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Highly connected rigidity matroids have unique underlying graphs

Tibor Jordán and Viktória E. Kaszanitzky

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Tibor Jordán* and Viktória E. Kaszanitzky**

Abstract

In this note we consider the following problem: is there a (smallest) integer k_d such that every graph G is uniquely determined by its d-dimensional rigidity matroid $\mathcal{R}_d(G)$, provided that $\mathcal{R}_d(G)$ is k_d-connected? Since $\mathcal{R}_1(G)$ is isomorphic to the cycle matroid of G, a celebrated result of H. Whitney implies that $k_1 = 3$. We prove that if G is 7-vertex-connected then it is uniquely determined by $\mathcal{R}_2(G)$. We use this result to deduce that $k_2 \leq 11$, which gives an affirmative answer for $d = 2$.

1 Introduction

Let $\mathcal{M}$ be a matroid on ground set E with rank function r and let k be a positive integer. We say that a partition (X, Y) of E is a *k-separation* if

$$\min\{|X|, |Y|\} \geq k, \quad \text{and}$$

$$r(X) + r(Y) \leq r(E) + k - 1.$$

The *connectivity* of $\mathcal{M}$, denoted by $\lambda(\mathcal{M})$, is defined to be the smallest integer j for which $\mathcal{M}$ has a j-separation. Note that $\lambda(\mathcal{M}) \geq 1$ for all matroids $\mathcal{M}$. We say that $\mathcal{M}$ is *h-connected* if $\lambda(\mathcal{M}) \geq h$ holds. We refer the reader to [7] for more details on matroids and matroid connectivity.

The following problem was recently proposed by Brigitte and Herman Servatius [1, Problem 17].

Problem Let G be a graph and $\mathcal{R}_d(G)$ its d-dimensional generic rigidity matroid. Is there a (smallest) constant k_d such that G is uniquely determined by $\mathcal{R}_d(G)$ provided that $\mathcal{R}_d(G)$ is k_d-connected?

*Department of Operations Research, Eötvös University, Pázmány sétány 1/C, 1117 Budapest, Hungary. Supported by the MTA-ELTE Egerváry Research Group on Combinatorial Optimization and the Hungarian Scientific Research Fund grant no. K81472. e-mail: jordan@cs.elte.hu

**Department of Operations Research, Eötvös University, Pázmány sétány 1/C, 1117 Budapest, Hungary.

The d-dimensional *generic rigidity matroid* (or simply *rigidity matroid*) $\mathcal{R}_d(G)$ of graph $G = (V, E)$ is defined on the edge set of G, see [2, 8]. It is not hard to see that $\mathcal{R}_1(G)$ is isomorphic to the cycle matroid of G, which implies, by a theorem of H. Whitney [9], that $k_1 = 3$. The two-dimensional rigidity matroid was characterized by G. Laman [5], who proved that a set $F \subseteq E$ is independent in $\mathcal{R}_2(G)$ if and only if

$$|E(G[X]) \cap F| \leq 2|X| - 3 \quad \text{for all} \quad X \subseteq V \quad \text{with} \quad |X| \geq 2, \tag{1}$$

where $G[X]$ denotes the subgraph of G induced by X. Note that $r_2(E) \leq 2|V| - 3$ by (1). In this note we shall prove that k_2 exists and provide an explicit bound $k_2 \leq 11$. It is a major open problem to find a good characterization for independence in d-dimensional rigidity matroids, for $d \geq 3$. Thus the problem for higher dimensions is probably substantially harder.

We shall consider graphs without loops and isolated vertices. Henceforth we shall assume that $d = 2$ and omit the subscripts referring to the dimension.

