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Tree-compositions and submodular flows

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Abstract

Tree-compositions were defined in [5] to describe the obstacles to k-edge-connected orientability of mixed graphs. Here we describe a structural result on tree-compositions that gives rise to a simple algorithm for computing an obstacle when the orientation does not exists.

1 Introduction

Submodular flows were introduced and investigated by Edmonds and Giles in [2], while the paper [5] described a theorem on existence of feasible submodular flows. This result can be used to describe various graph orientation theorems. There are several algorithms to compute a feasible subflow if one exists. In this paper we describe a method to compute a simple obstacle if no feasible subflow exists.

1.1 Definitions

Two sets A and $B \subseteq V$ are **crossing** if $A-B \neq \emptyset,\ B-A \neq \emptyset,\ A\cap B \neq \emptyset,\ V-(A\cup B) \neq \emptyset$. We say that a family $\mathcal{F}$ on a ground set V is a **crossing family** if $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$ for any two crossing members $A, B \in \mathcal{F}$. We call a family $\mathcal{F}$ on a ground set V **cross-free** if there are not any crossing pairs in $\mathcal{F}$.

A $t\overline{s}$**-set** is a set that contains t but does not contain s. For a given family $\mathcal{F}$ we usually use the term $t\overline{s}$-set only for the members of $\mathcal{F}$. Let S and T be disjoint sets. We call a family $\mathcal{F}$ on the ground set $S \cup T$ ST**-separating** if for any $s \in S$ and $t \in T$ there is an $t\overline{s}$-set in $\mathcal{F}$. We say that an ST-separating family $\mathcal{F}$ is **minimal** if no proper subfamily of $\mathcal{F}$ is ST-separating.

Let $\{Z_1, Z_2, ..., Z_t\}$ be a partition of $Z \subseteq S$, and let $\{Z_i^1, Z_i^2, ..., Z_i^{t_i}\}$ be a partition of $S - Z_i$. Then the set-system $\mathcal{D} := \{S - Z_i^j : 1 \leq i \leq t, q \leq j \leq t_i\}$ is called the **double-partition of** Z.

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Let $\varphi : V \to U$ be a map, and $F = (U, A)$ an oriented tree. For $e \in A$ let V_e be $\varphi^{-1}(U_e)$, where U_e is the component of $F - e$ that contains the head of e. Let S and T be disjoint non-empty sets. We call a family $\mathcal{F}$ on a ground set $V = S \cup T$ **S-T-tree-composition** if there is an oriented tree $F = (U = U_S \cup U_T, A)$ and a surjective map $\varphi : V \to U$ for which every edge of F has a tail in U_S and a head in U_T and $\mathcal{F} = \{V_e : e \in A\}$. By convention a family on a ground set V is a **tree-composition on a set** $\emptyset \neq Z \subseteq V$ if it is a $(V - Z)$-Z-tree-composition. The **ground-degree** of a double-partition or tree-composition $\mathcal{F}$ is $\min\{d_{\mathcal{F}}(v) : v \in V\}$. Observe that the difference between the ground-degree and the maximum degree of a double-partition or a tree-composition is one.

Let $G = (S, T, E)$ be a bipartite graph with color classes S and T. We say that an **S-T-tree-composition $\mathcal{F}$ belongs to the graph** if for any $s \in S,\ t \in T,\ st \in E$ the image $\varphi(s)\varphi(t) \in A$, where $F = (U, A)$ is the oriented tree belonging to $\mathcal{F}$ with the surjective map $\varphi : V \to U$.

Let $G = (V, \mathcal{E})$ be a graph. For $v \in V$ we denote by $d(v)$ or $d_G(v)$ the **degree** of v. For $X \subseteq V$ we denote by $i(X)$ the **number of edges induced by** X. For an oriented graph $D = (V, A)$ we denote by $\rho(v)$ or $\rho_D(v)$ the **in-degree** of the node v, and we denote by $\delta(v)$ or $\delta_D(v)$ the **out-degree** of the node v. We use $d(X),\ \rho(X),\ \delta(X)$ or $d_G(X),\ \rho_D(X),\ \delta_D(X)$ to denote the **degree, in-degree or out-degree of a subset** $X \subseteq V$. For $X, Y \subseteq V$ we denote by $d(X, Y)$ or $d_G(X, Y)$ the number of edges that have got an endpoint in both of $X - Y$ and $Y - X$. For a partition $\mathcal{P}$ of the node set V we denote by $e(\mathcal{P})$ the number of edges that endpoints belong to two elements of $\mathcal{P}$.

A digraph $D = (V, A)$ is **k-edge-connected** if $\delta_D(X) \geq k$ for any $X \subseteq V$. Using Menger's theorem this is equivalent to the following: for any two node u and v there is k edge disjoint oriented path from u to v. A digraph $D = (V, A)$ with a root node $r_0 \in V$ is called **r_0-rooted (k, ℓ)-edge-connected** if $\delta_D(X) \geq k$ and $\rho_D(X) \geq \ell$ for every set $r_0 \in X \subset V$. Using Menger's theorem this is equivalent to the following: there are k edge-disjoint paths from r_0 to every other node and there are ℓ edge-disjoint paths from every node to r_0.

We call a set-function $b : 2^S \to \mathbb{R} \cup \{\infty\}$ **(crossing) submodular** if $b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for every (crossing) $X, Y \subseteq S$. We call a set-function $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ **(crossing) supermodular** if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ for every (crossing) $X, Y \subseteq S$.

