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Sink-Stable Sets of Digraphs

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Abstract

We introduce the notion of sink-stable sets of a digraph and prove a min-max formula for the maximum cardinality of the union of k sink-stable sets. The results imply a recent min-max theorem of Abeledo and Atkinson [1] on the Clar number of bipartite plane graphs and a sharpening of Minty's coloring theorem [18]. We also exhibit a link to min-max results of Bessy and Thomassé [3] and of Sebő [19] on cyclic stable sets.

1 Introduction

It is well-known that the problem of finding a stable set of maximum cardinality is **NP**-complete in a general undirected graph but nicely tractable, for example, for comparability graphs. A comparability graph is the underlying undirected graph of a comparability digraph, which is, by definition, an acyclic and transitive digraph. Such a digraph can also be considered as one describing the relations between the pair of elements of a partially ordered set. A subset S of nodes of a directed graph $D = (V, A)$ is defined to be stable if S is stable in the underlying undirected graph of D. Dilworth's theorem [9] states in these terms that the maximum cardinality of a stable set of a comparability digraph is equal to the minimum number of cliques covering V. Greene and Kleitman [14] extended this result to a min-max theorem on the maximum cardinality of the union of k stable sets of a comparability digraph.

The present investigations have three apparently unrelated sources. In solving a long-standig conjecture of Gallai [13], Bessy and Thomassé [3] introduced a special type of stable sets, called cyclic stable sets, and proved a min-max result on the maximum cardinality of a cyclic stable set. They also derived a theorem on the minimum number of cyclic stable sets required to cover all nodes. These two results were unified and extended by Sebő [19] who proved a min-max formula for the the largest union of k cyclic stable sets. His theorem is an extension of the theorem of Greene and Kleitman [14]. Another source is a recent min-max result of Abeledo and Atkinson on the Clar number of plane bipartite graphs. The third source is a colouring theorem of Minty [18]. It will be shown that these remote results have, in fact, a root in common.

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To this end, we introduce and study another special kind of stable sets of an arbitrary digraph D. A **directed cut** or **dicut** of a digraph is a subset of edges entering a subset Z of nodes provided no edge leaves Z.

A node of D will be called a **sink node** (or just a sink) if it admits no leaving edges. A node is a **source node** if it admits no entering edge. A subset of nodes is a **sink set** if each of its elements is a sink. Clearly, a sink set is always stable. We say that a subset S of nodes of D is **sink-stable** if there are edge-disjoint directed cuts of D so that reorienting the edges of these dicuts S becomes a sink set. Obviously, any subset of a sink-stable set is also sink-stable. A source-stable set is defined analogously. Observe that a subset $S \subseteq V$ is sink-stable if and only if S is source-stable. Note that a node of a directed circuit never belongs to a sink-stable set since a dicut and a di-circuit are always disjoint. In the acyclic digraph with node-set $\{a, b, c, d\}$ and edge-set $\{ab,\ bc,\ cd,\ ad\}$ every single node is a one-element sink-stable set while the stable set $\{a, c\}$ is not sink-stable. Note that D is not transitive and hence D is not a comparability digraph.

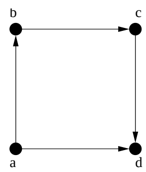

Figure 1: Every node is a sink-stable set of size 1, while the stable set $\{a, c\}$ is not sink-stable.

Proposition 1.1. *In a comparability digraph $D = (V, A)$, every stable set is sink-stable.*

Proof. Let S be a stable set. We may assume that S is maximal. Let Z denote the set of nodes of $V - S$ that can be reached along a dipath from S. We claim that no edge can leave Z. Indeed, if uv does, then v is also reachable form s and hence v must be in S. Since D is acyclic, $v \neq s$. Since D is transitiv, there is an edge $sv \in A$, contradicting the stability of S. It follows that the edges entering Z form a dicut and reorienting this dicut S becomes a sink set. •

In the sequel, we shall also use a strongly related notion of special stability. For a subset F of edges of a digraph D, a subset S of nodes is F**-stable** if S is sink-stable in the digraph D_F arising from D by reversing the elements of F. It can be checked that S is sink-stable in D if and only of S is F-stable in the digraph arising from D by adding the reverse of each edge of D where F denotes the set of these reversed edges. It will turn out that in some cases it is easier to work out a result for sink-stable sets and use then this to derive the corresponding result for F-stable sets. For example, in characterizing sink-stable and F-stable sets we shall follow this path. There are other cases when the reverse approach is more convenient. For example, we shall prove first a min-max formula for F-stable sets and use then this to derive the corresponding min-max theorem for sink-stable sets.

The property of sink-stability is in **NP** in the sense that the set of disjoint dicuts whose reorientation turns a subset S into a sink set is a fast checkable certificate for S to be sink-stable. Theorem 3.1 will describe a co-**NP** characterization for sink-stability. We shall also characterize for any integer $k \geq 2$ the union of k sink-stable sets, and as a main result, a min-max formula will be proved for the largest union of k sink-stable sets.

The result for $k = 1$ shall imply a recent min-max theorem of Abeledo and Atkinson [1] on the Clar number of a 2-connected bipartite plane graph G. Here the Clar number is defined to be the maximum number of disjoint bounded faces of G whose removal leaves a perfectly matchable graph. This notion was originally introduced in chemistry for hexagonal plane graphs to capture the behaviour of characteristic chemical and physical properties of aromatic benzenoids.

We will also derive a sharpening of Minty's colouring theorem [18] by proving a min-max formula for the minimum number of sink-stable sets to cover V, and show how this result implies a theorem of Bondy [4] stating that the chromatic number of a strongly connected digraph is at most the length of its largest directed circuit.

Finally, an interesting link will be explored to a recent min-max theorem of Bessy and Thomassé [3] on so-called cyclic stable sets of strongly connected digraphs, a result that implied a solution of a conjecture of Gallai. A min-max theorem of Sebő [19] on the largest union of k cyclic stable sets will also be a consequence.

To conclude this introductory section, we introduce some definitions and notation. For a function $m : V \to \mathbf{R}$ (or vector $m \in \mathbf{R}^V$), we define a set-function $\widetilde{m}$ by $\widetilde{m}(X) = \sum[m(v) : v \in X]$ where $X \subseteq V$. For a number x, let $x^+ := \max\{x, 0\}$. By a **multi-set** Z, we mean a collection of elements of V where an element of V may occur in more than one copy. The indicator function $\underline{\chi}_Z : V \to \{0, 1, 2, \dots\}$ of Z tells that $\underline{\chi}_Z(v)$ copies of an element v of V occurs in Z. A multiset is sometimes identified with its indicator function which is a non-negative integer-valued function on V.

Let $D = (V, A)$ be a digraph. For function $x : A \to \mathbf{R}$, the in-degree and out-degree functions ϱ_x and δ_x are defined by $\varrho_x(Z) = \sum[x(uv) : uv \in A,\ u \in V - Z,\ v \in Z]$ for $Z \subseteq V$ and by $\delta_x(Z) := \varrho_x(V - Z)$. x is a **circulation** if $\varrho_x = \delta_x$. A function $\pi : V \to \mathbf{R}$ is often called a potential. For a potential π, the **potential difference** $\Delta_\pi : A \to \mathbf{R}$ is defined by $\Delta_\pi(uv) := \pi(v) - \pi(u)$ where $uv \in A$. A function arising in this way is called a tension.

By a **topological ordering** of a digraph, we mean an ordering $v_1, \dots, v_n$ of the nodes so that every edge a of D goes forward, that is, a is of type v_iv_j where $i < j$.

A **circuit** is a connected undirected graph in which the degree of every node is 2. Typically, we use the convention for a circuit C that C also denotes the edge-set of the circuit while $V(C)$ denotes its node-set. A directed graph is also called a circuit if it arises from an undirected circuit by arbitrarily orienting its edges. Every circuit C of at least three nodes has two ways to traverse its elements. For a graph or digraph H on node-set V we fix an ordering of the elements of V. For every circuit C of H with at least three nodes, the three smallest indexed nodes of C uniquely determine a traversal direction of C called **clockwise direction** of C. When H is a digraph, the clockwise edges of C will also be called **forward** edges while the anti-clockwise edges of C are the **backward** edges.

In a digraph $D = (V, A)$, by a **walk** W we mean a sequence $(v_0, e_1, v_1, e_2, \dots, e_k, v_k)$ consisting of not necessarily distinct nodes and edges where e_i is either a $v_{i-1}v_i$-edge (called **forward** edge) or a v_iv_{i-1}-edge (called **backward** edge). If every edge is forward, we speak of a **one-way** walk. If $v_0 = v_k$, we speak of a **closed walk**. If the terms of a closed walk are distinct apart form v_0 and v_k, we speak of a simple closed walk. Therefore a simple closed walk with at least one edge can be identified with a circuit having a specified a traversal direction. Note that the closed walk consisting of a single node and no edge is not a circuit.

