major open problems on k-restricted 2-matchings are the 4-restricted 2-matching problem and the weighted 3-restricted 2-matching problem. In this paper, we have proved that the set of the degree sequences of all 4-restricted 2-matchings is a jump system. So, as Cunningham [7] conjectured, we anticipate that the 4-restricted 2-matching problem can be solved in polynomial time.

Related results reported recently are due to Bérczi and Kobayashi [2], Bérczi and Végh [3], Hartvigsen and Li [19], and Kobayashi [22]. Bérczi and Végh [3] presented a min-max formula and a polynomial algorithm for the $K_{t,t}$-free t-matching problem, the K_{t+1}-free t-matching problem and the $\{K_{t,t}, K_{t+1}\}$-free t-matching problem in graphs all of whose vertices are incident to at most $t+1$ edges. In particular, the 4-restricted 2-matching problem is solved in polynomial time in *subcubic graphs*, in which each vertex is incident to at most three edges.

Bérczi and Kobayashi [2] extended Corollary 1.3 to a weighted version for a special case where $t = 2$ and the graph is subcubic. That is, they proved that the weighted 2-matchings without C_4 in a subcubic graph induce an M-concave function on a constant-parity jump system if the weight function is vertex induced on every C_4. Moreover, based on a general framework of maximizing an M-concave function on a constant-parity jump system [30, 31, 36], they gave a polynomial algorithm for the problem of finding a maximum-weight 2-matching without C_4 in subcubic graphs if the weight function is vertex induced on every C_4.

Hartvigsen and Li [19] and Kobayashi [22] presented polynomial algorithms for the weighted 3-restricted 2-matching problem in subcubic graphs. Hartvigsen and Li [19] devised a primal-dual algorithm for the weighted 3-restricted 2-matching problem in subcubic graphs by a polyhedral approach. Kobayashi [22] solved the same problem from the viewpoint of discrete convex analysis. By Corollary 1.2, we know that the degree sequences of the 3-restricted 2-matchings form a jump system. Kobayashi extended this result to a weighted version in subcubic graphs, that is, he proved that the weighted 3-restricted 2-matchings induce an M-concave function on a constant-parity jump system if the graph is subcubic. He then presented an algorithm for the weighted 3-restricted 2-matching problem in subcubic graphs, which is based on the general framework of maximizing M-concave functions on a constant-parity jump system.

All of these resent developments are established in certain classes of graphs (in subcubic graphs, mainly). It would be interesting to consider whether these results can be extended to general graphs.

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References

[1] N. Apollonio and A. Sebő, *Minsquare factors and maxfix covers of graphs*, Proc. 10th IPCO, LNCS 3064, Springer-Verlag, 2004, pp. 388–400.

[2] K. Bérczi and Y. Kobayashi, *An algorithm for $(n-3)$-connectivity augmentation problem: jump system approach*, Tech. Report METR 2009-12, University of Tokyo, 2009.

[3] K. Bérczi and L. Végh, *Restricted b-matchings in degree-bounded graphs*, to appear in the 14th IPCO, 2010.

[4] A. Bouchet, *Greedy algorithm and symmetric matroids*, Math. Program., 38 (1987), pp. 147–159.

[5] A. Bouchet and W. H. Cunningham, *Delta-matroids, jump systems, and bisubmodular polyhedra*, SIAM J. Discrete Math., 8 (1995), pp. 17–32.

[6] R. Chandrasekaran and S. N. Kabadi, *Pseudomatroids*, Discrete Math., 71 (1988), pp. 205–217.

[7] W. H. Cunningham, *Matching, matroids, and extensions*, Math. Program., 91 (2002), pp. 515–542.

[8] W. H. Cunningham and J. F. Geelen, *Vertex-disjoint dipaths and even dicircuits*, unpublished, 2001.

[9] G. Cornuéjols and W. R. Pulleyblank, *A matching problem with side conditions*, Discrete Math. 29 (1980), pp. 135–159.

[10] A. W. M. Dress and T. Havel, *Some combinatorial properties of discriminants in metric vector spaces*, Adv. Math., 62 (1986), pp. 285–312.

[11] A. W. M. Dress and W. Wenzel, *Valuated matroid: A new look at the greedy algorithm*, Appl. Math. Lett., 3 (1990), pp. 33–35.