For a vector $x \in \mathbb{R}^S$ and $X \subseteq S$ $\tilde{x}(X) := \sum_{v \in X} x(v)$.

For a (crossing) submodular function b we define $B(b)$, as $B(b) := \{x \in \mathbb{R}^S : \tilde{x}(X) \leq b(X)(\forall X \subseteq S), b(S) = \tilde{x}(S)\}$. If b is submodular, then we say that $B(b)$ is the **base-polyhedron of** b. For a (crossing) supermodular function p we define $B'(p)$, as $B'(p) := \{x \in \mathbb{R}^S : \tilde{x}(X) \geq p(X)(\forall X \subseteq S), p(S) = \tilde{x}(S)\}$. If p is supermodular, then we say that $B'(p)$ is the **base-polyhedron of** p.

For a graph $G = (V, E)$ a set-function $h : 2^V \to \mathbb{R}$ is **(crossing) G-supermodular** if $h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) + d_G(X, Y)$ for every (crossing) $X, Y \subseteq V$.

Let $b : 2^S \to \mathbb{R} \cup \{\infty\}$ be an arbitrary set function. Then the **lower truncation**

of b is a set function $b^\vee : 2^S \to \mathbb{R} \cup \{\infty\}$, for which

$$b^\vee(X) := \min\{b(\mathcal{P}) : \mathcal{P} \text{ a partition of } X\} \tag{1}$$

holds for any $X \subseteq S$. Let $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ be an arbitrary set function. Then **the upper truncation of** p is a set function $p^\wedge : 2^S \to \mathbb{R} \cup \{-\infty\}$ for which

$$p^\wedge(X) := \max\{p(\mathcal{P}) : \mathcal{P} \text{ a partition of } X\} \tag{2}$$

holds for any $X \subseteq S$.

Let $b : 2^S \to \mathbb{R} \cup \{\infty\}$ be a crossing submodular function. It is known that if $B(b) \neq \emptyset$, then it is a base-polyhedron with a unique submodular border function. We call this submodular function the **full (lower) truncation or bi-truncation** of b and we denote it by $b^\downarrow$. It is also known that if $b(S) = 0$ and $B(b) \neq \emptyset$, then $b^\downarrow$ can be computed by the following formula.

$$b^\downarrow(Z) = \min\Big\{ \sum_{X \in \mathcal{D}} b(X) : \mathcal{D} \text{ a double-partition of } Z \Big\}. \tag{3}$$

Let $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ be a crossing supermodular function. It is well known that if $B'(p) \neq \emptyset$, then it is a base-polyhedron, whit unique submodular border function. We call this submodular function the **full (upper) truncation or bi-truncation** of p and we denote it by $p^\uparrow$. It is also known that if $B'(p) \neq \emptyset$, then $p^\uparrow$ can be computed by the following formula.

$$p^\uparrow(Z) = \max\Big\{ \sum_{X \in \mathcal{D}} p(X) - \Delta p(S) : \mathcal{D} \text{ a double-partition of } Z, \text{ with ground-degree } \Delta \Big\} \tag{4}$$

(see [8], [5]). These formulas were simplified by [5]. We will give a new proof to these simplified versions.

Let $D = (V, A)$ be a directed graph, $f : A \to \mathbb{Z} \cup \{-\infty\}$, $g : A \to \mathbb{Z} \cup \{\infty\}$ two integer-valued bounding functions for which $f \leq g$. Moreover, we are given a crossing submodular set-function $b : 2^V \to \mathbb{Z} \cup \{\infty\}$ for which $b(\emptyset) = 0$ and $b(V)$ is finite. A function (or vector) $x : A \to \mathbb{R}$ is called a **submodular flow** or **subflow** confined by b if $\Psi_x(Z) := \rho_x(Z) - \delta_x(Z) \leq b(Z)$ for every $Z \subseteq V$, where $\rho_x(Z) := \sum\{x(uv) : u \in V - Z, v \in Z\}, \delta_x(Z) := \sum\{x(uv) : u \in Z, v \in V - Z\}$. Since $\Psi_x(V) = 0$ we can assume that $b(V) = 0$. A subflow x is feasible if $f \leq x \leq g$. The set $Q(f, g; b)$ of feasible subflows is called a **submodular flow polyhedron**.

Let $D = (V, A)$ be a digraph and $R \subseteq V$. We denote by D/R the digraph obtained by **shrinking** R into a single node r_R and deleting the loops but keeping the multiple edges. $D[R]$ denotes the digraph **induced by** R in D.

2 The main result

Our main result is the following theorem.

Theorem 2.1. *A family $\mathcal{T}$ on the ground set $S \cup T$ is an S-T-tree-composition if and only if it is a minimal cross-free ST-separating family.*

Proof. By the definition of the S-T-tree-composition if $\mathcal{T}$ is an S-T-tree-composition, then it is a minimal cross-free ST-separating family. First we show the other direction for laminar families.