The number of forward and backward edges of a circuit C of a digraph are denoted by $\varphi(C)$ and $\beta(C)$, respectively, while their minimum will be called the η**-value** of C or sometimes simply the **value** of the circuit. The value of C is denoted by $\eta(C)$. When $\eta(C) = 0$, we speak of a **di-circuit**. We emphasize the difference between a circuit whose edges are just directed edges and a di-circuit. An edge of a digraph is **cyclic** if it belongs to a di-circuit. For a function $x : A \to \mathbf{R}$, $\varphi_x(C)$ denotes the sum of the x-values over the forward edges of circuit C while $\beta_x(C)$ is the sum of the x-values over the backward edges. Clearly, $\varphi_x(C) + \beta_x(C) = \widetilde{x}(C)$. For a subset B of edges, $\varphi_B(C)$ denotes the number of forward edges of C belonging to B, while $\beta_B(C)$ is the number of backward edges of C belonging to B.

A function $x : A \to \mathbf{R}$ is **conservative** if $\widetilde{c}(K) \geq 0$ for every di-circuit K. A potential π is c**-feasible** or just feasible if $\Delta_\pi \leq c$.

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2 Dicut equivalence and sink-stable sets

Lemma 2.1 (Gallai, [12]). *A cost function $c : A \to \mathbf{R}$ on the edge-set of a digraph $D = (V, A)$ is conservative if and only if there is a feasible potential. Moreover, if c is integer-valued, then π can also be selected to be integer-valued.* •

The lemma immediately implies for an integer-valued tension x that there is an integer-valued potential π for which $x = \Delta_\pi$.

Lemma 2.2. *For a subset $F \subseteq A$ of edges of a digraph $D = (V, A)$, the following are equivalent.*

(A) *F is the union of disjoint dicuts.*

(B) *$\varphi_F(C) = \beta_F(C)$ for every circuit C of D.*

(C) *There is an integer-valued potential $\pi : V \to \mathbf{Z}$ for which $\underline{\chi}_F = \Delta_\pi$.*

Proof. (A)$\to$(B) Let B be a dicut defined by a subset Z of nodes for which $\delta(Z) = 0$ and C a circuit. If we go around C clockwise, and a node $v \in V - Z$ follows a node $u \in Z$, then vu is an edge of D, while if a node $y \in Z$ follows a node $x \in Z$, then xy is an edge of D. Therefore the edges in $C \cap B$ are alternately forward and backward edges of C and hence $\varphi_B(C) = \beta_B(C)$. Consequently, $\varphi_F(C) = \beta_F(C)$ holds if F is the union of disjoint dicuts.

(B)$\to$(C) Let $x := \underline{\chi}_F$. Add the opposite edge e' of each edge e of D and define $x(e') := -x(e)$. Then (B) implies that x is conservative on the enlarged digraph. By Gallai's lemma, there is an integer-valued feasible potential π. For every edge $e = uv \in A$ and for its opposite edge $e' = vu$, we have $\pi(v) - \pi(u) \leq x(e)$ and $\pi(u) - \pi(v) \leq x(e') = -x(e)$ from which $\pi(v) - \pi(u) = x(e)$, and hence $\underline{\chi}_F = \Delta_\pi$.

(C)→(A) Let $\pi : V \to \mathbf{Z}$ be a potential for which $\underline{\chi}_F = \Delta_\pi$. We may assume that D is connected and also that the smallest value of π is zero. Let $0 = p_0 < p_1 < \cdots < p_q$ denote the distinct values of π and let $Z_i := \{v : \pi(v) \geq p_i\}$ for $i = 1, \ldots, q$. No edge uv can leave Z_i, for otherwise $\pi(v) - \pi(u) \leq -1$ but Δ_π is $(0, 1)$-valued. Let B_i denote the set of edges entering Z_i. Since Δ_π is $(0, 1)$-valued and D is connected, it follows that $p_i = i$. We claim that $F = \cup B_i$. Indeed, if $e = uv \in F$, then $\pi(v) - \pi(u) = 1$ and hence e belongs to B_i where $i = \pi(v)$ while if $e \in A - F$, then $\pi(v) - \pi(u) = 0$ and e does not belong to any B_i. •

Let $F \subseteq A$ be a subset of edges of $D = (V, A)$. We say that a cut B of D is F**-clean** if every edge of B in one direction belongs to F and every edge of B in the other direction belongs to $A - F$.

Claim 2.3. *Let F' be the symmetric difference of a subset $F \subseteq A$ and an F-clean cut B. Then $\varphi_F(C) = \varphi_{F'}(C)$ for every circuit C of D.*

Proof. If we go around C, then the edges of $C \cap B$ are alternately forward and backward edges of C. Since B is F-clean, $\varphi_F(C) = \varphi_{F'}(C)$ follows. •

Two orientations D and D' of an undirected graph are called **dicut equivalent** if D' may be obtained from D by reorienting a set of disjoint dicuts of D'. Obviously, in this case D can also be obtained from D' by reorienting disjoint dicuts of D'. The next lemma shows, among others, that dicut equivalence is an equivalence relationship.

Lemma 2.4. *Let $D = (V, A)$ and $D' = (V, A')$ be two orientations of an undirected graph G. The following are equivalent.*

(A1) *D and D' are dicut equivalent.*

(A2) *D' can be obtained from D by a sequence of dicut reorientations where each time a dicut of the current member of the sequence is reoriented. There is a sequence where the number of dicuts to be reoriented is at most $n - 1$.*

(A3) *D' can be obtained from D by a sequence of reorienting current source-nodes.* (Reorienting a source-node v means that we reorient all edges leaving v.) *There is a sequence where the number of reorientations is at most $(n - 1)^2$.*

Proof. (A1)→(A2) is immediate form the definition.

(A2)→(A1) Suppose that D' arises from D as described in (A2) and let $D_0 = D,\ D_1, \ldots, D_q = D'$ be a sequence of digraphs in which each D_i arises from D_{i-1} by reorienting a dicut B_{i-1} of D_{i-1}. Let F' denote the subset of those edges of D which are reversed in D'. We are going to show that F' is the union of disjoint dicuts of D. Let F denote the subset of those edges of D which are reversed in D_{q-1}. By induction, F is the union of disjoint dicuts of D. Let B denote the cut of D corresponding to the dicut B_{q-1} of D_{q-1}. Then B is F-clean and F' is the symmetric difference of B and F. By Claim 2.3, $\varphi_F(C) = \varphi_{F'}(C)$ holds for every circuit of D. This implies that $\beta_F(C) = \beta_{F'}(C)$ and, by Lemma 2.2, F' is the union of disjoint dicuts of D.

(A2)→(A3) It suffices to show that the reorientation of a single dicut can be obtained by a sequence of reorientations of current source-nodes. To this end, let Z be a subset of nodes so that no edge enters Z, that is, the set B of edges leaving Z is a dicut. There is a topological ordering $\{v_1, v_2, \ldots, v_n\}$ so that the nodes of Z precede the nodes outside of Z, that is, $Z = \{v_1, \ldots, v_j\}$ where $j = |Z|$. Reorient first the source-node v_1. Then v_2 becomes a source-node. Reorient now v_2 and continue in this way until the current source-node v_j gets reoriented. Since each edge induced by Z are reoriented exactly twice while the edges leaving Z are reoriented exactly once, this sequence of reorientations of current source-nodes results in a digraph that arises from D by reorienting the dicut B.

(A3)→(A2) is obvious. •

Note that an acyiclic tournament D on n nodes shows that the bound $(n-1)^2$ in Property (C) is sharp when D' is the reverse of D.

The property of dicut equivalence is in **NP** in the sense that for two orientations of G it can be certified by exhibiting the disjoint dicuts. The next result provides a **co-NP** characterization.

THEOREM 2.5. *Two orientations $D = (V, A)$ and $D' = (V, A')$ of an undirected graph G are dicut equivalent if and only if*

$$\varphi(C) = \varphi(C') \text{ for every circuit } C \text{ of } D \tag{1}$$

where C' denotes the circuit of D' corresponding to C.

Proof. Since the reorientation of a dicut does not change the number of forward edges of a circuit, $\varphi(C) = \varphi(C')$ holds if D and D' are dicut equivalent.

Conversely, suppose that $\varphi(C) = \varphi(C')$ for every pair of corresponding circuits. Let F denote the set of edges of D that are oppositely oriented in D'. Then $\varphi_F(C) = \beta_F(C)$ for every circuit C of D and Lemma 2.2 implies that F is the union of disjoint dicuts. Therefore D' and D are dicut equivalent. •

A subset L of edges of a digraph D is called **circuit-flat** or just **flat** if every cyclic edge of D belongs to a di-circuit of D containing exactly one element of L. We say that $L \subseteq A$ is a **transversal** or **covering** of di-circuits of D if L covers all di-circuits. A subset $F \subseteq A$ of edges of a digraph D is called a **flat covering** or **flat transversal** of di-circuits if F covers $\mathcal{C}$ and every cyclic edge of D belongs to a di-circuit covered exactly once by F.

Lemma 2.6 (Knuth lemma, [16]). *Every digraph $D = (V, A)$ admits a flat transversal of di-circuits.* •

Knuth formulated this result only for strongly connected digraphs but applying his version to the strong components of D, one obtains immediately the lemma. Knuth's proof is not particularly difficult but Iwata and Matsuda [15] found an even simpler proof based on the ear-decomposition of strong digraphs.

3 Characterizing the k-union of sink-stable and F-stable sets

By the k-**union** of sink-stable sets, we mean a subset U of nodes that can be partitioned into k sink-stable sets. U is also called k-**sink-stable**. A k-union of F-stable sets is defined analogously. In this section, we characterize these types of sets. We start the investigation with $k = 1$ since it behaves a bit differently from the case $k \geq 2$.