2 Highly connected graphs

We first show that if G is highly connected then its rigidity matroid uniquely determines G. We need some more definitions. Let $G = (V, E)$ be a graph. We say that G is *rigid* if $r(E) = 2|V| - 3$ and that G is *redundantly rigid* if $G - e$ is rigid for all $e \in E$. A *k-vertex separation* of a graph $H = (V, E)$ is a pair (H_1, H_2) of edge-disjoint subgraphs of G each with at least $k+1$ vertices such that $H = H_1 \cup H_2$ and $|V(H_1) \cap V(H_2)| = k$. The graph is said to be *k-vertex-connected* if it has at least $k+1$ vertices and has no j-vertex separation for all $0 \leq j \leq k-1$.

We shall also need the following three results from combinatorial rigidity.

Lemma 2.1. *[2, Theorem 4.7.2], [4, Lemma 3.1] Suppose that $\mathcal{R}(G)$ is 2-connected. Then G is redundantly rigid.*

Theorem 2.2. *[6, Theorem 2] Every 6-vertex-connected graph is redundantly rigid.*

Theorem 2.3. *[4, Theorem 3.2] Suppose that G is 3-vertex-connected and redundantly rigid. Then $\mathcal{R}(G)$ is 2-connected.*

The proof method of our first result is motivated by a proof for (a special case of) Whitney's theorem, due to J. Edmonds (see [7]). Let $J \subseteq E$ be a set of elements in matroid $\mathcal{M}$. We say that J is a 2-*hyperplane* of $\mathcal{M}$ if $r(J) = r(E) - 2$ and for all $e \in E - J$ we have $r(J + e) = r(E) - 1$.

Theorem 2.4. *Let G and H be two graphs and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If G is 7-vertex-connected then G is isomorphic to H.*

Proof. We say that a 2-hyperplane J of $\mathcal{R}(G)$ is 2-connected if the matroid restriction of $\mathcal{R}(G)$ to J is 2-connected. Since G is 7-vertex-connected, Theorems 2.2 and 2.3 imply that G is rigid and $E(G - v)$ (i.e. the edge set E minus the vertex bond of v) is a 2-connected 2-hyperplane of $\mathcal{R}(G)$ for all $v \in V(G)$.

Now consider an arbitrary 2-connected 2-hyperplane J of $\mathcal{R}(G)$. By Lemma 2.1 the subgraph $L = (V(J), J)$ of G on the set of end vertices of J is rigid. Thus $r(J) = 2|V(J)| - 3$ and, since 2-hyperplanes are closed sets, it follows that L is an induced subgraph of G. By using the fact that G is rigid, we obtain $|V(G)| = |V(J)|+1$. Thus the complement of J corresponds to a vertex bond of G.

It follows that there is a bijection between $V(G)$ and the 2-connected 2-hyperplanes of $\mathcal{R}(G)$ and that $\mathcal{R}(G)$ uniquely determines the vertex-edge incidencies in G.

By the assumption of the theorem $\mathcal{R}(G)$ and $\mathcal{R}(H)$ are isomorphic. It follows from Theorems 2.2 and 2.3 that $\mathcal{R}(G)$ is 2-connected. Thus $\mathcal{R}(H)$ is also 2-connected and hence H is rigid by Lemma 2.1. This implies that $2|V(G)| - 3 = r(G) = r(H) = 2|V(H)| - 3$ and hence $|V(G)| = |V(H)|$. Thus $\mathcal{R}(H)$ has $|V(H)|$ 2-connected 2-hyperplanes. So G and H are isomorphic, as claimed. □

2.1 Examples

The bound on the connectivity of G in Theorem 2.4 might be improved to 6, but it cannot be replaced by 5. To prove this claim we recall the following family of graphs from [6]: let G be a 5-regular 5-vertex-connected graph on k vertices. Split every vertex of G into 5 vertices of degree one, and identify these 5 vertices with the vertices of a complete graph K_5 on 5 vertices. See Figure 1 for two (non-isomorphic) examples with $k = 8$.