Claim 2.2. *If $\mathcal{F}$ is a minimal laminar ST-separating family, then it is an S-T-tree-composition, and the tree representing $\mathcal{F}$ is a star with the center node $\varphi(S)$.*

Proof. A minimal laminar ST-separating family is just a partition of T because if it has an element F for which $s \in F \cap S$, then since for all $t \in F \cap T$ there must be an $t\overline{s}$ that must be the subset of F by the laminarity. Hence we can delete F from $\mathcal{F}$ without violating that $\mathcal{F}$ is ST-separating that contradicts to the minimality. A partition of T is obviously represented by a star with the center node $\varphi(S)$. □

Now we are ready to prove the theorem. We will give the representing tree. Let the vertices of the tree be the **atoms** that are the maximal disjoint subsets of $S \cup T$, from which every $F \in \mathcal{F}$ can be built up. If there is a subset of $S \cup T$ that is not included in any member of $\mathcal{F}$, then $\mathcal{F}$ is laminar and we are done by Claim 2.2. So we can assume that the atoms cover $S \cup T$. Since $\mathcal{F}$ is an ST-separating family there is not any atom that intersects both of S and T, because if it contains an $s \in S$ and a $t \in T$, then there is not any $t\overline{s}$-set in $\mathcal{F}$. Let $\varphi : V \to U$ be a surjective map which maps every atoms to a single node. To define the edges we need the following claims.

Claim 2.3. *For any $F \in \mathcal{F}$ there uniquely exists an atom $A_F \subseteq T \cap F$ such that $A_F \nsubseteq F'$ for all $F' \in \mathcal{F}$ that is induced by F.*

Proof. There must exist such an atom by minimality, otherwise we could omit F from $\mathcal{F}$, hence we only need to prove the uniqueness.

Case 1: If $F \cap S = \emptyset$, then we can prove that F is an atom. For $F' \subsetneq F$ $F' \notin \mathcal{F}$ always holds, otherwise F' would be omittable from $\mathcal{F}$. If $F' \in \mathcal{F}$, then $S \nsubseteq F'$ because otherwise F' is omittable. So if F' intersects F, then it must contain F because we have seen that F could not contain F' and they cannot cross. So in this case the atom $A_F = F$ is unique.

Case 2: If $F \cap S \neq \emptyset$, then assume for a contradiction that there are two atoms A_1 and A_2 with the property described in the claim. There must be a set $F_1 \in \mathcal{F}$ that separates A_1 and A_2 because A_1 and A_2 are different atoms. We can assume by changing the indices that $A_2 \in F_1$ and $A_1 \cap F_1 = \emptyset$. Since A_1 and A_2 hold the property described in the claim this set could not be contained by F. Moreover $A_2 \subseteq F_1 \cap F \neq \emptyset$ and $A_1 \subseteq F \nsubseteq F_1$. $S - F_1 \neq \emptyset$ because otherwise F_1 is omittable. So $(S \cup T) - F \subseteq F_1$ and $(S \cup T) - F_1 \subseteq F$ because $\mathcal{F}$ is cross-free. For all $s \in S - F_1$ there is a set $F_s \in \mathcal{F}$ that avoids s and contains A_1. We can prove that $(S \cup T) - F \subseteq F_s$ as for F_1. Therefore $F_1 \cap F_s \subseteq (S \cup T) - F$. $s \notin F_1 \cup F_2$ and $A_1 \in F_s - F_1$, therefore since $\mathcal{F}$ is cross-free $F_1 \subseteq F_s$. Since this holds for any $s \in S - F_1$ and for any $t \in T \cap F_1$ F_s is also a $t\overline{s}$-set, hence we can omit F_1 from $\mathcal{F}$ that is a contradiction. □

Applying the above claim to the set-system that we get by complementing every element of $\mathcal{F}$ we get the following:

Claim 2.4. *For any $F \in \mathcal{F}$ there uniquely exists an atom $B_F \subseteq S - F$, such that $B_F \subseteq F'$ for all $F' \in \mathcal{F}$ that induces F.*

Let us define the edge set to be $A := \{e_F = \overrightarrow{\varphi(B_F)\varphi(A_F)} : F \in \mathcal{F}\}$, so we get an oriented graph $D = (U, A)$.

Claim 2.5. *The only edge of D that enters to a set $F \in \mathcal{F}$ is e_F.*

Proof. Assume that $e_{F'}$ also enters $\varphi(F)$. It means that $B_{F'} \subseteq S - (F \cup F')$ and $A_{F'} \subseteq F \cap F' \cap T$. Since F and F' do not cross $F \subset F'$ or $F' \subset F$ holds. If $F \subset F'$, then $A_{F'} \not\subseteq F$ by definition. Therefore $e_{F'}$ does not enters $\varphi(F)$, a contradiction. If $F' \subset F$, then $B_{F'} \subseteq F$ by definition. Therefore $e_{F'}$ does not enters $\varphi(F)$, a contradiction. □