[12] A. W. M. Dress and W. Wenzel, *A greedy-algorithm characterization of valuated Δ-matroids*, Appl. Math. Lett., 4 (1991), pp. 55–58.

[13] A. W. M. Dress and W. Wenzel, *Valuated matroids*, Adv. Math., 93 (1992), pp. 214–250.

[14] A. Frank, *Restricted t-matchings in bipartite graphs*, Discrete Appl. Math., 131 (2003), pp. 337–346.

[15] S. Fujishige, *Submodular Functions and Optimization*, 2nd ed., Annals of Discrete Mathematics, 58, Elsevier, 2005.

[16] J. F. Geelen, *The C_6-free 2-factor problem in bipartite graphs is NP-complete*, unpublished, 1999.

[17] D. Hartvigsen, *Extensions of Matching Theory*, Ph.D. thesis, Carnegie Mellon University, 1984.

[18] D. Hartvigsen, *Finding maximum square-free 2-matchings in bipartite graphs*, J. Combin. Theory, Ser. B, 96 (2006), pp. 693–705.

[19] D. Hartvigsen and Y. Li, *Triangle-free simple 2-matchings in subcubic graphs (extended abstract)*, Proc. 12th IPCO, LNCS 4513, Springer-Verlag, 2007, pp. 43–52.

[20] S. N. Kabadi and R. Sridhar, *Δ-matroid and jump system*, J. Appl. Math. Decision Sci., 9 (2005), pp. 95–106.

[21] Z. Király, *C_4-free 2-matchings in bipartite graphs*, Tech. Report TR-2001-13, Egerváry Research Group, 1999.

[22] Y. Kobayashi, *A simple algorithm for finding a maximum triangle-free 2-matching in subcubic graphs*, Tech. Report METR 2009-26, University of Tokyo, 2009.

[23] Y. Kobayashi and K. Murota, *Induction of M-convex functions by linking systems*, Discrete Appl. Math., 155 (2007), pp. 1471–1480.

[24] Y. Kobayashi, K. Murota, and K. Tanaka, *Operations on M-convex functions on jump systems*, SIAM J. Discrete Math., 21 (2007), pp. 107–129.

[25] Y. Kobayashi and K. Takazawa, *Even factors, jump systems, and discrete convexity*, J. Combin. Theory, Ser. B, 99 (2009), pp. 139–161.

[26] L. Lovász, *The membership problem in jump systems*, J. Combin. Theory, Ser. B, 70 (1997), pp. 45–66.

[27] M. Makai, *On maximum cost $K_{t,t}$-free t-matchings of bipartite graphs*, SIAM J. Discrete Math., 21 (2007), pp. 349–360.

[28] K. Murota, *Convexity and Steinitz's exchange property*, Adv. Math., 124 (1996), pp. 272–311.

[29] K. Murota, *Discrete Convex Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[30] K. Murota, *M-convex functions on jump systems: a general framework for minsquare graph factor problem*, SIAM J. Discrete Math., 20 (2006), pp. 213–226.

[31] K. Murota and K. Tanaka, *A steepest descent algorithm for M-convex functions on jump systems*, IEICE Trans. Fundamentals of Electr., Commun., Computer Sci., E89-A (2006), pp. 1160–1165.

[32] J. G. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[33] G. Pap, *Restricted b-matchings in bipartite graphs*, Tech. Report TR-2005-13, Egerváry Research Group, 2005.

[34] G. Pap, *Combinatorial algorithms for matchings, even factors and square-free 2-factors*, Math. Program., 110 (2007), pp. 57–69.

[35] M. Russell, *Restricted Two-Factors*, Master's Thesis, University of Waterloo, 2001.

[36] A. Shioura and K. Tanaka, *Polynomial-time algorithms for linear and convex optimization on jump systems*, SIAM J. Discrete Math., 21 (2007), 504–522.

[37] K. Takazawa, *A weighted $K_{t,t}$-free t-factor algorithm for bipartite graphs*, Math. Oper. Res., 34 (2009), pp. 351–362.

[38] D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

[39] W. Wenzel, *Pfaffian forms and Δ-matroids*, Discrete Math., 115 (1993), pp. 253–266.

[40] W. Wenzel, *Δ-matroids with the strong exchange conditions*, Appl. Math. Lett., 6 (1993), pp. 67–70.

[41] H. Whitney, *On the abstract properties of linear dependence*, Amer. J. Math., 57 (1935), pp. 509–533.