It is easy to see that the resulting graph G' on $5k$ vertices is 5-vertex-connected. It is also easy to verify that G' has rank at most $\frac{19}{2}k$, hence G' is not rigid when $k \geq 8$, see [6]. Furthermore, by using the Henneberg inductive construction to verify independence, one can also show that the rank of G' is exactly $\frac{19}{2}k$ and that the deletion of an arbitrary edge connecting distinct K_5's decreases the rank by one. Thus $\mathcal{R}(G')$ is the direct sum of k copies of $\mathcal{R}(K_5)$ and $\frac{5}{2}k$ copies of $\mathcal{R}(K_2)$, for any choice of the initital graph G. Our claim follows, since there exist non-isomorphic 5-regular 5-vertex-connected graphs on $k \geq 8$ vertices for all $k \geq 8$.

We also have similar examples with rigid graphs, but with smaller connectivity. The graphs on Figure 2 are non-isomorphic 3-vertex-connected rigid graphs of the same size. Their rigidity matroids are isomorphic, since the edge set of both graphs is a circuit in the corresponding rigidity matroid. This implies that 7-vertex-connected cannot be replaced by 3-vertex-connected in Theorem 2.4, even if we add the assumption that G is rigid.

3 Highly connected matroids

In this section we show that highly connected rigidity matroids have unique underlying graphs. We shall need the following two lemmas and Theorem 2.4. Let $d(v)$ denote the degree of vertex v in G and let $\delta(G) = \min\{d(v) : v \in V(G)\}$ denote the minimum degree of G.

Lemma 3.1. *Let $G = (V, E)$ be a rigid graph on at least three vertices and suppose that $\mathcal{R}(G)$ is k-connected for some $k \geq 1$. Then $\delta(G) \geq k + 1$.*

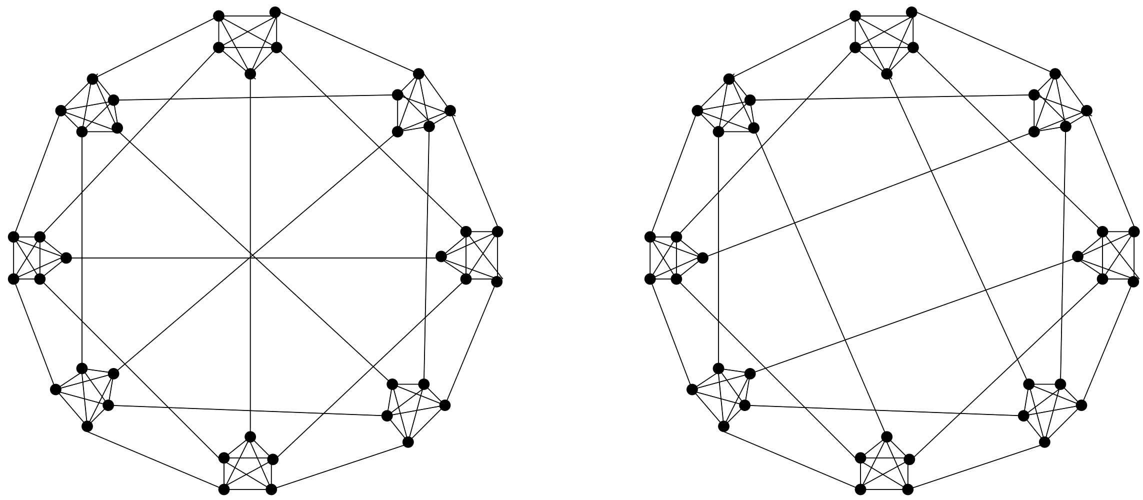

Figure 1: Two non-isomorphic 5-vertex-connected graphs with isomorphic rigidity matroids.

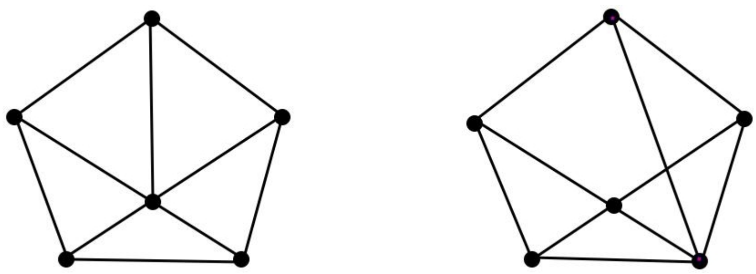

Figure 2: Two non-isomorphic 3-vertex-connected rigid graphs whose edge sets are circuits in their rigidity matroids.