Claim 2.6. *No edge of D leaves a set $F \in \mathcal{F}$.*

Proof. Assume that $e_{F'}$ leaves $\varphi(F)$ for an $F' \in \mathcal{F}$. First we show that $B_F \subset F'$. To do this we show that if $B_F \not\subset F'$, then $B_{F'} \not\subset F$, hence $e_{F'}$ does not leave $\varphi(F)$. By the definition of $B_{F'}$ every set $F'' \in \mathcal{F}$ that includes F' also includes $B_{F'}$, hence it intersects F. If B_F would not be a subset of F', then we could choose F'' not to contain B_F, since $B_F \neq B_{F'}$. So $F \cup F'' \neq S \cup T$. Since F and F' intersect each other and they do not cross, $F \subseteq F''$ or $F'' \subseteq F$. But since $e_{F'}$ leaves $\varphi(F)$, $A_{F'} \subseteq F' - F \subseteq F'' - F$, hence the second case could not hold. In the first case F'' would be an element of $\mathcal{F}$ that contains F, but does not contain B_F that contradicts to the definition of B_F. Therefore $B_F \subseteq F'$. For any $s \in B_F$ there is a $G \in \mathcal{F}$ that contains $A_{F'}$ but avoids s because $\mathcal{F}$ is ST-separating. $G \not\subseteq F'$ by the definition of $A_{F'}$. Since $\mathcal{F}$ is cross-free G does not cross F'. Therefore $(S \cup T) - F' \subset G$. G does not cross F also. Since $B_{F'} \subseteq G \cap F$ and they does not contain s, $F \subset G$. But in this case G would be an element of $\mathcal{F}$ that contains F but does not contain B_F that contradicts to the definition of B_F. □

The above two claims show that the underlying unoriented graph of D is a forest, since we gave a set for every edge e of D that has only e as in-edge and no out-edge. So to show that D is an oriented tree we only need to show that its vertex number is at least one more than its edge number. We know that the edge number is $|\mathcal{F}|$ by the definition of the edges.

Claim 2.7. *A cross-free family $\mathcal{F}$ on the ground set V has at most one more atoms than elements.*

Proof. We do induction on $|\mathcal{F}|$. The claim is obvious for $|\mathcal{F}| = 1$. Let $\mathcal{F}' := \mathcal{F} \cup \{F\}$, where $\mathcal{F}$ is cross-free. We will show that $\mathcal{F}'$ has at most one more atoms, or $\mathcal{F}'$ is not cross-free. $\mathcal{F}'$ could have at least two more atoms than $\mathcal{F}$ only if F intersects two atoms of $\mathcal{F}$, and does not contain them, and they do not contain F. Let these atoms be A and B. $\mathcal{F}$ has an element F' that contains exactly one of them, we can assume that it contains A. But then F and F' are crossing because $\emptyset \neq F \cap A \subseteq F \cap F'$, $\emptyset \neq A - F \subseteq F' - F$, $\emptyset \neq B - F \subseteq V - (F \cup F')$. $\emptyset \neq F \cap B \subseteq F - F'$. □

Therefore D is an oriented tree with edges from $\varphi(S)$ to $\varphi(T)$, and for any of its edge $V_{e_F} = F$. This proves the theorem. □

We will show that a crossing ST-separating family always includes a cross-free ST-separating family. Let $M_{t\overline{s}}$ be the minimal $t\overline{s}$-set for any $s \in S$, $t \in T$. Note that since $\mathcal{F}$ is crossing, the intersection of $t\overline{s}$-sets in $\mathcal{F}$ is also in $\mathcal{F}$ therefore $M_{t\overline{s}}$ really exists. Since the minimal (or the maximal) $t\overline{s}$-sets do not form always a cross-free family we need to work a bit more. For any $s \in S$ let $\mathcal{H}_s$ be the hypergraph on the node set $S \cup T$ and with the edge set $\mathcal{E}_s := \{M_{t\overline{s}} : t \in T\}$. Let $\mathcal{F}_s$ be the family of connected components of $\mathcal{H}_s$ that intersect T. Note that since $\mathcal{F}$ is crossing the members of $\mathcal{F}_s$ will be members of $\mathcal{F}$ also.

Lemma 2.8. *Let $\mathcal{F}$ be an ST-separating crossing family. Then the subfamily $\mathcal{G} := \bigcup_{s\in S} \mathcal{F}_s$ of $\mathcal{F}$ is cross-free ST-separating.*

Proof. By the definition of $\mathcal{G}$ it is easy to see that $\mathcal{G}$ is ST-separating, because $\mathcal{F}_s$ contains an $t\overline{s}$-set for all $t \in T$ that is the component of t in $\mathcal{H}_s$. We get the components of $\mathcal{H}_s$ by the union of crossing sets of $\mathcal{F}$, therefore $\mathcal{G} \subseteq \mathcal{F}$ is also clear.

Since the elements of $\mathcal{F}_s$ are disjoint, we only need to prove that for two $s_1 \neq s_2$ elements of S a set $F_1 \in \mathcal{F}_{s_1}$ will not cross a set $F_2 \in \mathcal{F}_{s_2}$. For a contradiction assume that F_1 and F_2 are crossing.

Case 1: If $s_2 \notin F_1$, then for $t \in F_1 \cap T$ $M_{t\overline{s_1}} = M_{t\overline{s_2}}$ because $M_{t\overline{s_1}} \subseteq F_1 \not\ni s_2$, so $M_{t\overline{s_1}}$ is a $t\overline{s_2}$-set also, but if $M_{t\overline{s_2}}$ is smaller than $M_{t\overline{s_1}}$, then it would be a smallest $t\overline{s_1}$-set. That means there is a component of H_{s_2} that contains F_1, hence F_1 and F_2 are disjoint or $F_1 \subseteq F_2$ that is a contradiction.