Sink-stability was introduced as an **NP**-property. The first goal of this section is to show that sink-stability is also in co-**NP**. That is, the next result provides an easily checkable tool to certify that a given stable set is *not* sink-stable. Recall that $\eta(C)$ denoted the minimum of the number of forward edges and the number of backward edges of a circuit C.

THEOREM 3.1. *Let $D = (V, A)$ be a digraph. A stable set $S \subseteq V$ is sink-stable if and only if*

$$|S \cap V(C)| \leq \eta(C) \text{ for every circuit } C \text{ of } D. \tag{2}$$

Proof. If C is a circuit and $v \in V(C)$ is a sink node of D, then one of the two edges of C entering v is a forward edge and the other one is a backward edge of C. Therefore the value $\eta(C)$ is as large as the number of sink nodes in $V(C)$. Since reorienting a dicut does not change $\eta(C)$, we conclude that every circuit C can contain at most $\eta(C)$ elements of a sink-stable set, that is, (2) is necessary.

To see sufficiency, assume the truth of (2). Let $s \in S$ be an element of S. We can assume by induction that the elements of $S - s$ are all sink nodes. If no edge enters s, then the edges leaving s form a dicut B. By reorienting B, the whole S becomes a sink set and we are done.

Therefore we can assume that at least one edge enters s. Let T denote the set of nodes u for which $us \in A$. Let D' denote an auxiliary digraph arising from D in such a way that we add to D the opposite of all edges of D entering an element of $S - s$. Let $Z \subseteq V$ denote the set of nodes reachable in D' from s. There are two cases.

Case 1 $Z \cap T \neq \emptyset$, that is, D' includes a directed path P from s to a node t in T. Now $P + ts$ is a di-circuit of D'. This di-circuit determines a circuit C of D. If we go around C in the direction of the edge st, then there are exactly $|(S - s) \cap V(C)|$ oppositely oriented edges of C and there are at least $|(S - s) \cap V(C)|$ edges in the direction of st. Since s belongs to C, we conclude that $|S \cap V(C)| \geq \eta(C) + 1$, contradicting (2).

Case 2 $Z \cap T = \emptyset$. Since no edge of D' leaves Z, no edge of D can leave Z either. In addition, no edge entering $S - s$ can enter Z, since the opposite of such an edge leaves Z and belongs to D'. Therefore the set of edges entering Z is a dicut B of D. By reorienting B, every element of $S - s$ remains a sink node. Furthermore, s becomes a source-node since the head of each edge leaving s is in Z while the tail of each edge entering s is not in Z. Finally, by reorienting the edges leaving the source-node s, s also becomes a sink node, that is, the whole S will be a sink set. •

Note that the proof can easily be turned to a polynomial algorithm that either finds a circuit C violating (2) or transforms S into a sink set by reorienting a (polynomially long) sequence of (current) dicuts, showing in this way that S is a sink-stable set.

How can one characterize k-sink-stable sets when $k \geq 2$? Before answering this question, we recall a pretty theorem of Minty [18]. By the chromatic number $\chi(D)$ of a digraph D, we simply mean the chromatic number of the underlying undirected graph. Minty provided an interesting upper bound for $\chi(D)$.

THEOREM 3.2 (Minty). *Let $D = (V, A)$ be a digraph and $k \geq 2$ an integer. If*

$$|C| \leq k\eta(C) \text{ for every circuit } C \text{ of } D, \tag{3}$$

then $\chi(D) \leq k$, that is, the node-set of D can be partitioned into k stable sets. •

The theorem shows that (3) is a sufficient condition for k-colourability. As a sharpening, we prove that (3) is actually a necessary and sufficient condition for the existence of a partition of V into k sink-stable sets. In fact, we prove a bit more.

THEOREM 3.3. *Let $D = (V, A)$ be a digraph and $k \geq 2$ an integer. A subset $S \subseteq V$ is k-sink-stable if and only if*

$$|S \cap V(C)| \leq k\eta(C) \text{ for every circuit } C \text{ of } D. \tag{4}$$

Proof. We have already observed in Theorem 3.1 that a circuit C can contain at most $\eta(K)$ elements of a sink-stable set from which the necessity of (4) follows.

To see sufficiency, consider the digraph $D^* = (V, A \cup A')$ arising from D by adding the reverse of every edge of D. Define a cost function c on $A \cup A'$ as follows. For an edge a of D, let $c(a) = k$ and for the reverse a' of a let $c(a') = 0$. For a two-element di-circuit K consisting of edges a and a' we have $|S \cap V(K)| \leq 2\widetilde{c}(K)$ holds since $k \geq 2$. Hence (4) is equivalent to the following condition.

$$|S \cap V(K)| \leq \widetilde{c}(K) \text{ for every di-circuit } K \text{ of } D^*. \tag{5}$$

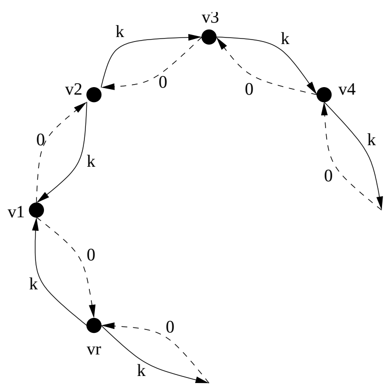

Figure 2: Graph $D^*(V, A \cup A')$. A' contains every edge in A in reversed direction. For every edge $a \in A : c(a) = k$, for every edge $a' \in A' : c(a') = 0$.

Revise now c in such a way that $c(e)$ is reduced by 1 for every edge of D^* for which the head is in S. Let c^* denote the resulting cost function. Observe that the $\widetilde{c}^*$-cost of a di-circuit K of D^* is equal to $\widetilde{c}(K)$ minus the number of edges of K having their head in

S, that is, $\widetilde{c}^*(K) = \widetilde{c}(K) - |S \cap V(K)|$. Therefore (5) is equivalent to requiring that c^* is conservative.

By Lemma 2.1 there is an integer-valued c^*-feasible potential π. Since π can be translated by a constant, we can assume that the smallest value of π is 0. Let M denote the maximum value of π and consider the following sets for $i = 0, \dots, M$.

$$P_i := \{v : \pi(v) = i\} \quad \text{and} \quad U_i := P_0 \cup P_1 \cup \dots \cup P_i.$$

Moreover, define for $j = 0, \dots, k-1$ the following sets.

$$V_j := \{v : \pi(v) \equiv j \bmod k\} \quad \text{and} \quad S_j := V_j \cap S.$$

For each $uv \in A$, we have $\pi(v) \geq \pi(u)$ since $c^*(vu) \leq 0$ from which $\pi(u) - \pi(v) \leq c^*(vu) \leq 0$. Therefore no edge of D enters any U_i, that is, the set B_i of edges of D leaving U_i is a dicut of D. Obviously, the sets S_j partition S. We are going to prove that each S_j is a sink-stable set from which the theorem will follow. To this end, consider the dicuts $B_j, B_{j+k}, B_{j+2k}, \dots$. These are disjoint since $\pi(v) - \pi(u) \leq c^*(uv) \leq k$ holds for each edge $uv \in A$.

Let $z \in S_j$. For any edge $uz \in A$ entering z, we have $\pi(z) - \pi(u) \leq c^*(uz) = k - 1$ and hence uz is not in any of the dicuts $B_j, B_{j+k}, B_{j+2k}, \dots$. For any edge $zv \in A$ leaving z, we have $c^*(vz) = -1$ from which $\pi(z) - \pi(v) \leq c^*(vz) = -1$ and hence $\pi(v) - \pi(z) \geq 1$. Therefore zv belongs to one of the dicuts $B_j, B_{j+k}, B_{j+2k}, \dots$. Consequently, each node $z \in S_j$ is a sink node in D_B where B is the union of the dicuts $B_j, B_{j+k}, B_{j+2k}, \dots$ and D_B denotes the digraph arising from D by reversing B. •

It is known that there is a polynomial time algorithm for an arbitrary cost function c that either finds a c-feasible potential or finds a negative di-circuit. Therefore the proof of Theorem 3.3 above gives rise to an algorithm that either finds a partition of S into k sink-stable sets or finds a circuit C of D violating (4).

Remark It is useful to observe that for $k = 1$ the statement in Theorem 3.3 fails to hold: in a digraph consisting of two nodes and a single edge, (4) holds automatically since there is no circuit at all but V is not a stable set. This is why we assumed a priori in Theorem 3.1 that S is a stable set. We also remark that the proof technique of Theorem 3.3 can be used for $k = 1$, as well, to obtain an alternative proof for the non-trivial direction of Theorem 3.1 since in the latter case the stability of S is part of the assumption.

Next, we describe a characterization for the union of k F-stable sets. Since F-stability was defined through sink-stability, it is a straightforward task to translate the theorems above on characterizing k-unions of sink-stable sets to those on characterizing k-unions of F-stable sets. The resulting necessary and sufficient condition is a certain inequality required to hold for *all* circuits of the digraph. In the applications, however, the digraph in question is strongly connected, and in this case it turns out that it suffices to require an inequality to hold only for every di-circuit.