Proof. Since G is rigid, G is 2-vertex connected and $\delta(G) \geq 2$. Let X be the set of edges obtained from the vertex bond of some vertex v of degree $d(v)$ by deleting an arbitrary edge. Let $Y = E - X$. The 2-vertex connectivity of G implies that $|Y| = |E(G-v)| + 1 \geq |V(G)| - 1 + 1 \geq d(v)$. Thus $\min\{|X|, |Y|\} \geq d(v) - 1$ holds. Since X is a co-circuit of $\mathcal{R}(G)$, we have

$$r(X) + r(Y) \leq d(v) - 1 + r(E) - 1 = r(E) + d(v) - 2.$$

Hence (X, Y) is a $(d(v) - 1)$-separator of $\mathcal{R}(G)$, which implies $\delta(G) \geq k + 1$, as required. □

Lemma 3.2. *Let $G = (V, E)$ be a graph and suppose that $\mathcal{R}(G)$ is $(2k-3)$-connected for some $k \geq 3$. Then G is k-vertex connected.*

Proof. The hypothesis of the lemma implies that $\mathcal{R}(G)$ is 2-connected. Thus G is rigid by Lemma 2.1. Hence $r(E) = 2|V| - 3$ and, by Lemma 3.1, we have $\delta(G) \geq 2k - 2$ and $|V| \geq 2k - 1 \geq k + 1$.

For a contradiction suppose that G has a j-vertex separation (G_1, G_2) for some $j \leq k-1$. Let $X = E(G_1)$ and $Y = E(G_2)$. Since $\delta(G) \geq 2k-2$, we must have $\min\{|X|, |Y|\} \geq 2k-2$. By using (1) we can now deduce that

$$r(X)+r(Y) \leq 2|V(G_1)|-3+2|V(G_2)|-3 = 2(|V|+j)-6 \leq 2|V|+2k-8 = r(E)+2k-5.$$

Hence (X, Y) is a $(2k-4)$-separator of $\mathcal{R}(G)$, a contradiction. This proves the lemma. □

Note that a highly vertex-connected graph G does not necessarily have a highly connected rigidity matroid. The existence of a complete graph K_4 in G (whose edge set is a circuit in $\mathcal{R}(G)$) implies that $\lambda(\mathcal{R}(G)) \leq 6$, even if G is highly vertex-connected.

The main result of this section is now a direct corollary of Theorem 2.4 and Lemma 3.2.

Theorem 3.3. *Let G and H be two graphs and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If $\mathcal{R}(G)$ is 11-connected then G is isomorphic to H.*

Theorem 3.3 implies that $k_2 \leq 11$. By the example of Figure 2 we have $k_2 \geq 3$.

We remark that the proofs and results in this section can easily be extended to *vertical connectivity*, which is another natural form of matroid connectivity [7]. In particular, we can replace 11-connected by vertically 11-connected in Theorem 3.3.

4 Concluding remarks

In this note we have shown that a highly connected two-dimensional rigidity matroid uniquely determines its underlying graph. Since no good characterization is known for independence in the three-dimensional rigidity matroid, the question whether k_3 exists seems more difficult. We note that three-dimensional versions of some of the key results that we used in the proofs exist as conjectures: Lovász and Yemini [6] conjecture that 12-vertex-connected graphs are rigid in three-space, while Jackson and Jordán [3] conjecture that if G is 5-vertex-connected and $\mathcal{R}_3(G)$ is 2-connected then G is redundantly rigid. The bounds on the vertex connectivity would be best possible in both conjectures.

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