Case 2: If $s_2 \in F_1$, then there is a $t \in F_1 \cap T$ for which $s_2 \in M_{t\overline{s_1}}$. $M_{t\overline{s_1}} \cup M_{t\overline{s_2}} \subseteq F_1 \cup F_2 \neq S \cup T$ because F_1 and F_2 are crossing sets by our assumption. We also know that $t \in M_{t\overline{s_1}} \cap M_{t\overline{s_2}} \neq \emptyset$ that means $M_{t\overline{s_1}}$ and $M_{t\overline{s_2}}$ are crossing sets or $s_2 \notin M_{t\overline{s_2}} \subset M_{t\overline{s_1}} \ni s_2$. Therefore we got that $s_2 \notin M_{t\overline{s_1}} \cap M_{t\overline{s_2}} \in \mathcal{F}$ is a smaller $t\overline{s_1}$-set than $s_2 \in M_{t\overline{s_1}}$ that is a contradiction. □

Using the lemma we get the following useful corollary of Theorem 2.1 by minimizing $\mathcal{G}$.

Corollary 2.9. *Let $\mathcal{F}$ be an ST-separating crossing family. Then there is an S-T-tree-composition $\mathcal{T}$ such that $\mathcal{T} \subseteq \mathcal{F}$.*

The proof above give rise to a polynomial algorithm to compute a subfamily of a crossing ST-separating family that is an S-T-tree-composition.

2.1 Another proof

In this section we will give another proof to Corollary 2.9, because instead of Theorem 2.1 we could use the following result of Edmonds and Giles [2]:

Lemma 2.10. *For every cross-free family $\mathcal{F}$ on a ground set V, there exists a directed tree $T = (U, A)$, along with a map $\varphi : V \to U$ so that the sets in $\mathcal{F}$ and the edges of T are in a one-to-one correspondence, as follows. For every edge $e \in A$ the corresponding set of $\mathcal{F}$ is $\varphi^{-1}(V_e)$.*

We will modify the cross-free family $\mathcal{G}$ given in Lemma 2.8 until its tree-representation is such as wanted. We will also use the following observation:

Remark 2.11. It can be proved by induction that a directed tree $T = (U, A)$ admits a **level function** $\pi : U \to \{0, 1, 2, ...\}$ so that $\pi(v) - \pi(u) = 1$ for every $uv \in A$. One can see that if (T, φ) is the tree representation of the cross-free family $\mathcal{F}$ on the ground set V, and π is a level function of T, then $\pi(\varphi(v)) - \pi(\varphi(v')) = d_{\mathcal{F}}(v) - d_{\mathcal{F}}(v')$.

Theorem 2.12. *Let $\mathcal{F}$ be an ST-separating crossing family. Then there is an S-T-tree-composition $\mathcal{T}$ such that $\mathcal{T} \subseteq \mathcal{F}$.*

Proof. Using Lemma 2.8 we get a cross-free subfamily $\mathcal{G} \subseteq \mathcal{F}$, that is ST-separating, moreover all the elements of T are in $\Delta = |S|$ elements of $\mathcal{G}$, and all the elements of S are in less than $\Delta = |S|$ elements of $\mathcal{G}$. By Lemma 2.10 and Remark 2.11 $\mathcal{G}$ has a tree representation $(F, \varphi) = (U, A, \varphi)$, with a level function π. Let L^* and L_* denote the highest and lowest level's nodes, respectively.

If $\varphi^{-1}(u)$ is empty for a node $u \in L^*$, then $\{V_e : e \in A, e = vu\}$ is a co-partition of V. Revise $\mathcal{G}$ by removing the members of this co-partition. Since a co-partition is a regular hypergraph, the revised $\mathcal{G}$ continues to cover the elements of T Δ times and the elements of S less then Δ times for some Δ, and by this it remains to be an ST-separating family.

If $\varphi^{-1}(u)$ is empty for a node $u \in L^*$, then $\{V_e : e \in A, e = uv\}$ is a partition of V. Revise $\mathcal{G}$ by removing the members of this partition. Since a partition is a regular hypergraph, the revised $\mathcal{G}$ continues to cover the elements of T Δ times, and the elements of S less then Δ times for some Δ, and by this it remains to be an ST-separating family.

Consider now the case when $\varphi^{-1}(u)$ is non-empty and F has more than two levels. Then $d_{\mathcal{G}}(s) \leq \Delta - 2$ for every $s \in \varphi^{-1}(u) \subseteq S$ since the difference of the highest and the lowest levels is at least two. Now the members of $\mathcal{G}$ corresponding to the tree edges leaving u form a partition of $V - \varphi^{-1}(u)$. Revise again $\mathcal{G}$ by removing the members of this subpartition. The revised $\mathcal{G}$ continues to cover the elements of T Δ' times, and the elements of S less then Δ' times for $\Delta' = \Delta - 1$, because the elements of $\varphi^{-1}(u)$ were in at most $\Delta - 2$ elements of the original $\mathcal{G}$, and the other elements will be in one less element of the revised $\mathcal{G}$, then they were in the original.

In this way we arrive at a family $\mathcal{G} \subseteq \mathcal{F}$ for which the representing directed tree has two levels and no empty nodes, which means that $\mathcal{G}$ is a subfamily of $\mathcal{F}$ that is an S-T-tree-composition. □

Since the proof of Lemma 2.10 is algorithmic, the above proof also gives us a polynomial algorithm.

3 Computing the full truncation

Here we will give an algorithm to compute the full truncation of a crossing submodular function b. We will use the following well-known theorem, see for example in [1].