For this simplification, we shall need a lemma. Let $F \subseteq A^*$ be a subset of edges of a digraph $D^* = (V, A^*)$. For a closed walk W, let

$$\sigma_F(W) := \varphi_F(W) + \beta_{A^*-F}(W)$$

where $\varphi_F(W)$ and $\beta_{A^*-F}(W)$ denote the number of forward F-edges and the backward $(A^* - F)$-edges, respectively. Therefore $\sigma_F(W) = |F \cap W|$ for a one-way walk W and, in particular, for a di-circuit. Recall that F was called flat if every cyclic edge of D^* belongs to a di-circuit covered by exactly one element of F.

Lemma 3.4. *If $F \subseteq A^*$ is a flat subset of a digraph $D^* = (V, A^*)$, then the node-set $V(C)$ of every circuit C of D^* consisting of cyclic edges can be covered by di-circuits $K_1, \dots, K_q$ for which $\sum_i |F \cap K_i| = \sum_i \sigma_F(K_i) \leq \sigma_F(C)$.*

Proof. It is more convenient to prove the more general statement asserting that the node-set of a closed walk W consisting of cyclic edges of D^* can be covered by di-circuits $K_1, \dots, K_q$ for which $\sum_i |F \cap K_i| = \sum_i \sigma_F(K_i) \leq \sigma_F(W)$. We use induction on the number of backward edges of W. If this number is zero, that is if W is a one-way walk, then by traversing all the edges of W, we obtain di-circuits $K_1, \dots, K_q$ for which $\sum_i |F \cap K_i| = \sum_i \sigma_F(K_i) = \sigma_F(C)$.

Suppose now that $e = uv \in A^*$ is a backward edge of W. By the hypothesis, e belongs to a di-circuit K containing one F-edge, that is, there is a directed path P from v to u. Replace e in the walk W by P and let W' denote the closed walk obtained in this way. Obviously $V(W) \subseteq V(W')$ and W' has one less backward edge than W. We claim that $\sigma_F(W') \leq \sigma_F(W)$. Indeed, if $e \in F$, then e contributes to $\sigma_F(W)$ by zero and P contains no F-edge from which $\sigma_F(W') = \sigma_F(W)$. If $e \in A^* - F$, then e contributes to $\sigma_F(W)$ by 1. Furthermore, since P contains one F-edge, we confer $\sigma_F(W') = \sigma_F(W)$ from which the lemma follows by induction. •

THEOREM 3.5. *Let $F \subseteq A^*$ be a flat subset of edges of a strongly connected digraph $D^* = (V, A^*)$ and let $k \geq 1$ be an integer. A subset $S \subseteq V$ is the union of k F-stable sets if and only if*

$$|S \cap V(K)| \leq k|F \cap K| \textit{ for every di-circuit } K \textit{ of } D^*. \tag{6}$$

In particular, if F is a flat transversal of di-circuits, then the minimum number of F-stable sets covering S is equal to

$$\max \left\{ \left\lceil \frac{|S \cap V(K)|}{|F \cap K|} \right\rceil : K \textit{ a di-circuit of } D^* \right\}. \tag{7}$$

Proof. Let $D' = D^*_F$ denote the digraph arising from D^* by reversing F. Suppose that S is the k-union of F-stable sets, that is, S is k-sink-stable in D'. Let K be a di-circuit of D^* and let K' denote the corresponding circuit of D'. Then $|S \cap V(K)| \leq k\eta(K') \leq k|F \cap K|$, from which the necessity of (6) follows.

Suppose now that (6) holds. For every set X of edges of D^*, the corresponding set in D' will be denoted by X'.

Claim 3.6. $|S \cap V(C')| \leq k\eta(C')$ *for every circuit C' of D'.*

Proof. We may assume that $\eta(C') = \beta(C') \leq \varphi(C')$. Let C be the circuit of D^* corresponding to C'. By applying Lemma 3.4 to D^* and to C, we obtain that $V(C) = V(C')$ can be covered by di-circuits $K_1, \dots, K_q$ of D^* for which $\sum_i |F \cap K_i| = \sum_i \sigma_F(K_i) \leq \sigma_F(C)$.

By applying (6) to di-circuits K_i, we see that $|S \cap V(K_i)| \leq k|F \cap K_i|$. Hence $|S \cap V(C')| = |S \cap V(C)| \leq \sum_i |S \cap V(K_i)| \leq \sum_i k|F \cap K_i| \leq k\sigma_F(C) = k[\varphi_F(C) + \beta_{A^*-F}(C)] = k\beta(C') = k\eta(C')$, as required. •

If $k = 1$, then every edge of D^* belongs to a di-circuit K for which $|F \cap K| = 1$ and hence S is stable by (6). Theorem 3.1, when applied to D', implies that S is sink-stable in D', that is, S is F-stable in D^*. If $k \geq 2$, then we can apply Theorem 3.3 to D'. By Claim 3.6 above, S is k-sink-stable in $D' = D^*_F$, showing that S is the k-union of F-stable sets in D^*. • •

Note that, unlike the corresponding situation with sink-stable sets where we formulated the characterization of k-unions of sink-stable sets separately for $k = 1$ and for $k \geq 2$, in the formulation of Theorem 3.5 these cases are not separated. It is the proof of the theorem where the two cases were handled separately.

Corollary 3.7 (Bondy, [4])**.** *The chromatic number $\chi(D)$ of a strongly connected digraph $D^* = (V, A^*)$ is at most the length of the longest di-circuit of D^*.*

Proof. Let F be a flat transversal of di-circuits ensured by Lemma 2.6. By applying Theorem 3.5 to $S := V$, we confer that $\chi(D) \leq$ the minimum number of F-stable sets covering $V = \max\{\lceil \frac{|K|}{|F \cap K|} \rceil : K$ a di-circuit of $D^*\} \leq \max\{|K| : K$ a di-circuit of $D^*\}$, as required. •

In the last section, we will point out a link between F-stable sets and so-called cyclic stable sets introduced by Bessy and Thomassé [3]. It was their paper that first showed how Bondy's theorem follows from results on cyclic stable sets.

Remark We showed above how Theorems 3.1 and 3.3 gave rise to Theorem 3.5. But the reverse derivation is also possible. To obtain, for example, the non-trivial sufficiency part of Theorem 3.3, suppose that (4) holds. Then every di-circuit of D is disjoint from S. Let F denote the set of reverse edges of D and consider the strong digraph $D^+ = (V, A + F)$. Then F is clearly flat since each original edge of D and its reverse edge form a di-circuit covered once by F.

Let K be a di-circuit of D^+. We are going to show that (6) holds. If K consists of edges of D, that is, if $F \cap K = \emptyset$, then S is disjoint from $V(K)$ and hence $|S \cap V(K)| = 0 \leq k|F \cap K|$. Suppose now that $F \cap K \neq \emptyset$. If $|K| = 2$, then $|K| = 2 \leq k|F \cap K|$. Finally, if $|K| \geq 3$, then K determines a circuit C of D so that the backward edges of C correspond to the elements of $F \cap K$. Since $|S \cap V(C)| \leq k\eta(C)$ by (4), we obtain that $|S \cap V(K)| \leq k|F \cap K|$.

By applying Theorem 3.5 to D^* in place of D, we obtain that S is the union of k F-stable sets. By definition a set Z is F-stable if S is sink-stable in D^+_F. Since D^+_F arises from D by duplicating each edge in parallel, Z is sink-stable in D, as well, and hence S is indeed the union of k sink-stable sets.

Theorem 3.1 follows from the special case $k = 1$ of Theorem 3.5 in an analogous (and even simpler) way.

4 Optimal sink-stable and F-stable sets

Our next goal is to investigate sink-stable and F-stable sets of largest cardinality and, more generally, of maximum weight. The main device to obtain min-max theorems for these parameters is a result of Gallai.

A theorem of Gallai

Let $D = (V, A)$ be a digraph and $c : A \to \mathbf{Z}_+$ a non-negative integer-valued function. The c**-value** of a circuit C is the sum of the c-values of the edges of C, that is, $\widetilde{c}(C)$. We say that a multiset of nodes is c**-independent** if it contains at most $\widetilde{c}(K)$ nodes of every di-circuit K of D. A multiset can be identified with a non-negative integral vector $x : V \to \mathbf{Z}_+$ and then the c-independence of x means that $\widetilde{x}(V(K)) \leq \widetilde{c}(K)$ for every di-circuit K.

Let $w : V \to \mathbf{Z}_+$ be a weight function. For a function $y \geq 0$ defined on the set of di-circuits of D, we say that y **covers** w if

$$\sum[y(K) : v \in V(K), K \text{ a di-circuit}] \geq w(v) \text{ for every } v \in V. \tag{8}$$

A circulation $z \geq 0$ is said to cover w if $\varrho_z(v) \geq w(v)$ holds for every node $v \in V$. The following lemma describes a simple and well-known relationship between circulations and families of di-circuits covering w.

Lemma 4.1. *If $y \geq 0$ is a function on the set of di-circuits covering w, then the function $z : A \to \mathbf{Z}_+$ defined by $z(e) := \sum[y(K) : K$ a di-circuit and $e \in K]$ is a non-negative circulation covering w for which $cz = \sum[y(K)\widetilde{c}(K) : K$ a di-circuit$]$. Furthermore, if y is integer-valued, then so is z. Conversely, a circulation $z \geq 0$ covering w can be expressed as a non-negative linear combination of di-circuits, and if $z = \sum y(K)\underline{\chi}(K)$ is such an expression, then y covers w and $cz = \sum[y(K)\widetilde{c}(K) : K$ a di-circuit$]$. Furthermore, if z is integer-valued, then y can also be chosen integer-valued.* •

The following result of Gallai [12] appeared in 1958. We cite it in its original form because the literature does not seem to know about it.

(3.2.7) SATZ. *Ist Γ endlich und gilt $\psi[k] \geq 0$ für jeden positiven Kreis k gibt ist ferner zu jedem Punkt X mit $\varphi(X) > 0$ einen positiven Kreis, der X enthält, so ist das Minimum der ψ-Werte der punktfüllende positiven Kreissysteme gleich dem Maximum der φ-Werte der kreisaufnehmbaren Punktsysteme.*

That is: *If Γ is finite and $\psi[k] \geq 0$ holds for every positive circuit, and if, furthermore, every point X with $\varphi(X) > 0$ is included in a positive circuit, then the minimum ψ-value of point-covering positive circuit-systems is equal to the maximum φ-value of circuit-independent point-systems.*

Here ψ and φ are integer-valued functions on the edge-set and on the node-set, respectively, of the digraph Γ, and a positive circuit means a di-circuit. In the present context, we use functions c and w in place of ψ and φ, respectively, and the theorem can be formulated as follows.

THEOREM 4.2 (Gallai, Theorem (3.2.7) in [12]). *Let $c : A \to \mathbf{Z}_+$ and $w : V \to \mathbf{Z}_+$ be non-negative functions on the edge-set and on the node-set, respectively, of a digraph $D = (V, A)$, and assume that for each $v \in V$ with $w(v) > 0$ that v belongs to a di-circuit. Then the minimum total sum of c-values of a system of di-circuits covering w is equal to the maximum w-weight of a c-independent multiset of nodes of D, or more formally, to*

$$\max\{wz : z \in \mathbf{Z}_+^V,\ z \textit{ c-independent}\}. \tag{9}$$

Note that in the original version cited above only the conservativeness of c $(= \psi)$ was assumed and not its non-negativity. But for a conservative c there is a feasible potential π and then the cost function c_π defined by $c_\pi(uv) = c(uv) - \pi(v) + \pi(u)$ is non-negative for which $\widetilde{c}(K) = \widetilde{c}_\pi(K)$ holds for every di-circuit K. That is, the theorem for conservative c follows from its special case for non-negative c.

In the special case of Theorem 4.2, when w is $(0, 1)$-valued, that is, when $w := \underline{\chi}_U$ for a subset $U \subseteq V$ of nodes, we have the following.

Corollary 4.3. *Let $U \subseteq V$ be a specified subset of nodes belonging to some di-circuits. The minimum total c-value of di-circuits covering U is equal to the maximum number of* (not-necessarily distinct) *c-independent elements of U.* •

For completeness, we outline a proof of Theorem 4.2. Let Q denote the di-circuits versus nodes incidence matrix of a digraph D. That is, Q is a $(0, 1)$-matrix with rows corresponding to the di-circuits and columns corresponding to nodes. An entry corresponding to a di-circuit C and a node v is 1 or zero according to whether v is in $V(C)$ or not. A fundamental theorem of Edmonds and Giles [10] states that a polyhedron R is integral provided that R is described by a totally dually integral system and both the constraint matrix and the bounding vector is integral. Combined with the l.p. duality theorem, the following result is equivalent to Theorem 4.2.

THEOREM 4.4. *Let Q be the di-circuits versus node incidence matrix of a digraph D. Let $\widetilde{c}$ denote a vector the components of which correspond to the rows of Q* (that is, to the di-circuits of D) *and the value of a component corresponding to a di-circuit K is $\widetilde{c}(K)$. Then the linear system*

$$\{Qx \leq \widetilde{c},\ x \geq 0\} \tag{10}$$

is totally dual integral.

Proof. Let w be an integer-valued function on V. Consider the following linear program:

$$\min\{\textstyle\sum[y(K)\widetilde{c}(K) : K \text{ a di-circuit}] : yQ \geq w,\ y \geq 0\}. \tag{11}$$

What we have to show is that this program has an integral-valued optimum if it has an optimum at all. We may assume that w is non-negative. We may also assume that each $v \in V$ with $w(v) > 0$ belongs to a di-circuit for otherwise (11) has no feasible solution at all.

By Lemma 4.1, it suffices to show that the linear system $\min\{cz : z \geq 0$ a circulation covering $w\}$ has an integer-valued optimum. But this follows from the integrality of the circulation polyhedron by appplying the standard node-duplicating technique. Indeed, replace each node v of D by nodes v' and v'', replace each edge $uv \in A$ by a new edge $u'v''$ (with lower capacity 0 and cost $c(uv)$), and finally add a new edge $v''v'$ (with lower capacity $w(v)$ and cost 0) for every original node $v \in V$. In the resulting digraph D', a feasible circulation z' defines a non-negative circulation z of D which covers w (that is, $\varrho_z(v) \geq w(v)$ for $v \in V$) and $c'z' = cz$. •

With some work, this proof can be used to turn a min-cost circulation algorithm to one that computes the optima in Theorem 4.2 in polynomial time.

On a solved conjecture of Gallai

Before turning to F-stable sets, we make a detour and show how Gallai's theorem from 1958 and and Knuth's lemma from 1974 imply immediately the following conjecture of Gallai [13] that was first proved by Bessy and Thomassé [3]. Recall that $\alpha(D)$ denotes the stability number of D while $\gamma(D)$ denotes the minimum number of di-circuits fo D covering V.

THEOREM 4.5 (Bessy and Thomassé)**.** *Let $D = (V, A)$ be a strongly connected digraph with at least two nodes. Then $\gamma(F) \leq \alpha(D)$, that is, V can be covered by $\alpha(D)$ di-circuits.*

Proof. By Lemma 2.6 D has a flat covering F of di-circuits. By applying Corollary 4.3 to $U := V$ and to $c := \underline{\chi}_F$, we obtain that the minimum total c-weight γ_F of di-circuits covering V is equal to the maximum number α_F of c-independent elements of V. Since F covers every di-circuit K, the c-weight of K is at least 1 and hence $\gamma \leq \gamma_F$. Since F is a flat covering of di-circuits, every node belongs to a di-circuit of c-cost 1. Therefore a c-independent multiset S is actually a set. The c-independence also implies that S is a stable set. Hence $\gamma(D) \leq \gamma_F = |S| \leq \alpha(D)$. •

Note that the same argument shows that the following extension also holds.

THEOREM 4.6. *Let $D = (V, A)$ be a strongly connected digraph and U a subset of nodes of D. Then U can be covered by α_U di-circuits where α_U denotes the maximum cardinality of a stable subset of U.* •

We note that Cameron and Edmonds [5] (not knowing of the paper of Gallai [12]) proved an extension of Theorem 4.4 asserting that the linear system $\{Qx \leq \widetilde{c}\}$ is actually box-TDI (see, Theorem 6.1 below). Based on this, they derived Theorem 4.5 in [6].

Optimal F-stable and sink-stable sets

Let F be a flat subset of edges of a strongly connected digraph $D^* = (V, A^*)$. Here F is not necessarily a transversal of di-circuits. Let Q denote the di-circuit versus node incidence matrix of D^* and let c_F be a vector whose components correspond to the di-circuits of D^* and $c_F(K)$ is the F-value $|F \cap K|$ of K for a di-circuit K.

THEOREM 4.7. *Let F be a flat subset of edges of a strongly connected digraph $D^* = (V, A^*)$ and let $w : V \to \mathbf{Z}_+$ be an integer-valued weight-function on the node-set of D^*. The maximum w-weight of an F-stable set of D^* is equal to the minimum total F-value of di-circuits of D^* covering w. For a given subset $U \subseteq V$, the maximum cardinality of an F-stable subset of U is equal to the the minimum total F-value of di-circuits of D^* covering U.*

Proof. Apply Gallai's theorem to c_F. Since every edge of D^* belongs to a di-circuit K for which $c_F(K) = 1$, a c_F-independent multiset of nodes of D^* is actually a set $S \subseteq V$, and by Theorem 3.5 S is F-stable. Therefore the result is a direct consequence of Theorem 4.2. The second half of the theorem follows by applying the the first one to $w := \underline{\chi}_U$. •

THEOREM 4.8. *Let $D = (V, A)$ be a digraph with no isolated nodes and let $w : V \to \mathbf{Z}_+$ be an integer-valued weight-function on the node-set of D. The maximum w-weight of a sink-stable set of D is equal to the minimum total value of circuits and edges of D covering w where the value of a circuit C is $\eta(C)$ while the value of an edge is 1. For a given subset $U \subseteq V$, the maximum cardinality of a sink-stable subset of U is equal to the minimum total value of circuits and edges of D covering U.*

Proof. For a sink-stable subset S, an edge can cover at most one element of S. In Theorem 3.1 we already observed that a circuit C can cover at most $\eta(C)$ elements of a sink-stable set from which $\max \leq \min$ follows.