Theorem 3.1. *Let b be a submodular function. Then $b(Z) = \max\{\tilde{m}(Z) : m \in B(b)\}$. Let p be a supermodular function. Then $p(Z) = \min\{\tilde{m}(Z) : m \in B'(p)\}$. If b or p is integer valued, then the maximum or minimum could be taken by an integer vector.*

We give an algorithmic proof to the following theorem of [5].

Theorem 3.2. *Let b be a crossing submodular function for which $b(S) = 0$ and $B(b) \neq \emptyset$. Then*

$$b^{\downarrow}(Z) = \min\Big\{\sum_{F\in\mathcal{F}} b(F) : \mathcal{F} \text{ a tree-composition on } Z\Big\}.$$

Proof. Since a tree-composition is a special double-partition by definition, (3) implies that $b^{\downarrow}(Z) \leq \min\{\sum_{F\in\mathcal{F}} b(F) : \mathcal{F}$ is a tree-composition on $Z\}$, therefore we need to show a tree-composition for which equality holds.

$B(b) = B(b^{\downarrow})$ by definition. Theorem 3.1 implies that there is an element m of $B(b)$ for which $\tilde{m}(Z) = \sum_{z\in Z} m(z) = b^{\downarrow}(Z)$. Call a subset $X \subset S$ tight if $\tilde{m}(X) = b(X)$ and let $\mathcal{F}$ be the family of tight sets. Then $\mathcal{F}$ is a crossing system by submodularity: $\tilde{m}(X) + \tilde{m}(Y) = b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \geq \tilde{m}(X \cap Y) + \tilde{m}(X \cup Y) = \tilde{m}(X) + \tilde{m}(Y)$ whenever $X, Y \subseteq S$ are crossing.

Claim 3.3. *There exists a tight $t\bar{s}$-set for every $s \in S - Z$, $t \in Z$, so $\mathcal{F}$ is $(S - Z)Z$-separating.*

Proof. If there is an $s \in S - Z$, $t \in Z$ for which no tight $t\bar{s}$-set exists, then for $\varepsilon := \min\{b(X) - \tilde{m}(X) : X$ is an $t\bar{s}$-set$\}$ the vector m' for which $m'(s) = m(s) - \varepsilon$, $m'(t) = m(t) + \epsilon$, $m'(v) = m(v)$ $(v \in S - \{s, t\})$ would belong to $B(b)$ but would not belong to $B(b^{\downarrow})$ that is a contradiction. □

By Corollary 2.9 there is an $(S-Z)$-Z-tree-composition $\mathcal{T}$ of tight sets. Let Δ be the ground-degree of $\mathcal{T}$. Then $\sum_{X\in\mathcal{T}} b(X) = \sum_{X\in\mathcal{T}} \tilde{m}(X) = \Delta\tilde{m}(S-Z)+(\Delta+1)\tilde{m}(Z) = \Delta\tilde{m}(S) + \tilde{m}(Z) = \Delta b(S) + \tilde{m}(Z) = 0 + b^{\downarrow}(Z)$. □

With a similar proof one can get the following.

Theorem 3.4. *Let p be a crossing supermodular function for which $B'(p) \neq \emptyset$. Then*

$$p^{\uparrow}(Z) = \max\Big\{\sum_{F\in\mathcal{F}} p(F) - \Delta p(S) : \mathcal{F} \text{ a tree-composition on } Z \text{ with ground-degree } \Delta\Big\}.$$

4 Feasible submodular flows

In this section we give an algorithmic proof to the following theorem of [5] on existence of submodular flows. Now we will define the tree-composition of the ground set and the empty set also, as follows. We say that $\mathcal{F}$ is a tree-composition of V or $\emptyset$ if $\mathcal{F}$ is a partition or a co-partition of V.

Theorem 4.1. *Let b be a crossing submodular function for which $b(V) = 0$. There is an integer feasible submodular flow if and only if*

$$\rho_f(Z) - \delta_g(Z) \leq \sum_{X \in \mathcal{F}} b(X) \tag{5}$$

for every subset $Z \subseteq V$ and every tree-composition $\mathcal{F}$ of Z.

Proof. First we check whether $B(b)$ is empty or not. This can be done for example by the algorithm of [8] (see also [13]). If $B(b) = \emptyset$, then the algorithm returns a partition or a co-partition $\mathcal{F}$ of V for which $0 > \sum[b(X) : X \in \mathcal{F}]$. In this case the algorithm for testing $Q = Q(f, g; b)$ for emptiness concludes that Q is empty and returns $\mathcal{F}$ as a tree-composition of V which violates (5) since $\rho_f(V) - \delta_g(V) = 0$.

If $B(b)$ is non-empty, then the algorithm computes an element $m \in B(b)$. In this case $b^{\downarrow}$ exists and by definition $B(b) = B(b^{\downarrow})$. Obviously $Q(f, g; b) = Q(f, g; b^{\downarrow})$, therefore we need an algorithm for finding a feasible flow for a fully submodular functions. Fujishige and Zhang [10] developed such an algorithm, for a simplified version see [7]. There are two issues to solve.

First, the algorithm require a subroutine for computing $\Delta_{b^{\downarrow}-\tilde{m}}(u, v) := \min\{b^{\downarrow}(X) - \tilde{m}(X) : u \in X \subseteq V - v\}$. The next lemma shows that it suffices to have a subroutine for computing $\Delta_{b-\tilde{m}}(u, v) := \min\{b(X) - \tilde{m}(X) : u \in X \subseteq V - v\}$. This is simpler since $\Delta_{b-\tilde{m}}(u, v)$ is a function depending on b and not on its full truncation $b^{\downarrow}$. (In applications to orientation problems, the above subroutine is typically available via a max-flow min-cut computation.)