The proof of the reverse direction $\max \geq \min$ can be made separately for the components of D, and hence we can assume that D is weakly connected. Let $D^* = (V, A \cup A')$ be the digraph arising from D by adding the reverse of each edge of D. Here A' denotes the set of reverse edges of D. D^* is clearly strongly connected and $F := A'$ is flat since each edge $e \in A$ and its reverse $e' \in A'$ form a 2-element di-circuit covered once by F.

There are two types of di-circuits of D^*. Type I is of form $K = \{e, e'\}$ where $e = uv \in A$ and $e' = vu \in A'$, and in this case $\widetilde{c}_F(K) = 1$. A Type II di-circuit K arises from a circuit C of D by reversing its forward edges or by reversing its backward edges. Therefore if K is such a di-circuit of D^*, then the reverse $\vec{K}$ of K is also a di-circuit of D^*, and $\eta(C) = \min\{\widetilde{c}_F(K),\ \widetilde{c}_F(\vec{K})\}$. For notational convenience, we will assume that $\eta(C) = \widetilde{c}_F(K)$.

Let S be a sink-stable set of D with maximum w-weight. Since D^*_F is a digraph that can be obtained from D by doubling each edge of D in parallel, a subset Z of nodes is sink-stable in D if and only of Z is F-stable in D^*. Therefore S is an F-stable set in D^* of maximum w-weight.

In order to prove $\max \geq \min$, we are going to show that there is a family of circuits and edges of D covering w for which the total value is $\widetilde{w}(S)$. Since S is a maximum w-weight F-stable set, Theorem 4.7 implies the existence of a family $\mathcal{C}^*$ of di-circuits of D^* covering w for which the total F-value is $w(S)$. As mentioned above, a di-circuit in D^* of Type I determines an edge of D while a di-circuit K of Type II determines a circuit C of D for which $\eta(C) = c_F(K)$. Therefore $\mathcal{C}^*$ defines a family of edges and circuits of D covering w for which the total value is the total c_F-value of $\mathcal{C}^*$, that is $\widetilde{w}(S)$, and hence the requested direction $\max \geq \min$ follows.

The second half of the theorem follows by applying the the first one to $w := \underline{\chi}_U$. •

Remark One may be wondering whether the minimal covering of U in Theorem 4.8 can perhaps be realized only by circuits, without using edges. The following example shows, however, that the use of edges is anavoidable. Let $U := V := \{a, b, c, d, e\}$ and let the edges of D be $\{ab, ac, ad, eb, ec, ed\}$. In this digraph $S = \{b, c, d\}$ is a largest sink-stable set. On the other hand, each circuit C of D has 4 edges and $\eta(C) = 2$. Therefore the total value of the best covering of V by only circuit is 4. An optimal covering consists of a circuit with edge-set $\{ab, be, ec, ca\}$ and of an edge ad with total value 3.

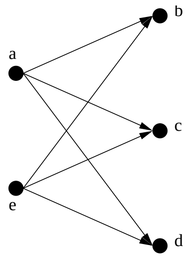

Figure 3: The largest sink-stable set in this graph is $S = \{b, c, d\}$. An optimal covering in this graph consist of a circuit $C = \{ab, be, ec, ca\}$ and an edge ad, the total value of this covering is 3.

5 Clar number of plane bipartite graphs

As an application of Theorem 4.7, we derive a recent min-max theorem of Abeledo and Atkinson [1] on the Clar number of bipartite plane graphs. Let $G = (S, T; E)$ be a perfectly matchable 2-connected bipartite plane graph. The expression plane graph means that G is planar and we consider a fixed embedding in the plane. Note that the embedding subdivides the plane into regions, among them exactly one is unbounded. The bounded regions will be referred to as faces of G. Since G is 2-connected, each region is bounded by a circuit of G.

We call a set of faces **resonant** if the faces are disjoint and their deletion leaves a perfectly matchable graph. Here the deletion of a face means that we delete all the nodes of the circuit bounding the face. The **Clar number** of G is defined to be the maximum cardinality of a resonant set of faces. For example, if G is the graph of a cube, then the Clar number is 2, independently of the embedding. It is not difficult to find an example where the Clar number does depend on the embedding. This notion was originally introduced in chemistry by E. Clar [8] for hexagonal graphs (where each face is a 6-circuit) to capture the behaviour of characteristic chemical and physical properties of aromatic benzenoids.

Before stating the result of Abeledo and Atkinson on the Clar number, we introduce some notation that will be used in the theorem and also in its proof. With the bipartite graph $G = (S, T; E)$, we associate a digraph $D = (V, A)$, where $V = S \cup T$, arising from G by orienting all edges from S to T. Clearly, D is acyclic. A subset of edges of D corresponding to a subset X of edges of G will be denoted by $\vec{X}$. For a digraph H and subset F of its edges, H_F denotes the digraph arising from H by reversing F. For a subset $Z \subseteq V$, the set $B = \Delta(Z)$ of edges connecting Z and $V - Z$ will be called a cut of G determined by Z. We call such a cut of G **feasible** if it determines a dicut in the associated digraph D. The **value** val(B) of B is defined to be the absolute value of $|S \cap Z| - |S \cap T|$. It is an easy exercise to see that val$(B) = d_M(Z) = |M \cap B|$ for an arbitrary perfect matching M of G. In particular, this means for a feasible cut $\Delta(Z)$ that $d_M(Z)$ is independent of the choice of perfect matching M.

Let $G^* = (V^*, E^*)$ denote the planar dual of G. There is a one-to-one correspondence between the regions of G and the nodes of G^* and also there is a one-to-one correspondence between the edges of G and the edges of G^*. We use the convention that for a subset $X \subseteq E$ of edges, the corresponding subset of edges of G^* is denoted by X^*. It is well-known that X is a circuit of G if and only if X^* is a bond of G^*. (A bond of a graph is a minimal cut. A useful property is that a cut $B = \Delta(Z)$ of a connected graph is a bond if and only if both Z and $V - Z$ induce a connected subgraph, and another one is that every cut can be partitioned into bonds.)

We also need the planar dual digraph $D^* = (V^*, A^*)$ of D. This is an orientation of G^* in such a way that, for a pair $e \in E$ and $e^* \in E^*$ of corresponding edges, if the directed edge $\vec{e}$ of D is represented in the plane by a vertical line segment oriented downward, then the corresponding horizontal dual edge $\vec{e^*}$ of D^* is oriented from right to left. It follows that a subset X of edges of D is a minimal dicut if and only if the corresponding subset X^* of edges of D^* is a di-circuit. Similarly, a subset X of edges of $D_{\vec{M}}$ is a di-circuit if and only if the corresponding set X^* of edges of $D^*_{\vec{M}^*}$ is a minimal dicut of $D^*_{\vec{M}^*}$. In particular, the circuit bounding a region of $D_{\vec{M}}$ is a di-circuit oriented clockwise if and only if the corresponding node of $D^*_{\vec{M}^*}$ is a sink node. Since D is acyclic, D^* is strongly connected. Since each node in T determines a dicut of D, each edge of D belongs to a dicut covered exactly once by $\vec{M}$. Therefore $\vec{M}^*$ is flat in D^*. Note, however, that $\vec{M}$ need not cover all dicuts of D and hence $\vec{M}^*$ is not necessarily a transversal of di-circuits of D^*.

THEOREM 5.1 (Abeledo and Atkinson)**.** *Let $G = (S, T; E)$ be a 2-connected perfectly matchable plane bipartite graph. The Clar number of a G is equal to the minimum total value of feasible cuts intersecting all faces of G.*

Proof. Let M be an arbitrary perfect matching of G. Consider the digraph D^*_F which is the planar dual digraph of $D_{\vec{M}}$, where $F := \vec{M}^*$.

Lemma 5.2. *A set $\mathcal{S}$ of disjoint faces of G is resonant if and only if the corresponding set S of nodes of D^* is F-stable.*

Proof. By definition, $\mathcal{S}$ is resonant if there is a perfect matching M' of G so that the bounding circuit of each member of $\mathcal{S}$ is M'-alternating. This is equivalent to requiring that these bounding circuits are directed circuits in $D_{\vec{M}'}$. By reorienting such a di-circuit if necessary, we can assume that the bounding circuits of the members of $\mathcal{S}$ are clockwise oriented di-circuits in $D_{\vec{M}'}$. Since the symmetric difference of two perfect matchings consists of disjoint alternating circuits, $D_{\vec{M}'}$ arises from D_M by reorienting disjoint di-circuits of D_M. Therefore $\mathcal{S}$ is resonant if and only if it is possible to reorient disjoint di-circuits of $D_{\vec{M}}$ so that the members of $\mathcal{S}$ will be bounded by clockwise oriented di-circuits. This is, in turn, equivalent to requiring that it is possible to reorient disjoint dicuts of $D^*_{\vec{M}^*}$ so that the members of S becomes sink nodes, that is, S is an F-stable set of D^*. •

By Lemma 5.2, Theorem 4.7 when applied to $D^* = (V^*, A^*)$ and to $F := \vec{M}^*$ implies the theorem. • •

In the last section, we extend the theorem of Abeledo and Atkinson by deriving a min-max formula, for the maximum number of faces that can be partitioned into k resonant sets.

6 Optimal k-union of sink-stable and F-stable sets

In the preceding section, a result of Gallai was used to prove a min-max formula for the maximum w-weight of the k-union of sink-stable sets and F-stable sets for the special case

$k = 1$. Now we solve the case $k \geq 2$ with the help of an extension of Theorem 4.2, due to Cameron and Edmonds [5].

THEOREM 6.1 (Cameron and Edmonds)**.** *Let Q be the di-circuits versus nodes incidence matrix of a digraph D. Let $f : V \to \mathbf{Z}_+ \cup \{-\infty\}$ and $g : V \to \mathbf{Z}_+ \cup \{\infty\}$ be functions for which $f \leq g$. The linear system $\{Qx \leq \widetilde{c},\ f \leq x \leq g\}$ is TDI.* •

In the special case when $f \equiv 0$ and $g \equiv 1$, Theorem 6.1 and the l.p. duality theorem immediately gives rise to the following min-max formula.

THEOREM 6.2. *Let $D^* = (V, A^*)$ be a digraph in which every node belongs to a di-circuit and let $\mathcal{K}^*$ denote the set of di-circuits of D^*. Let $w : V \to \mathbf{Z}_+$ and $c : A^* \to \mathbf{Z}_+$ be functions. Then*

$$\max\{\widetilde{w}(S) : S \subseteq V,\ |S \cap V(K)| \leq \widetilde{c}(K) \textit{ for every } K \in \mathcal{K}^*\} = \tag{12}$$

$$\min_{y:\mathcal{K}^* \to \mathbf{Z}_+} \left\{ \sum_{K \in \mathcal{K}^*} y(K)\widetilde{c}(K) + \sum_{v \in V} \left(w(v) - \sum_{K \in \mathcal{K}^*, v \in V(K)} y(K) \right)^+ \right\}. \bullet \tag{13}$$

We apply this result in the special case when c is the indicator function of a flat subset of edges.

THEOREM 6.3. *Let F be a flat subset of edges of a strongly connected digraph $D^* = (V, A^*)$ with $|V| \geq 2$ and let $\mathcal{K}^*$ denote the set of di-circuits of D^*. Let $k \geq 2$ be an integer and $w : V \to \mathbf{Z}_+$ an integer-valued weight-function on the node-set of D^*. The maximum w-weight of the k-union of F-stable sets of D^* is equal to*

$$\min_{y:\mathcal{K}^* \to \mathbf{Z}_+} \left\{ k \sum_{K \in \mathcal{K}^*} y(K)|F \cap K| + \sum_{v \in V} \left(w(v) - \sum_{K \in \mathcal{K}^*, v \in V(K)} y(K) \right)^+ \right\}. \tag{14}$$

In particular, for a specified subset U of nodes, the maximum cardinality of the k-union of F-stable subsets of U is equal to

$$\min_{\mathcal{K} \subseteq \mathcal{K}^*} \left\{ k \sum_{K \in \mathcal{K}} |F \cap K| + |U - \cup(V(K) : K \in \mathcal{K})| \right\}. \tag{15}$$

Proof. Apply Theorem 6.2 to $c' := k\underline{\chi}_F$ and observe that (13) transforms to (14). By Theorem 3.5, a subset S of nodes satisfies the properties in (12) for c' if and only if S is the k-union of F-stable sets. Hence the first part is a consequence of Theorem 6.2. The second part follows by applying the first one in the special case $w := \underline{\chi}_U$. •

In the special case when F is a flat transversal of di-circuits, Theorem 6.3 is equivalent to a min-max result of Sebő [19] concerning the maximum weight of k-unions of cyclic-stable sets. Cyclic stability was introduced by Bessy and Thomasse [3] who proved a min-max result on the maximum cardinality of a cyclic stable set. See the last section for details.

Corollary 6.4 (Greene and Kleitman, [14]). *In a transitive and acyclic digraph $D' = (U, A')$ the maximum cardinality of the union of k stable sets is eqaul to* $\min\{k \sum_i |V(P_i)| + |U - \cup_i V(P_i)| : \{P_1, \dots, P_q\}$ *a set of disjoint di-paths*$\}$.

Proof. We prove only the non trivial direction $\max \geq \min$. Extend D' by a new node z, add a pair of opposite edges zu and uz for every $u \in U$. Let $D = (V, A)$ denote the resulting digraph and let F be the set of edges entering z. Then F is a flat transversal of di-circuits of D. Apply the second half of Theorem 6.3. Observe that each di-circuit of D contains exactly 1 F-edge and that the di-circuits in the optimal covering of U can be made pairwise disjoint in U by the transitivity of D' and hence their restrictions to U are disjoint di-paths of D. •

It should be noted that the Greene-Kleitman theorem was derived by Cameron and Edmonds directly from Theorem 6.1.

We show now how Theorem 6.3 gives rise to a min-max formula for the maximum weight of the k-union of sink-stable sets.

THEOREM 6.5. *Let $D = (V, A)$ be a digraph with no isolated nodes, $k \geq 2$ an integer, and $w : V \to \mathbf{Z}_+$ an integer-valued weight-function on the node-set of D. Then*

$$\max\{\widetilde{w}(S) : S \subseteq V \text{ a } k\text{-sink-stable set}\} =$$

$$\min_{y:\mathcal{C}_D \to \mathbf{Z}_+} \left\{ k \sum_{C \in \mathcal{C}_D} y(C)\eta(C) + \sum_{v \in V} \left(w(v) - \sum_{C \in \mathcal{C}_D, v \in V(C)} y(C) \right)^+ \right\} \tag{16}$$

where $\mathcal{C}_D$ denotes the set of circuits of D. More concisely, the linear system $\{Qx \leq k\eta,\ 0 \leq x \leq \underline{1}\}$ is totally dual integral where Q is the circuit versus node incidence matrix of D while η is a vector the component of which corresponding to a circuit C is $\eta(C)$.

Proof. Let S be a k-sink-stable set. By Theorem 3.3, $|S \cap V(C)| \leq k\eta(C)$ for every circuit C of D and hence $z = \underline{\chi}_S$ satisfies the primal constraints $\{Qx \leq k\eta,\ 0 \leq x \leq \underline{1}\}$. The trivial direction $\max \leq \min$ of the l.p. duality theorem implies that $\max \leq \min$ holds in the theorem.

To prove the reverse direction, consider the digraph $D^* = (V, A \cup A')$ arising from D by adding the reverse of each edge of D. Here A' denotes the set of reverse edges of D. Clearly, $F := A'$ is a flat subset since the pair $\{a, a'\}$ is a two-element di-circuit for every $a \in A$.

Claim 6.6. *A subset S is a sink-stable set of D if and only if S is an F-stable set of D^*.*

Proof. Let D_2 be the digraph arising from D by duplicating in parallel each edge of D. Clearly, S is sink-stable in D if and only if it is sink-stable in D_2. On the other hand, D_2 can be obtained from D^* by reorienting $F = A'$, and hence S is sink-stable in D_2 if and only if it is F-stable in D^*. •

It follows from the claim that a subset S of nodes is the k-union of sink-stable sets of D if and only if it is the k-union of F-stable sets of D^*. By Theorem 6.3, the maximum

w-weight of a k-union of F-stable sets of D^* is equal to

$$\min_{y:\mathcal{K}\to\mathbf{Z}_+}\left\{k\sum_{K\in\mathcal{K}^*} y(K)|F\cap K| + \sum_{v\in V}\left(w(v)-\sum_{K\in\mathcal{K}^*,v\in V(K)} y(K)\right)^+\right\} \tag{17}$$

where $\mathcal{K}^*$ denotes the sets of di-circuits of D^*

Recall that D^* has two types of di-circuits. A Type I di-circuits is of form $K_a = \{a, a'\}$ where $a \in A$ while Type II di-circuits arise from circuits of D by replacing each forward edge by its reverse or by replacing each backward edge by its reverse.

Consider an optimal (integer-valued) solution y^* to (17) and let $z^*(v) := w(v) - \sum[y^*(K):\ K\in\mathcal{K}^*, v\in V(K)]$. Now $z'(v) = z^*(v)$ for $v\in V-\{s,t\}$, $z'(s) = z^*(s)+\alpha$, and $z'(t) = z^*(t)+\alpha$. Hence

$$\sum_{v\in V}(z'(v))^+ \le \sum_{v\in V}(z^*(v))^+ + 2\alpha.$$

Furthermore, $|F\cap K_a| = 1$ implies that $\sum_{K\in\mathcal{K}^*} y'(K)|F\cap K| = k\sum_{K\in\mathcal{K}^*} y^*(K)|F\cap K| - k\alpha$. By combining these observations with the assumption $k\ge 2$, we obtain that

$$k\sum_{K\in\mathcal{K}^*} y'(K)|F\cap K| + \sum_{v\in V}\left(w(v)-\sum_{K\in\mathcal{K}^*,v\in V(K)} y'(K)\right)^+ \le$$

$$k\sum_{K\in\mathcal{K}^*} y^*(K)|F\cap K| + \sum_{v\in V}\left(w(v)-\sum_{K\in\mathcal{K}^*,v\in V(K)} y^*(K)\right)^+.$$

Since y^* is an optimal solution to (17), so is y' (and we must have $k=2$), contradicting the special choice of y^*. This contradiction shows that $y^*(K)=0$ for every di-circuit in D^* of Type

Suppose now that K is a di-circuit of D^* of Type II. By reversing the F-edges of K, we obtain a circuit C of D for which $\eta(C)\le|F\cap K|$. Let $y_0(C) := y^*(K)$. If $y^*(K)>0$ for a di-circuit of D^*, then we must actually have $\eta(C) = |F\cap K|$. Indeed, for if $\eta(C) < |F\cap K|$, then $|F'\cap K| = \eta(C) < |F\cap K|$ holds for the reverse di-circuit K' of K and then by increasing $y^*(K)$ with $y^*(K)$ and reducing $y^*(K)$ to 0 we would obtain a solution to (17) which is better than the optimal y^*. It follows from Theorem 6.3 that y_0 is a solution to (16) for which $\max\{w(S)\ :\ S \text{ a } k\text{-sink-set}\} = k\sum_{C\in\mathcal{C}_D} y_0(C)\eta(C) + \sum_{v\in V}(w(v) - \sum_{C\in\mathcal{C}_D,v\in V(C)} y_0(C)\ )^+$ from which the min-max result of the theorem follows. • •

In the special case $w = \underline{\chi}_U$, we obtain the following.