Lemma 4.2. *Let b be a crossing submodular function for which $b(V) = 0$ and m a member of $B(b)$. Then $\Delta_{b^{\downarrow}-\tilde{m}}(u, v) = \Delta_{b-\tilde{m}}(u, v)$.*

Proof. First we prove a bit different claim:

Claim 4.3. *Let h be a non-negative crossing submodular function on ground set V for which $h(V) = 0$. Then $\Delta_h = \Delta_{h^{\downarrow}}$.*

Note that $B(h)$ is non-empty since the non-negativity of h implies that the origin is in $B(h)$. Hence the full truncation of h is submodular.

Proof. Since $h^{\downarrow} \leq h$ we have $\Delta_{h^{\downarrow}} \leq \Delta_h$. Let $u, v \in V$ and let Z be a $u\bar{v}$-set for which $\Delta_{h^{\downarrow}}(u, v) = h^{\downarrow}(Z)$. By Theorem 3.2 there exists a tree-composition $\mathcal{F}$ of Z such that $h^{\downarrow}(Z) = \sum_{X \in \mathcal{F}} h(X)$. Since a tree-composition on Z is a $(V - Z)Z$-separating family there is a $u\bar{v}$-set X in $\mathcal{F}$ and by the non-negativity $h(X) \leq h^{\downarrow}(Z)$ from which $\Delta_h(u, v) \leq \Delta_{h^{\downarrow}}(u, v)$. □

We can use the above claim for $h = b - \tilde{m}$, since it holds the conditions required because $m \in B(b)$. We need to prove that $(b-\tilde{m})^{\downarrow} = b^{\downarrow} - \tilde{m}$. First we prove that $B(b-\tilde{m}) = B(b^{\downarrow} - \tilde{m})$. To this end let x be a vector with $\tilde{x}(V) = 0$. We have $x \in B(b-\tilde{m}) \Leftrightarrow x \leq b - \tilde{m} \Leftrightarrow \tilde{x} + \tilde{m} \leq b \Leftrightarrow x + m \in B(b) \Leftrightarrow x + m \in B(b^{\downarrow}) \Leftrightarrow \tilde{x} + \tilde{m} \leq b^{\downarrow} \Leftrightarrow \tilde{x} \leq b^{\downarrow} - \tilde{m} \Leftrightarrow x \in B(b^{\downarrow} - \tilde{m})$, as we wanted. As noted above $B(b-\tilde{m})$ is non-empty, so the full truncation $(b-\tilde{m})^{\downarrow}$ of $(b-\tilde{m})$ exists, it is submodular and $B(b-\tilde{m}) = B((b-\tilde{m})^{\downarrow})$ by definition. By this and the first statement we get $B((b-\tilde{m})^{\downarrow}) = B(b^{\downarrow} - \tilde{m})$. Since both function are submodular we get by Theorem 3.1 our statement. Therefore the lemma follows by the claim. □

The second problem arises when the subflow polyhedron is empty and the algorithm terminates by returning a subset Z for which $\rho_f(Z) - \delta_g(Z) > b^{\downarrow}(Z)$. Then we can use the algorithm outlined in the proof of Theorem 3.2 for computing a tree-composition $\mathcal{F}$ of Z for which $b^{\downarrow}(Z) = \sum_{X \in \mathcal{F}} b(X)$. □

5 Orientations

Theorem 4.1 implies the following theorem (see in [5]). Hence we got an algorithm to show if a given graph is orientable, as follows.

For an edge $e = uv$ and for a family $\mathcal{F}$ let $w_e(\mathcal{F})$ be the maximum of the number of $u\overline{v}$-sets and the number of the $v\overline{u}$-sets.

Theorem 5.1. *Let $G = (V, E)$ be a graph, and let $h : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing G-supermodular function for which $h(\emptyset) = 0$. G has an orientation covering h that is an orientation for which $\rho(X) \geq h(X)$ for all $X \subseteq V$, if and only if*

$$\sum_{e \in E} w_e(\mathcal{F}) \geq \sum_{i=1}^{q} h(V_i) = \tilde{h}(\mathcal{F}), \tag{6}$$

holds for all tree composition $\mathcal{F} = \{V_1, V_2, ..., V_q\}$ of any subset of V.

The strength of the above theorem is shown in [5], so here we just recall that by the above algorithm we got a simpler algorithm for the following problems. Given a mixed graph $M = (V, E, A)$. Orient E such that the resulting digraph $\overrightarrow{M} = (V, \overrightarrow{E} \cup A)$ is k-edge-connected. In the second problem let r be a fixed node of M. Then we want to orient E such that the resulting digraph is r-rooted (k, ℓ)-edge-connected.

For another application of our new algorithm to the full truncation recall the following theorem of [4]:

Theorem 5.2. *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function for which $h(\emptyset) = 0$. Suppose that G has an orientation covering h that is an orientation for which $\rho(X) \geq h(X)$ for all $X \subseteq V$, if and only if*

$$e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V_i) \text{ and } e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V - V_i) \tag{7}$$

hold for all partition $\mathcal{P} = \{V_1, V_2, ..., V_q\}$ of V.

Theorem 5.2 can be proved using Fujishige's theorem [9] on non-emptiness of the base-polyhedron and the Orientation lemma of Hakimi [11] (see in [6]). Let $p(X) = h(X) + i(X)$. It can be shown that p is crossing supermodular and that an integer vector in the base-polyhedron $B'(p)$ belongs to the in-degree vector of an orientation covering h. (With a little calculation one can see that from (7) the conditions of Fujishige's theorem follow completing the proof.)

Let $X \subseteq V$. Observe that an integer vector $x \in B'(p)$, for which $\tilde{x}(X) = \min\{\tilde{m}(X) : m \in B'(p)\}$, belongs to an in-degree vector of an orientation covering h for which $\rho(X)$ is minimal. So by Theorem 3.1 and Theorem 3.4 $\min\{\rho_{\overrightarrow{G}}(X) : \overrightarrow{G}$ covers $h\} = p^{\uparrow}(X) - i(X) = \max\big\{\sum_{F\in\mathcal{F}} p(F) - \Delta p(V) :$ $\mathcal{F}$ a tree-composition on X with ground-degree $\Delta\big\} - i(X) = \max$ $\big\{\sum_{F\in\mathcal{F}}(h(F)+i(F)) - \Delta|E| - i(X) : \mathcal{F}$ a tree-composition on X with ground-degree $\Delta\big\}$. We got the following theorem:

Theorem 5.3. *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function for which $h(\emptyset) = 0$. Suppose that G has an orientation covering h where h meets* (7)*. Let $X \subseteq V$. Then*

$$\min\Big\{\rho_{\overrightarrow{G}}(X) : \overrightarrow{G} \text{ covers } h\Big\} = \max\Big\{\sum_{F\in\mathcal{F}}(h(F) + i(F)) - \Delta|E| - i(X) :$$

$$\mathcal{F} \text{ a tree-composition on } X \text{ with ground-degree } \Delta\Big\}.$$

Since the proof was algorithmic we have got an algorithm to find an orientation of G covering h such that the in-degree of X is minimal. The only difficult part is to find an orientation covering h that uses generally the algorithm described in [8, 13]. This will be simpler for r-rooted $(k, 1)$-edge-connected orientations by the algorithm given in [12].

We can use Theorem 5.3 to the most important corollaries of Theorem 5.2. For example we can find that k-edge-connected orientation of a $2k$-edge-connected graph in which the in-degree of a given subset of nodes $X \subseteq V$ is minimal. We get the following theorem.

Theorem 5.4. *Let $G = (V, E)$ be a $2k$-edge connected graph, and let $X \subseteq V$. Then*

$$\min\Big\{\rho_{\overrightarrow{G}}(X) : \overrightarrow{G} \text{ is } k\text{-edge-connected}\Big\} = \max\Big\{\sum_{F\in\mathcal{F}} i(F) + k|\mathcal{F}| - \Delta|E| - i(X) :$$

$$\mathcal{F} \text{ a tree-composition on } X \text{ with ground-degree } \Delta\Big\}.$$

Consider the case when $G = (S, T, E)$ is a bipartite graph, $k = 1$ and $X = T$. In this case we can show that there is a minimizing S-T-tree-composition that belongs to the graph. Note that one edge of G can be induced in at most Δ elements of an S-T-tree-composition $\mathcal{F}$ and if $\mathcal{F}$ is a tree-composition that belongs to the graph, then every edge of G is induced in exactly Δ elements of $\mathcal{F}$ where Δ is the ground-degree of $\mathcal{F}$. Therefore $\sum_{F\in\mathcal{F}} i(F) - \Delta|E|$ is the sum of the deficits. Note also that $i(T) = 0$.

Assume for the minimizing S-T-tree-composition $\mathcal{F}$ and for $s \in S,\ t \in T,\ st \in E$ that the image to the representating oriented tree $F = (U = U_S \cup U_T, A)$ $\varphi(s)\varphi(t) \notin A$ and the deficit of $\mathcal{F}$ is minimal. Let P be the unoriented path of length $2t+1$ between $\varphi(s)$ and $\varphi(t)$. Shrink the set $V(P) \cap U_T$ in F, and let F' be the resulting tree and $\mathcal{F}'$ the S-T-tree-composition that belongs to F' with ground-degree Δ'. It is easy to see that $|\mathcal{F}'| = |\mathcal{F}| - t$ and $\sum_{F\in\mathcal{F}'} i(F) - \Delta'|E| \leq \sum_{F\in\mathcal{F}} i(F) - \Delta|E| - t$, since the deficit of the edge st becomes 0 from t and the deficit of the other edges does not increase. Therefore $\sum_{F\in\mathcal{F}} i(F) + |\mathcal{F}| - \Delta|E| \geq \sum_{F\in\mathcal{F}'} i(F) + |\mathcal{F}'| - \Delta'|E|$, so $\mathcal{F}'$ is also a minimizing S-T-tree-composition with smaller deficit, a contradiction. We get the following theorem.

Theorem 5.5. *Let $G = (S, T, E)$ be a 2-edge connected bipartite graph. Then*

$$\min\left\{\rho_{\overrightarrow{G}}(T) : \overrightarrow{G} \text{ strongly edge-connected}\right\} =$$

$$= \max\left\{|\mathcal{F}| : \mathcal{F} \text{ a tree-composition that belongs to } G\right\}.$$

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