THEOREM 6.7. *Let $D=(V,A)$ be a digraph with no isolated nodes, $k\ge 2$ an integer, and $U\subseteq V$ a prescribed subset of nodes. The maximum cardinality of a k-sink-stable subset of U is equal to*

$$\min\{k\sum[\eta(C) : C\in\mathcal{C}] + |U - \cup(V(C) : C\in\mathcal{C})| : \mathcal{C} \textit{ a set of circuits}\}.\ \bullet \tag{18}$$

The same way as the theorem of Abeledo and Atkinson (Theorem 5.1) was derived from Theorem 4.7, the following result can be obtained from the second part of Theorem 6.3.

THEOREM 6.8. *Let $G = (S, T; E)$ be a 2-connected perfectly matchable bipartite plane graph and $k \geq 2$ an integer. The maximum number of faces that can be partitioned into k resonant sets is equal to the minimum of*

$$k \sum_i \textit{val}(B_i) + \textit{the number of faces avoided by each } B_i$$

where the minimum is taken over all choices of feasible cuts $B_1, \ldots, B_q$. In particular, the set of faces of G can be partitioned into k resonant sets if and only if for any set $\mathcal{B}$ of feasible cuts the number of faces intersected by $\mathcal{B}$ is at most k times the total value of $\mathcal{B}$. •

Corollary 6.9. *The faces of a 2-connected perfectly matchable bipartite plane graph can be partitioned into k resonant sets if and only if, for every feasible cut B of G, the number of faces intersected by B is at most $k\mathrm{val}(B)$.* •

7 Link to cyclic stable sets

Suppose that $D = (V, A)$ is a strongly connected loopless digraph on $n \geq 2$ nodes and consider a linear order $\mathcal{L} = [v_1, \ldots, v_n]$ of the nodes of D. An edge e of D is a forward edge if its tail precedes its head, otherwise e is a backward edge.

Let P be a regular n-gon in a horizontal plane and assign the nodes of V to the vertices of P in this order. In this way, we arrive at a cyclic order $\mathcal{O} = (v_1, \ldots, v_n)$ of D. A set of consecutive elements is called an **interval** of $\mathcal{O}$. For example, both $\{v_2, v_3, v_4\}$ and $\{v_{n-1}, v_n, v_1, v_2\}$ are intervals. Each edge uv of D can be represented in the plane by an arc going clockwise outside P. Clearly, the linear order $[v_i, \ldots, v_n, v_1, \ldots, v_{i-1}]$ defines the same cyclic order for each v_i. Each of these n linear orders is called an opening of $\mathcal{O}$. The backward edges of the opening $\mathcal{L}' = [v_i, \ldots, v_n, v_1, \ldots, v_{i-1}]$ of $\mathcal{O}$ is called the edge-set **belonging** to the opening.

Let K be a di-circuit of D. Starting from a node v of K and going along K we arrive back to v. In the plane, this simple closed walk goes around P one ore more times. This number is called the winding number or the index of K and is denoted by $\mathrm{ind}(K)$. It follows from this definition that if F denotes the set of edges belonging to an opening of $\mathcal{O}$, then

$$\mathrm{ind}(K) = |F \cap K|. \tag{19}$$

For example, if D itself is a di-circuit K consisting of the edges $\{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$, then the index of K with respect to the cyclic order $(v_1, \ldots, v_n)$ is 1 while $\mathrm{ind}(K) = n - 1$ with respect to the reverse cyclic order $(v_n, \ldots, v_1)$.

These notions were introduced by Bessy and Thomassé [3] who called a cyclic order of D **coherent** if each edge of D belongs to a di-circuit of index 1. They proved that every strong digraph has a coherent ordering. Let $\mathcal{O}$ be a cyclic ordering and F the set of edges belonging to an opening of $\mathcal{O}$.

Iwata and Matsuda [15] observed the following link between flat transversals of di-circuits and coherent cyclic orderings.

Lemma 7.1 (Iwata and Matsuda). *Let $D = (V, A)$ be a strongly connected digraph. A subset F of edges is a flat transversal of di-circuits if and only of F belongs to an opening of a coherent ordering of D.*

Proof. (Outline) If F belongs to an opening of a cyclic order $\mathcal{O}$, then F is clearly a transversal of di-circuits. If $\mathcal{O}$ is, in addition, coherent, that is, if each edge belongs to a di-circuit of index 1, then F is flat since $\text{ind}(K) = |F \cap K|$ for every di-circuit.

Conversely, if F is a flat transversal of di-circuits, then F is certainly a minimal transversal with respect to inclusion. An easy excercise shows that the digraph D_F arising from D by reversing the elements of F is acyclic. Hence any topological ordering $\mathcal{L}$ of D_F has the property that the elements of F (in D) are precisely the backward edges. Therefore the cyclic order determined by $\mathcal{L}$ is coherent. •

Due to this correspondence, the existence of a coherent cyclic order is equivalent to Knuth's lemma on the existence of a flat transversal of di-circuits.

Bessy and Thomasse called the exchange of two consecutive elements u and v in a cyclic order **elementary** if there is no edge (in either direction) between u and v. They called two cyclic orders equivalent if one can be obtained from the other by a sequence of elementary exchanges. Finally, a stable set of nodes is **cyclic stable** with respect to a given cyclic order $\mathcal{O}$ if there is an equivalent cyclic order where S forms an interval.

>From a complexity point of view, a slight disadvantage of this definition is that it does not show (as it is) that cyclic stability is an NP-property. Indeed, in principle it could be the case that a cyclic order can be obtained from an equivalent cyclic order only by a sequence of exponentially many elementary exchanges, and in such a case the definition would not provide a polynomally checkable certificate for cyclic stability. A. Sebő [20], however, pointed out that a cyclic ordering can always be obtained from an equivalent cyclic order by a sequence of at most n^2 elementary exchanges. Hence cyclic stability is an NP-property. Moreover, Sebő proved the following co-NP characterization of cyclic stability.

THEOREM 7.2 ([19], Statement (5)). *A subset S of nodes of a strongly connected digraph D is cyclic stable with respect to a coherent cyclic ordering if and only if $|S \cap V(K)| \leq$ind(K) for every di-circuit K of D.* •

The proof of this theorem provides a polynomial algorithm that either finds a sequence of elementary exchanges that transform S into an interval or else it finds a di-circuit K violating the inequality in the theorem.

The two main theorems of Bessy and Thomassé [3] are as follows.

THEOREM 7.3 (Bessy and Thomassé). *Given a strong digraph $D = (V, A)$ along with a coherent cyclic ordering, the maximum cardinality of a cyclic stable set of D is equal to the minimum total index of di-circuits covering V.* •

THEOREM 7.4 (Bessy and Thomassé). *Let $D = (V, A)$ be a strong digraph along with a coherent cyclic ordering and let $k \geq 2$ be an integer. The node-set of D can be partitioned into k cyclic stable sets if and only if $|K| \leq k\ \text{ind}(K)$ for every di-circuit K.* •

As a common generalization of the theorems of Bessy and Thomassé, Sebő [19] proved the following.

THEOREM 7.5 (Sebő). *Let $D = (V, A)$ be a strong digraph along with a coherent cyclic ordering. Let $k \geq 1$ be an integer and U a subset of nodes. The maximum cardinality of the union of k cyclic stable sets of D is equal to* $\min\{k \sum_i \text{ind}(K_i) + |U - \cup_i V(K_i)| :$ $\{K_1, \ldots, K_q\}$ *a set of di-circuits*$\}$.

Sebő actually proved this result in a more general form by providing a min-max formula for the maximum w-weight of the k-union of cyclic stable sets.

By combining Theorems 3.5 and 7.2, we obtain by (19) the following.

Lemma 7.6. *Let F be a flat transversal of di-circuit of a strong digraph D and let $\mathcal{O} = (v_1, \ldots, v_n)$ be a coherent cyclic order so that F belongs to an opening of $\mathcal{O}$. Then a subset S of nodes is F-stable if and only if S is cyclic stable.* •

This lemma implies that Theorem 7.5 is equivalent to the that special case of the second half of Theorem 6.3 when F is not only flat but it is a transversal of di-circuits, as-well. Note that requiring only the flatness of F in Theorem 6.3 allowed us to derive the theorem of Abeledo and Atkinson and its extension.

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